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A Note on Conjugacy Class Sizes of Finite Groups.

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Dedicated to prof. Cesarina Tibiletti on her eightieth birthday.

Let G be a finite group, and let

$$1 = k_0 < k_1 < \ldots < k_r$$

be all the distinct sizes of its conjugacy classes. The arithmetical nature of this sequence is far from being arbitrary, and clearly it has to enjoy rather special, if not yet understood, properties. Often, imposing hypotheses on the sequence, or just on some of its members, yields strong restrictions on the structure of the group G (e.g. as in Burnside's Lemma). For instance, it was first proved by Kazarin [3] (we are grateful to the referee for this reference), and later by Bertram, Herzog and Mann [1] and, independently, Dolfi [2], that if the set of sizes of the non central conjugacy classes of G can be partitioned in two non-empty subsets such that each member of one subset is coprime to any member of the other subset (fur further reference, a group with this property we call a *class separable* group), then G is metabelian and $G/Z(G)$ is a Frobenius group.

In this note we show that the same conclusion holds if one just assumes that the two largest sizes are coprime. More precisely, we prove

THEOREM. *Let G be a finite group, and $n < m$ be the two largest sizes of the non-central conjugacy classes of G. Let $a, b \in G$ whose con-*

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jugacy classes have sizes n, m *respectively, and set* $N = C_G(a)$, $H = C_G(b)$. *Assume that* $(n, m) = 1$; *then*

(i) N, H *are abelian and* $N \cap H = Z(G)$;

(ii) $G/Z(G)$ *is a Frobenius group with kernel* $N/Z(G)$ *and complement* $H/Z(G)$.

We adopt the following notations. If $x \in G$, then $[x]$ denotes the conjugacy class of x in G. If B is a conjugacy class, $B^{-1} = \{b^{-1} \mid b \in B\}$ is also a conjugacy class. If $S \neq \emptyset$ is a subset of G, following [1] we set

$$K_S = \{x \in G \mid xS = S\}.$$

Clearly, K_S is a subgroup of G, and it is normal if S is a normal subset. Also, S is the union of right cosets of K_S, whence $|K_S|$ divides $|S|$. The rest of our notation is standard.

In our arguments we will make use of some ideas of Bertram, Herzog and Mann, which prove to be quite useful when dealing with classes of coprime sizes, and that we introduce in the following Lemmas, the first of those goes back to Burnside.

LEMMA 1 ([1]). *Let* $A = [a]$, $B = [b]$ *be conjugacy classes of the finite group* G. *If* $(|A|, |B|) = 1$ *then* AB *is a conjugacy class and* $C_G(a)C_G(b) = G$.

LEMMA 2 ([4]). *Let* B *be a conjugacy class of largest size of the finite group* G. *Let* A *be a conjugacy class of* G *such that* $(|A|, |B|) = 1$. *Then* $\langle A^{-1}A \rangle \leq K_B$, *and* $\langle A \rangle$ *is a normal abelian subgroup of* G.

We now come to the proof of the Theorem. Let $n < m$ with $(n, m) = 1$ be the largest sizes of the non-central conjugacy classes of G. Let $a, b \in G$ with $|[a]| = n$, $|[b]| = m$; put $A = [a]$, $B = [b]$ and $N = C_G(a)$, $H = C_G(b)$. We proceed in a number of steps.

STEP 1. *Let* $x \in Z(H)$; *then either* $x \in Z(G)$ *or* $C_G(x) = H$.

PROOF. Let $x \in Z(H)$ and $X = [x]$. As $C_G(x) \geq H$, $|X|$ divides $|B|$. Suppose $C_G(x) \neq H$. Then $|X| < |B|$, so $|X| < |A|$ and $(|X|, |A|) = 1$. Then XA is a conjugacy class and, as $|XA| \geq |A|$, it follows that either $|XA| = |B|$ or $|XA| = |A|$. If it were $|XA| = |B|$ then, by Lemma 1, XAA^{-1} would be a conjugacy class containing X, so that $X = XAA^{-1}$; on the other hand $|XAA^{-1}| \geqslant |A|$ and so $|X| = |XAA^{-1}| \geqslant |A|$, a contradiction. Therefore $|XA| = |A|$; then by Lemma 2, $\langle (XA)^{-1}XA \rangle \leqslant K_B$, yielding $\langle X^{-1}X \rangle \leqslant K_B$ as K_B is normal. Also, since $|XA| = |A|$, $X^{-1}XA$ is a class, so $A = X^{-1}XA$ and $\langle X^{-1}X \rangle \leqslant K_A$. This implies $\langle X^{-1}X \rangle = \{1\}$ since $K_A \cap K_B = \{1\}$ (in fact $|K_A|$ divides $|A|$ and $|K_B|$ divides $|B|$), so we have $X = \{x\}$ and $x \in Z(G)$.

