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Quasi-isomorphism and $\mathbb{Z}_{(2)}$-Representations for a Class of Butler Groups.

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ABSTRACT - A $\mathcal{B}^{(1)}$-group is a finite rank torsion-free abelian group which is a homomorphic image, with rank 1 kernel, of a completely decomposable group. The study of these groups reduces to that of the special form $G = G[A_1, \ldots, A_n]$, which is the cokernel of the diagonal imbedding of $A_1 \cap \ldots \cap A_n$ into $A_1 \oplus \ldots \oplus A_n$, where the A_i's form an n-tuple of nonzero subgroups of the additive group of rational numbers. With any such group G we associate in a canonical manner a vector space representation $\mathcal{R}_G$, over the 2 element field $\mathbb{Z}_{(2)}$, of the poset typeset (G), consisting of the types realized by the nonzero elements of G. Let $H = G[B_1, \ldots, B_n]$ be another such group with the same typeset. We prove that G and H are quasi-isomorphic if and only if $\mathcal{R}_G$ and $\mathcal{R}_H$ are isomorphic representations.

1. - $\mathcal{B}^{(1)}$-Groups..

In this paper, we continue earlier investigations of the quasi-isomorphism problem for that special class of Butler groups which occur as homomorphic images, with rank 1 kernel, of finite rank completely decomposable groups. This is the class $\mathcal{B}^{(1)}$ of Fuchs and Metelli [11] and the «Butler dual» [6] of the class studied in a series of papers by Arnold and Vinsonhaler, namely, those groups that occur as corank 1 pure subgroups of finite rank completely decomposable groups. Indeed all results

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obtained here apply with equal force to this latter class via what is by now a routine dualization process. Two recent papers contain quite different characterizations of when strongly indecomposable $\mathcal{B}^{(1)}$-groups G and H are quasi-isomorphic. In [5] and [6] a characterization is given in terms of the equality of ranks of a finite collection of functorially defined fully-invariant subgroups of G and H; while in [11] the characterization involves properties of a certain $\{0, 1\}$-matrix $\mathcal{E}$ that transforms a distinguished set of types for G into a corresponding set of types for H. We shall give yet another characterization of the quasi-isomorphism problem for $\mathcal{B}^{(1)}$-groups and furthermore illuminate the heretofore rather mysterious nature of the Fuchs-Metelli matrix $\mathcal{E}$. In fact, when G and H are quasi-isomorphic, then $\mathcal{E}$ is just the matrix with entries in the field $\mathbb{Z}_{(2)} = \{0, 1\}$ associated with a linear transformation that exhibits an equivalence between canonically defined $\mathbb{Z}_{(2)}$-representations of the posets typeset (G) and typeset (H). Although our solution of the quasi-isomorphism problem for $\mathcal{B}^{(1)}$-groups in terms of $\mathbb{Z}_{(2)}$-representations is more in the spirit of Fuchs-Metelli rather than of Arnold-Vinsonhaler, we show that the characterization given in [5] is a consequence of the one obtained in [11].

As we shall shortly note, it suffices to deal with groups that arise from the following special construction: Let $\mathcal{A} = (A_1, \ldots, A_n)$ be an n-tuple of nonzero subgroups of the group $\mathbb{Q}$ of rational numbers, and let $G[\mathcal{A}] = G[A_1, \ldots, A_n]$ denote the cokernel of the diagonal map $\Delta_{\mathcal{A}}: \bigcap_{i=1}^{n} A_i \to \bigoplus_{i=1}^{n} A_i$. Thus we have a canonical epimorphism $\pi_{\mathcal{A}}: A_1 \oplus \oplus \ldots \oplus A_n \to G[A_1, \ldots, A_n]$ where $\pi_{\mathcal{A}}(x_1, \ldots, x_n) = 0$ if and only if $x_1 = = \ldots = x_n$. Since clearly Im $\Delta_{\mathcal{A}}$ is a rank 1 pure subgroup of $\bigoplus_{i=1}^{n} A_i$, $G[A_1, \ldots, A_n]$ is a $\mathcal{B}^{(1)}$-group of rank $n-1$. No real loss of generality is involved in limiting our attention to groups of the form $G[\mathcal{A}]$; for if $\pi : A_1 \oplus \ldots \oplus A_n \to G$ is an epimorhpism with Ker π a rank 1 pure subgroup, then it is easy to see that modulo a permutation there is an $m \leqslant n$ such that G is isomorphic to the direct sum of $A_{m+1}, \ldots, A_n$ and a group quasi-isomorphic to $G[A_1, \ldots, A_m]$. We shall henceforth write $\langle x_1, \ldots, x_n \rangle$ for $\pi_{\mathcal{A}}(x_1, \ldots, x_n)$, and so $\langle x_1, \ldots, x_n \rangle = \langle y_1, \ldots, y_n \rangle$ if and only if there is an r in $\cap_{i=1}^{n} A_i$ such that $(x_1 - y_1, \ldots, x_n - y_n) = r 1_n$ where 1_n denotes the n-vector with all components equal to the rational integer 1. Up to quasi-isomorphism, $G[A_1, \ldots, A_n]$ depends only on the isomorphism classes of the subgroups $A_1, \ldots, A_n$.

PROPOSITION 1.1. [15] *If* $A_i \simeq B_i$ *for* $i = 1, \dots, n$, *then the groups* $G = G[A_1, \dots, A_n]$ *and* $H = G[B_1, \dots, B_n]$ *are quasi-isomorphic.*

PROOF. Choose isomorphisms $\phi_i \colon A_i \to B_i$ for $i = 1, \dots n$, and suppose a is a nonzero element of $\bigcap_{i=1}^{n} A_i$. Then there exist nonzero integers $d_1, \dots, d_n$ such that $d_1 \phi_1(a) = \dots = d_n \phi_n(a)$ is an element of $\bigcap_{i=1}^{n} B_i$. Thus if $\Theta : \bigoplus_{i=1}^{n} A_i \to \bigoplus_{i=1}^{n} B_i$ is the direct sum of the maps $d_1 \phi_1, \dots, d_n \phi_n$, then $\Theta \Delta_{\mathcal{A}} \leqslant \Delta_{\mathcal{B}}$ and consequently there is an induced homomorphism $\Psi : G \to H$ such that there is a nonzero integer d with $dH \subseteq \Psi(G)$. It is routine to check that $d = d_1 \dots d_n$ serves this purpose. ■

We now fix further notation that will remain in effect throughout this paper. Henceforth, $G = G[A_1, \dots, A_n]$, for a given n-tuple $(A_1, \dots, A_n)$, with corresponding types $\tau_i = \text{type}(A_i)$ for all $i \in \overline{n}$, where $\overline{n} = \{1, 2, \dots, n\}$. The set $2^{\overline{n}}$ of all subsets of $\overline{n}$ is a boolean algebra with set-theoretical intersection as multiplication and the addition is symmetric difference; given by $I + J = (I \cap J') \cup (I' \cap J) = (I \cup J) \cap (I' \cup J')$, where, of course, $I' = \overline{n} \setminus I$ for each $I \in 2^{\overline{n}}$. Then $\emptyset$ and $\overline{n}$ are, respectively, the zero and identity elements of $2^{\overline{n}}$; and, as a vector space over $\mathbb{Z}_{(2)}$, the dimension of $2^{\overline{n}}$ is n. The importance of the ring $2^{\overline{n}}$ to the study of groups of the form $G[\mathcal{A}]$ is a consequence of the conspicuous description of the typeset of $G = G[A_1, \dots, A_n]$ as given in [11] and [12] (see Theorem 1.1 below). If I is a nonempty subset of $\overline{n}$, then we write τ_I for the type $\bigwedge_{i \in I} \tau_i$. As a matter of convention, $\tau_\emptyset = \infty$, the type determined by the rationals $\mathbb{Q}$.

Since we shall be treating the question of when two groups of the form $G[\mathcal{A}]$ are quasi-isomorphic, it is crucial that we have a general method for constructing homomorphisms between such groups. In fact it is a consequence of Theorem 4.1 below that whenever $G[\mathcal{A}]$ and $G[\mathcal{B}]$ are quasi- isomorphic, there exist monomorphisms between these groups that are induced by certain matrices. Following Fuchs-Metelli, we shall let $\mathcal{E}_k$ denote the matrix obtained from the $\{0, 1\}$-matrix $\mathcal{E}$ by replacing each 0 in the k^{th} column of $\mathcal{E}$ by 1. With a slight abuse of terminology, when $F \subseteq \overline{n}$ we say that $x = \langle x_1, \dots, x_n \rangle$ is *constant on* F provided there is an $a \in \bigcap_{i \in F} A_i$ such that $x_i = a$ for all $i \in F$; clearly this notion is independent of the representation of x.

PROPOSITION 1.2. *Suppose* $\mathcal{E} = [e_{ik}]$ *is an* $n \times m$ $\{0, 1\}$*-matrix with* 1_n *contained in the column space of* $\mathcal{E}$, *and, for each* $k \in \overline{m}$, *let* $E_k =$

$= \{i \in \overline{n}: e_{ik} = 1\}$. *If* $H = G[B_1, \ldots, B_m]$ *where* type $(B_k) \leq \tau_{E_k} \vee \tau_{E_k'}$ *for each* $k \in \overline{m}$, *then there is an induced homomorphism* $\Psi_{\mathcal{E}}: H \to G$. *Moreover, if* rank $\mathcal{E} = m$, *then* Ker $\Psi_{\mathcal{E}}$ *is quasi-equal to the subgroup of H consisting of all* $\langle b_1, \ldots, b_m \rangle$ *that are constant on* $F = \{k \in \overline{m}$: rank $\mathcal{E}_k = m\}$.