STEP 2. *We may assume* $|b| = p^k$ *for some prime* p.

PROOF. Let $x = b^t$ be a primary component of b such that $C_G(x) \neq G$. Then, by Step 1, $C_G(x) = H$, and we can substitute b by x.

STEP 3. *Let* $P \in Syl_p(H)$. *The* $H = P \times L$, *where* L *is an abelian* p'*-group.*

PROOF. Let y be a p'-element of H. As y permutes with b, and $(|y|, |b|) = 1$, we have $C_G(yb) = C_G(y) \cap C_G(b)$. By our choice of $B = [b]$ as a class of maximal size, we get $H = C_G(yb)$; hence $H \leqslant C_G(y)$, thus proving the assertion.

STEP 4. p *does not divide* m.

PROOF. By Lemma 1, $G = NH$. Suppose, by contradiction, that $p|m$. Then p does not divide $n = [G : N] = [H : H \cap N]$. Thus $P \leqslant N \cap H$ and so $a \in C_G(b) = H$ (as $b \in P$), yielding $L \leqslant C_G(a)$, and the contradiction $H = P \times L \leqslant N$.

STEP 5. $K_B \cap H = \{1\}$.

PROOF. Let $x \in K_B \cap H$. Then $xb \in B$, so $xb = b^g$ for some $g \in G$. As $x \in H = C_G(b)$ we get

$$x^{p^k} = x^{p^k} b^{p^k} = (xb)^{p^k} = (b^g)^{p^k} = (b^{p^k})^g = 1\ .$$

But $|K_B|$ divides $|B| = m$, and this, by Step 4, implies that p does not divide $|K_B|$ thus forcing $x = 1$.

STEP 6. *We can choose* a *to be a* p'*-element.*

PROOF. Let $a_1 = a^s$ be the p-component of a. Then $a_1 \in H^w$ for some $w \in G$. Now, $G = NH^w$ and every $g \in G$ can be written as $g = uv$ with

$u \in N$, $v \in H^w$. Then, being $\langle A \rangle$ normal and abelian

$$[a_1, g] = [a^s, g] = [a, g]^s = (a^{-1} a^g)^s \in K_B;$$

but we have also

$$[a_1, g] = [a_1, uv] = [a_1, v][a_1, u]^v = [a_1, v] \in H^w.$$

By Step 5, $[a_1, g] \in K_B \cap H^w = (K_B \cap H)^w = \{1\}$. Hence $a_1 \in Z(G)$ and we can replace a by aa_1^{-1}.

STEP 7. *Assume H not abelian; then $L \leqslant Z(G)$ and $|A| = p^l$ for some $l \in \mathbb{N}$.*

PROOF. Deny, and suppose the existence of a prime divisors q of $|L|$ and of a q-element $x \in L \backslash Z(G)$. Now, Step 1 assures that $C_G(x) = H$. But then if $Q \in Syl_q(L)$, arguing as in Step 3, we get $H = Q \times T$ with T abelian. This means in particular that $P \leqslant T$ is abelian, a contradiction.

STEP 8. $N \cap H = Z(G)$.

PROOF. Clearly, $Z(G) \leqslant N \cap H$. To prove the reverse inclusion, we consider separately the cases H abelian and H not abelian (we will see later on that the latter case does not occur). If H is abelian, and $y \in N \cap$ $\cap H$, then $C_G(y) \geqslant \langle H, a \rangle > H$, so, by Step 1, $y \in Z(G)$. Let H be not abelian. Then $L \leqslant Z(G)$ by Step 7. Let $y \in P \cap N$, then as a is a p'-element, $C_G(ya) = C_G(a) \cap C_G(y)$, so $C_G(ya) = C_G(a)$ and, arguing as in Step 6, we get $y \in Z(G)$.

We now set $K = \langle A \rangle$. By Lemma 2 and Step 6, K is a normal abelian p'-subgroup of G.

STEP 9. *Assume H not abelian. Then $P/P \cap Z(G)$ acts as a group of fixed point free automorphisms on $K/K \cap Z(G)$.*

PROOF. Write $P_0 = P \cap Z(G)$ and $K_0 = K \cap Z(G)$. By Steps 7 and 8, $P_0 = C_P(K)$ and $[P : P_0] = p^l = n$. Suppose, by contradiction, that there exists $x \in P \backslash P_0$ such that $C_K(x) > K_0$. Observe that p divides $|[x]|$; for otherwise, $x \in Z(P^c)$ for some $c \in G$ and so $x \in Z(H^c)$, whence $C_G(x) = H^c$ by Step 1, and $C_K(x) = H^c \cap K = (H \cap K)^c \leqslant Z(G) \cap K = K_0$, against our assumption on x.