PROOF. By Proposition 1.1, there is no loss of generality in assuming, for each $k \in \overline{m}$, that B_k is a subgroup of $\bigcap_{i \in E_k} A_i + \bigcap_{j \in E_k'} A_j$. Thus each $b_k \in B_k$ has a representation $b_k = c_k - d_k$ where $c_k \in \bigcap_{i \in E_k} A_i$ and $d_k \in \bigcap_{\psi \in E_k'} A_j$. This representation of b_k is not unique, but if $b_k = c_k' - d_k'$ is another such representation, then $c_k - c_k' = d_k - d_k'$ is an element of $\bigcap_{i \in \overline{n}} A_i$. From this observation, it follows that there is a well-defined monomorphism $\theta_k: B_k \to G$ given by $\theta_k(b_k) = \langle x_1, \ldots, x_n \rangle$ where $x_i = e_{ik} b_k + d_k$ for all i; that is $x_i = c_k$ if $i \in E_k$ and $x_i = d_k$ if $i \in E_k'$. From the hypothesis that 1_n lies in the column space of $\mathcal{E}$, it follows that there exists integers $a_1, \ldots, a_m$ and $r \neq 0$ such that $\sum_{k=1}^{m} e_{ik} a_k = r$ for all $i \in \overline{n}$. Now take $\Theta_{\mathcal{E}}: B_1 \oplus \ldots \oplus B_m \to G$ to be the map $\Theta_{\mathcal{E}}((b_1, \ldots, b_m)) = \sum_{k=1}^{m} a_k \theta_k(b_k)$.

If we write $\Theta_{\mathcal{E}}((b_1, \ldots, b_m)) = \langle x_1, \ldots, x_n \rangle$, then $x_i = \sum_{k=1}^{m} e_{ik} a_k b_k + dr$ where $d = a_1 d_1 + \ldots + a_m d_m$. Thus if $(b_1, \ldots, b_m)$ is in the image of $\Delta_{\mathcal{B}}$, then there is a $b \in \bigcap_{\kappa \in \overline{m}} B_k$ such that $b = b_k$ for each k and in this case, $x_i = \sum_{k=1}^{m} e_{ik} a_k b + d = rb + d$. Therefore $\Theta_{\mathcal{E}} \Delta_{\mathcal{B}}$ is the zero map and consequently there is an induced homomorphism $\Psi_{\mathcal{E}}: H \to G$ such that $\Psi_{\mathcal{E}} \pi_{\mathcal{B}} = \Theta_{\mathcal{E}}$.

Assume now that rank $\mathcal{E} = m$. Then a routine linear dependency argument shows that $a_k \neq 0$ if and only if rank $\mathcal{E}_k = m$. If $\langle b_1, \ldots, b_m \rangle$ is constant on $F = \{k \in \overline{m}$: rank $\mathcal{E}_k = m\}$, say, $b_k = b$ for all $k \in F$, then $\Psi_{\mathcal{E}}(\langle b_1, \ldots, b_m \rangle) = \langle x_1, \ldots, x_n \rangle$ where $x_i = rb + d$ for all $i \in \overline{n}$ and hence $\langle b_1, \ldots, b_m \rangle$ is in Ker $\Psi_{\mathcal{E}}$. Conversely, suppose $\langle b_1, \ldots, b_m \rangle$ is in Ker $\Psi_{\mathcal{E}}$. Then $\sum_{k=1}^{m} e_{ik} a_k b_k + d$ is a constant independent of i; that is, there is a rational number s such that $\sum_{k=1}^{m} e_{ik} a_k b_k = s$ for all k, so that, $b_k = 0$ for all $k \in F$. In case $s = 0$, the linear independence of the column vectors of $\mathcal{E}$ implies that $a_k b_k = 0$ for all k; thus, $b_k = 0$ for all $k \in F$. On the other hand, if $s \neq 0$, then again by the independence of the column vectors, $\frac{a_k b_k}{s} = \frac{a_k}{r}$ for all k; that is, $b_k = \frac{s}{r}$ for all $k \in F$. ■

COROLLARY 1.1. *Let $I_1, \ldots, I_m$ be a partition of $\overline{n}$ where $m \geq 2$. If $B_k = \bigcap_{i \in I_k} A_i + \bigcap_{j \in I_k'} A_j$ for each $k \in \overline{m}$, then there is a canonical monomorphism $\Psi : G[B_1, \ldots, B_m] \to G[A_1, \ldots, A_n]$ where Im Ψ is the pure subgroup consisting of all $\langle x_1, \ldots, x_n \rangle$ that are constant on each I_k.*

PROOF. Let $\mathcal{E} = [e_{ik}]$ be the $n \times m\{0, 1\}$-matrix where $e_{ik} = 1$ if and only if $i \in I_k$. Thus, in the notation of Proposition 1.2, $I_k = E_k$ for each $k \in \in \overline{m}$. Moreover, since $I_1, \ldots, I_m$ is a partition of $\overline{n}$, the column vectors of $\mathcal{E}$ are linearly independent and their sum is 1_n. Indeed, in the notation of the preceding proof, $a_1 = \ldots = a_m = r = 1$. Therefore $\Psi = \Psi_{\mathcal{E}}$ is a monomorphism. Furthermore, if $\langle x_1, \ldots, x_m \rangle = \Psi(\langle b_1, \ldots, b_m \rangle)$ and if $i \in I_k$, then $x_i = \sum_{k=1}^{m} e_{ik} b_k + d = b_k + d$; that is, $\langle x_1, \ldots, x_n \rangle$ is constant on I_k. Conversely, if $x = \langle x_1, \ldots, x_n \rangle$ is constant on each I_k, then there exist elements $c_1, \ldots, c_m$ such that $x_i = c_i$ whenever $i \in I_k$. Thus, for each $k \in \in \overline{m}$, $c_k \in \bigcap_{i \in I_k} A_i \subset B_k$ and $\langle c_1, \ldots, c_m \rangle$ is an element of $G[B_1, \ldots, B_m]$ such that $\Psi(\langle c_1, \ldots, c_k \rangle) = x$. The purity of Im Ψ is now easily verified. Indeed if $y = \langle y_1, \ldots, y_n \rangle$ is an element of $G = G[A_1, \ldots, A_n]$ and if p is a prime such that $py \in$ Im Ψ, then $py_i = py_j$ for all $i, j \in I_k$; that is, $y_i = y_j$ whenever $i, j \in I_k$ and consequently $y \in$ Im Ψ. ■

COROLLARY 1.2. *If $A_k' = A_k + \bigcap_{i \neq k} A_i$ for each $k \in \overline{n}$, then there is a natural isomorphism between $G[A_1', \ldots, A_n']$ and $G[A_1, \ldots, A_n]$. Furthermore, for each $k \in \overline{n}$, the canonical image of A_k' is pure in $G[A_1', \ldots, A_n']$.*

PROOF. With $I_k = \{k\}$ for each $k \in \overline{n}$, Corollary 1.1 yields a monomorphism $\Psi : G[A_1', \ldots, A_n'] \to G[A_1, \ldots, A_n]$ with Im Ψ pure in $G = = G[A_1, \ldots, A_n]$. But since rank (Im Ψ) $= n - 1$, Ψ is onto. Finally, because $A_k' = A_k' + \bigcap_{i \neq k} A_i'$, the canonical image $\pi_{\mathcal{C}'}(A_k')$ of A_k' is the pure subgroup of $G[A_1', \ldots, A_n']$ consisting of all $\langle x_1, \ldots, x_n \rangle$ that are constant on both $\{k\}$ and $\{k\}'$. ■

The upshot of Corollary 1.2 is that, after replacing the A_i's by the A_i''s, there is no loss of generality in assuming that $A_k = A_k + \bigcap_{i \neq k} A_i$ for all $k \in \overline{n}$ and that each $\tau_k =$ type (A_k) is in the typeset of G. (Under these circumstances, the n-tuple is said to be *cotrimmed* in [7]; while in [11], the group $G = G[A_1, \ldots, A_n]$ is said to be *regularly represented.*) The only advantage in this normalization process, however, seems to be a psychological one, and moreover it fails to be preserved when one passes to quasi- decompositions of G (see § 4). Therefore we shall generally avoid making any such restrictions on the n-tuple $(A_1, \ldots, A_n)$.

Our final corollary of Proposition 1.2 will prove to be of fundamental significance in § 2.

COROLLARY 1.3. *Suppose* $\mathcal{E} = [e_{ik}]$ *is an* $n \times n, \{0, 1\}$*-matrix such that all row sums of* $\mathcal{E}$ *have the same parity and det* $\mathcal{E}$ *is an odd integer. If, for each* $k \in \overline{n}$, type $(B_k) \leq \tau_{E_k} \vee \tau_{E_k'}$ *where* $E_k = \{i \in \overline{n} : e_{ik} = 1\}$, *then there is a monomorphism* $\Psi_{\mathcal{E}} : G[B_1, \ldots, B_n] \to G[A_1, \ldots, A_n]$.