Now, by coprime action, we have the decomposition, $K = [K, x] \times$

$\times C_K(x)$, and we can write $a = uw$ with $u \in [K, x]$, $w \in C_K(x)$. Take $g = wx$; then $C_G(g) = C_G(w) \cap C_G(x)$, so by what we observed above, p divides $|[g]|$.

Also, $C_G(g) \geqslant \langle P_0, x\rangle$, whence $|[g]| \neq p^l$. By our hypotheses on n, m we then have $|[g]| < n = [P : P_0]$. In particular, $[KP : C_{KP}(g)] < n$. But since K is normal and abelian, $C_{KP}(g) = C_K(x)D$, where $D = C_P(x) \cap \cap C_P(w)$. So

$$[P : P_0] = n > [K : C_K(x)][P : D] = |[K, x]|[P : D],$$

yielding $[D : P_0] > |[K, x]|$. As $D \leqslant C_P(x)$ acts on $[K, x]$, this implies that $C_D(u) > P_0$, so there exists $y \in C_D(u) \backslash P_0$. But $y \in D \leqslant C_P(w)$ and, consequently, $y \in C_P(u) \cap C_P(w) \leqslant C_P(uw) = C_P(a) = N$, whence $y \in H \cap N = Z(G)$, a contradiction.

STEP 10. *H is abelian.*

PROOF. By Step 3, $H = P \times L$, where L is abelian. By Step 9, $P/P \cap \cap Z(G)$ is cyclic or generalized quaternion. So, if P is not abelian, $p = 2$ and $P/P \cap Z(G)$ is generalized quaternion. Now, $b \in Z(P) \backslash Z(G)$, so there exists, in this case, an element $y \in P \backslash Z(P)$ such that $b = y^2 c$ with $c \in \in Z(G) \cap P$. But then $C_G(y) \leqslant C_g(b) = H$ which implies $C_G(y) = H$ yielding the contradiction $y \in Z(P)$.

STEP 11. *Conclusion.*

PROOF. Let π be the set of prime divisors of $m = [G : H] = [N : N \cap \cap H]$. By Step 8, $N \cap H = Z(G)$, hence $O^\pi(N) \leqslant Z(G)$, and so $N = R \times W$, where $R = O_\pi(N)$ and $W \leqslant Z(G)$.

Let $y \in R \backslash Z(G)$. Then $C_H(y) = Z(G)$; in fact, suppose there exists $x \in C_H(y) \backslash Z(G)$, then $C_G(x) = H$, whence $y \in H \cap N = Z(G)$, against the choice of y. Thus, setting $Y = [y]$, we get

$$|Y| \geqslant [H : C_H(y)] = [H : Z(G)] = n = |A|.$$

Since $C(y) \geqslant \langle Z(G), y\rangle$ and y is a non-central π-element, we have that the π-part of $|Y|$ is smaller than m, hence $|Y| = n$. Therefore, by Lemma 2, $\langle Y\rangle$ is a normal abelian π-group. As $(|R|, [G : N]) = 1$, it follows $\langle Y\rangle \leqslant O_\pi(G) \leqslant R$. The same argument holds for any $y \in R \backslash Z(G)$ and so we obtain that $R = O_\pi(G)$ is normal in G and nilpotent. Hence $N = RZ(G)$ is normal and nilpotent; it is now immediate to conclude that $G/Z(G)$ is a Frobenius group with kernel $N/Z(G)$ and complement

$H/Z(G)$. It remains to show that N is abelian. Let $x \in N \setminus Z(G)$; then

$$C_G(x)/Z(G) \leq C_{G/Z(G)}(xZ(G)) \leq N/Z(G),$$

thus $C_G(x) \leq N$ which, by our hypotheses on n, m implies $C_G(x) = N$, thus completing the proof.

As we mentioned at the beginning, by results in [1] and [2], the finite groups satisfying the conclusion of our Theorem are just the class-separable groups. So we have the following

COROLLAY 1. *A finite group G is class separable if and only if the sizes of the two largest non-central conjugacy classes of G are coprime.*

We also observe that such groups have just three class sizes, namely 1, $|N/Z(G)|$, $|H/Z(G)|$. Thus, we have also

COROLLARY 2. *Let $1 < k_1 < k_2 < \ldots < k_r$ be the sizes of the conjugacy classes of a finite group. Then, if $r \geq 3$, $(k_{r-1}, k_r) \neq 1$.*

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