PROOF. Since $\mathcal{E}$ is nonsingular, 1_n is in the column space of $\mathcal{E}$ and, by Proposition 1.2, it suffices to show that rank $\mathcal{E}_k = n$ for all $k \in \overline{n}$. In other words, we need to prove that det $\mathcal{E}_k \neq 0$ for all $k \in \overline{n}$. This end appears to be most easily achieved by means of linear algebra over the field $\mathbb{Z}_{(2)}$. To wit, if A is a $n \times n, \{0, 1\}$-matrix, then $\det_2 A$ denotes $\mathbb{Z}_{(2)}$-determinant of A. From the familiar axiomatic characterization of determinants, it follows that $\det_2 A = \theta(\det A)$ where $\theta : \mathbb{Z} \to \mathbb{Z}_{(2)}$ is the canonical ring homomorphism; that is, $\det_2 A = 1$ when $\det A$ is odd and $\det_2 A = 0$ when $\det A$ is even. In particular, $det_2 \mathcal{E} = 1$. Consequently, we need only show that $\det_2 \mathcal{E}_k \neq 0$ for each $k \in \overline{n}$.

Notice that the hypotheses on $\mathcal{E}$ imply that all row sums are necessarily odd; for otherwise the column vectors of $\mathcal{E}$ would not be linearly independent in $\mathbb{Z}^n_{(2)}$. Next observe by the linearity of $\det_2$ that $\det_2 \mathcal{E}_k = \det_2 \mathcal{E} + \det_2 \mathcal{F}$ where $\mathcal{F}$ is an $n \times n, \{0, 1\}$-matrix with all row sums of even parity. Hence the column vectors of $\mathcal{F}$ are linearly independent in $\mathbb{Z}^n_{(2)}$; that is, $\det_2 \mathcal{F} = 0$ and therefore $\det_2 \mathcal{E}_k = \det_2 \mathcal{E} \neq 0$, as needed. ■

Recall that typeset $(G) = \{\text{type }(x) : x \in G \text{ and } x \neq 0\}$ where type (x) is the type associated with the rank 1 pure subgroup of G generated by x. For $x = \langle x_1, \ldots, x_n \rangle$ an element of $G = G[A_1, \ldots, A_n]$, type (x) can be calculated explicitly in terms of the types of the A_i's and the subsets of $\overline{n}$ on x which is constant. We defer to [11] or [12] for a proof of the following elementary but indispensable result.

THEOREM 1.1. ([11], [12]) *If* $x = \langle x_1, \ldots, x_n \rangle$ *is a nonzero element of* $G = G[A_1, \ldots, A_n]$, *then*

$$\text{type }(x) = \bigwedge_{k \in \overline{s}} (\tau_{J_k} \vee \tau_{J_k'}) = \bigwedge_{1 \leq k < l \leq s} (\tau_{J_k} \vee \tau_{J_l'})$$

where $J_1, \ldots, J_s$ is the partition of $\overline{n}$ defined by the requirement that i and j lie in the same J_k if and only if $x_i = x_j$.

The preceding theorem, of course, yields a characterization of the typeset of G; namely, $\sigma \in$ typeset (G) if and only if there exists some partition $I_1, \ldots, I_m$ of $\overline{n}$ that *determines* σ in the sense that

$$(*) \quad \sigma = \bigwedge_{k \in \overline{m}} (\tau_{I_k} \vee \tau_{I_k'}) = \bigwedge_{1 \leq k < l \leq m} (\tau_{I_k} \vee \tau_{I_l}).$$

There may be more than one such partition of $\overline{n}$ that determines σ, but as we shall see in Theorem 1.2 below there is always a finest such partition determining σ. But first we shall work with the identity $(*)$ to establish a fundamental computational lemma (recall that $I + J$ represents the symmetric difference of the sets I and J).

LEMMA 1.1. *Let $I_1, \ldots, I_m$ and E be subsets of $\overline{n}$.*

(a) *If $E = I_1 + \ldots + I_m$, then $\tau_E \vee \tau_{E'} \geq \bigwedge_{k \in \overline{m}} (\tau_{I_k} \vee \tau_{I_k'})$.*

(b) *If $I_1, \ldots, I_m$ is a partition of E, then*

$$\bigwedge_{k \in \overline{m}} (\tau_{I_k} \vee \tau_{I_k'}) = (\tau_E \vee \tau_{E'}) \wedge \bigwedge_{1 \leq k \leq l \leq m} (\tau_{I_k} \vee \tau_{I_l}).$$

PROOF. (a). By induction, it suffices to consider the case $m = 2$. Assume $E = I + J = (I \cup J) \cap (I' \cup J')$. Then $E' = I + J' = (I \cup J') \cap \cap (I' \cup J)$ and hence

$$\tau_E \vee \tau_{E'} \geq (\tau_{I \cup J} \vee \tau_{I' \cup J'}) \vee (\tau_{I \cup J'} \vee \tau_{I' \cup J})$$

$$= (\tau_I \wedge \tau_J) \vee (\tau_{I'} \wedge \tau_{J'}) \vee (\tau_I \wedge \tau_{J'}) \vee (\tau_{I'} \wedge \tau_J)$$

$$= (\tau_I \vee \tau_{I'}) \wedge (\tau_J \vee \tau_{J'}).$$

(b). If $E = \overline{n}$, then the conclusion follows immediately from $(*)$. Otherwise, $I_1, \ldots, I_m, E'$ is a partition of $\overline{n}$ and $E = I_1 + \ldots + I_m$. Then by (a) and $(*)$,

$$\bigwedge_{k \in \overline{m}} (\tau_{I_k} \vee \tau_{I_k'}) = \bigwedge_{1 \leq k < l \leq m} (\tau_{I_k} \vee \tau_{I_l}) \wedge \bigwedge_{i \in m} (\tau_{I_i} \vee \tau_{E'}).$$

Notice finally that, since $E = \cup_{k \in \overline{m}} I_k$,

$$\bigwedge_{k \in \overline{m}} (\tau_{I_k} \vee \tau_{E'}) = \Big(\bigwedge_{k \in \overline{m}} \tau_{I_k}\Big) \vee \tau_{E'} = \tau_E \vee \tau_{E'}.$$

Recall that for an arbitrary type σ, $G(\sigma) = \{x \in G : \text{type}(x) \geqslant \sigma\}$ is a fully-invariant pure subgroup of the torsion-free group G. Our next result is an elaboration on a theorem of Wuyen Lee [15].

THEOREM 1.2. *Let $\sigma \in$ typeset (G) where $G = G[A_1, \ldots, A_n]$. Then there exists a partition $I_1, \ldots, I_m$ of $\overline{n}$ that satisfies the following conditions:*

1. $\sigma = \bigwedge_{k \in \overline{m}} (\tau_{I_k} \vee \tau_{I_k'})$.

2. *For an arbitrary partition $J_1, \ldots, J_s$ of $\overline{n}$, $\tau_{J_l} \vee \tau_{J_l'} \geqslant \sigma$ for all $l \in \overline{s}$ if and only if $J_l = \cup \{I_k : I_k \cap J_l \neq \emptyset\}$ for all $l \in \overline{s}$.*

3. *If $B_k = \bigcap_{i \in I_k} A_i + \bigcap_{j \in I_k'} A_j$ for each $k \in \overline{m}$, then $G(\sigma)$ is the image of the canonical map $\Psi : G[B_1, \ldots, B_m] \to G[A_1, \ldots, A_n]$, and consequently rank $G(\sigma) = m - 1$.*

PROOF. We begin with the following observation: If $K_1, \ldots, K_t$ is a partition of $\overline{n}$ that determines σ and if I, J is a partition of K_t such that $\tau_I \vee \tau_J \geqslant \sigma$, then the partition $K_1, \ldots, K_{t-1}, I, J$ also determines σ. Indeed, first observe that Lemma 1.1 (b) with $m = 2$ yields

$$(\tau_I \vee \tau_{I'}) \wedge (\tau_J \vee \tau_{J'}) = (\tau_{K_t} \vee \tau_{K_t'}) \wedge (\tau_I \vee \tau_J) \geqslant \sigma .$$

But since $\tau_{K_t} \vee \tau_{K_t'} \geqslant (\tau_I \vee \tau_{I'}) \wedge (\tau_J \vee \tau_{J'})$ by Lemma 1.1 (a), we have the desired conclusion that

$$\sigma = \bigwedge_{j \in \overline{t}} (\tau_{K_j} \vee \tau_{K_j'}) \geqslant$$

$$\geqslant (\tau_{K_1} \vee \tau_{K_1'}) \wedge \ldots \wedge (\tau_{K_{t-1}} \vee \tau_{K_{t-1}'}) \wedge (\tau_I \vee \tau_{I'}) \wedge (\tau_J \vee \tau_{J'}) \geqslant \sigma .$$

Therefore, given a partition of $\overline{n}$ that determines σ, we can by successive applications of the foregoing observation refine it it to a partition $I_1, \ldots, I_m$ that satisfies number 1 and enjoys the following special property:

(**) For each $k \in \overline{m}$, if I, J is a partition of I_k, then $\tau_I \vee \tau_J \ngeqslant \sigma$.

With regard to item 2, first note that if J is any subset of $\overline{n}$ which is the union of I_k's, then $\tau_J \vee \tau_{J'} \geqslant \sigma$ by Lemma 1.1 (a). Conversely, let $J_1, \ldots, J_s$ be a partition of $\overline{n}$ such that $\tau_{J_l} \vee \tau_{J_l'} \geqslant \sigma$ for all $l \in \overline{s}$. Given any J_l, there certainly exists some $k \in \overline{m}$ with $I_k \cap J_l \neq \emptyset$. The proof of 2 will be complete if we can show that $I_k \subseteq J_l$. Suppose to the contrary that $I_k \subseteq / J_l$. Then in contradiction to (* *), $I = I_k \cap J_l$ and $J = I_k \cap J_l'$ yields a partition of I_k such that $\tau_I \vee \tau_J \geq \tau_{J_l} \vee \tau_{J_l'} \geq \sigma$. Finally, observe that 3 is an immediate consequence of 2, Theorem 1.1 and the description of the image of Ψ as given in Corollary 1.1. ■

COROLLARY 1.4. *Let* $\sigma \in \text{typeset}(G)$ *where* $G = G[A_1, \ldots, A_n]$. *Then* $\text{rank}\, G(\sigma) = 1$ *if and only if there is a nonempty proper subset* E *of* $\overline{n}$ *that satisfies the following two conditions:*

(i) $\sigma = \tau_E \vee \tau_{E'}$.

(ii) *If* I, J *is a partition of either* E *or* E', *then* $\tau_I \vee \tau_J \ngeq \sigma$.

Furthermore, if conditions (i) *and* (ii) *are satisfied by some nonempty proper subset* E *of* $\overline{n}$ *and if* F *is a subset of* $\overline{n}$ *such that* $\tau_F \vee \tau_{F'} = \sigma$, *then either* $F = E$ *or* $F = E'$.

In the sequel, the (obviously unique) partition $I_1, \ldots, I_m$ of $\overline{n}$ described in Theorem 1.2 will be referred to as *the canonical partition* associated with σ.

2. – The $\mathbb{Z}_{(2)}$-representation associated with $G[A_1, \ldots, A_n]$..

In this section, we associate with each $G = G[A_1, \ldots, A_n]$ a contravariant $\mathbb{Z}_{(2)}$-representation $\mathcal{R}_G$ of the poset $T_G = \text{typeset}(G)$. Generally, if $(T, \leq)$ is a finite poset and κ is a field, then a contravariant κ-representation of T is a family $(V, V_i : i \in T)$ where V is a finite dimensional vector space over κ, each V_i is a subspace of V, and $V_i \supseteq V_j$ whenever $i \leq j$. These κ-representations of T form a category with finite coproducts where a morphism from $(V, V_i;\, i \in T)$ to $(U, U_i;\, i \in T)$ consists of a vector space map $\phi : V \to U$ such that $\phi(V_j) \subseteq U_j$ for all j. We say that the representations $(V, V_i\colon i \in T)$ and $(U, U_i\colon i \in T)$ are *equivalent*, and write $(V, U_i\colon i \in T) \simeq (U, U_i\colon i \in T)$, provided they are isomorphic in this category.

The categories of Q-representations and $\mathbb{Z}_{(p)}$-representations are intimately related to the structure of arbitrary Butler groups [10]; see [2] for a summary and a list of references. Since the Boolean ring $2^{\overline{n}}$ is a vector space over $\mathbb{Z}_{(2)}$, the close connection between the subsets of $\overline{n}$ and the typeset of $G[A_1, \ldots, A_n]$ suggests $\mathbb{Z}_{(2)}$-representations are a possible vehicle for the study of $B^{(1)}$-groups.

In fact, as we shall establish in this paper, $G = G[A_1, \ldots, A_n]$ and $H = G[B_1, \ldots, B_n]$ are quasi-isomorphic if and only if the corresponding canonically defined $\mathbb{Z}_{(2)}$-representations $\mathcal{R}_G$ and $\mathcal{R}_H$ are equivalent. In the representation $\mathcal{R}_G$, the base space will be $2^{\overline{n}}$ and the component subspaces are the unital subrings introduced in the following theorem.

THEOREM 2.1. *Let $G = G[A_1, \ldots, A_n]$ and suppose $I_1, \ldots, I_m$ is the canonical partition of $\overline{n}$ associated with $\sigma \in$ typeset (G). Then $\overline{n}_G(\sigma) = \{E \in 2^{\overline{n}} : \tau_E \vee \tau_{E'} \geq \sigma\}$ is a unital subring of $2^{\overline{n}}$ with* dim $\overline{n}_G(\sigma) =$ rank $G(\sigma) + 1 = m$. *Furthermore, if $\overline{\tau}_k$ denotes the type of $B_k = \bigcap_{i \in I_k} A_i + \bigcap_{j \in I_k'} A_j$ for each $k \in \overline{m}$, then the map $\Theta : \overline{n}_G(\sigma) \to 2^{\overline{m}}$ given by $E \mapsto \overline{E} = \{k \in \overline{m} : I_k \subseteq E\}$ is a ring isomorphism that satisfies the following two properties:*

(1) $\tau_E \vee \tau_{E'} = \overline{\tau}_{\overline{E}} \vee \overline{\tau}_{\overline{E}'} = \left(\bigwedge_{k \in \overline{E}} \overline{\tau}_k\right) \vee \left(\bigwedge_{l \in \overline{E}'} \overline{\tau}_l\right)$.

(2) $\Theta(\overline{n}_G(\mu)) = \overline{m}_{G(\sigma)}(\mu)$ for each type $\mu \geq \sigma$.

PROOF. That $\overline{n}_G(\sigma)$ is closed under addition follows from Lemma 1.1 (a); while closure under intersection is a consequence of the fact that each nonempty $E \in \overline{n}_G(\sigma)$ is, by Theorem 1.2 number 2, a union of I_k's. It is, of course, obvious from the definitions that $\overline{n}_G(\sigma)$ contains the minimal subring $\{\emptyset, \overline{n}\}$. That dim $\overline{n}_G(\sigma) = m$ will follow once we establish the assertion about $\Theta : \overline{n}_G(\sigma) \to 2^{\overline{m}}$.

It is computationally more convenient to work with the inverse of Θ and so we begin by defining a function $\Phi : 2^{\overline{m}} \to 2^{\overline{n}}$ by $\Phi(\overline{E}) = \cup\{I_k : k \in \overline{E}\} = E$. Clearly Φ is a one-to-one map and by Theorem 1.2 part 2, Φ maps $2^{\overline{m}}$ onto $\overline{n}_G(\sigma)$. It is trivial that $\Phi(\overline{E}') = E' = \Phi(\overline{E})'$. Suppose now that $E_1 = \Phi(\overline{E}_1)$ and $E_2 = \Phi(\overline{E}_2)$. Since the I_k's are pairwise disjoint, it is evident that $E_1 \cap E_2 = \Phi(\overline{E} \cap \overline{E}_2)$. That Φ preserves addition is a consequence of the following computation:

$$E_1 + E_2 = (E_1 \cup E_2) \cap (E_1' \cup E_2')$$

$$= \bigcup\{I_k : k \in \overline{E}_1 \cup \overline{E}_2\} \cap \bigcup\{I_l : l \in \overline{E}_1' \cup \overline{E}_2'\}$$

$$= \bigcup\{I_k \cap I_l : (k, l) \in (\overline{E}_1 \cup \overline{E}_2) \times (\overline{E}_1' \cup \overline{E}_2')\}$$

$$= \bigcup\{I_k : (k, k) \in (\overline{E}_1 \cup \overline{E}_2) \times (\overline{E}_1' \cup \overline{E}_2')\} = \bigcup\{I_k : k \in \overline{E}_1 + \overline{E}_2\}.$$

With regard to condition (1), notice that Lemma 1.1 (b) yields $\overline{\tau}_{\overline{E}} = (\tau_E \vee \tau_{E'}) \wedge \delta_E$ where

$$\delta_E = \bigwedge\{\tau_{I_k} \vee \tau_{I_l} : k, l \in \overline{E} \text{ and } k \neq l\} \geq \bigwedge\{\tau_{I_k} : k \in \overline{E}\} = \tau_E.$$

Similarly, $\bar{\tau}_{\bar{E}} = (\tau_{E_1} \vee \tau_E) \vee \delta_{E'}$ where $\delta_{E'} \geq \tau_{E'}$, and therefore $\bar{\tau}_{\bar{E}} \vee \vee \bar{\tau}_{\bar{E}'} = (\tau_E \vee \tau_E) \wedge (\delta_E \vee \delta_{E'}) = \tau_E \vee \tau_{E'}$, as desired. If μ is an element of typeset (G) with $\mu \geq \sigma$, then clearly $G(\mu) \subset G(\sigma)$ and $\bar{n}_G(\mu) \subseteq \bar{n}_G(\sigma)$. Recalling the identification of $G(\sigma)$ with $G[B_1, \ldots, B_n]$ (Theorem 1.2 number 3), we see that condition (1) implies that Θ maps $\bar{n}_G(\mu)$ onto $\bar{m}_{G(\sigma)}(\mu)$. ■

Clearly the definition of $\bar{n}_G(\sigma)$ given above is meaningful even when σ is not in T_G = typeset (G). Furthermore, if $\bar{n}_G(\sigma) \neq \{\emptyset, \bar{n}\}$, then it is easily seen that $\bar{n}_G(\sigma) = \bar{n}_G(\mu)$ where μ is the least type in T_G with $\mu \geq \sigma$. Indeed when σ and μ are so related, then $G(\sigma) = G(\mu)$ and the equation dim $\bar{n}_G(\sigma) =$ rank $G(\sigma) + 1$ remains valid. We shall henceforth let $\mathcal{R}_G$ denote the contraviant $\mathbb{Z}_{(2)}$-representation $(2^{\bar{n}}, \bar{n}_G(\sigma): \sigma \in T_G)$. We could, of course, replace T_G by a larger set of types, but generally we shall only need to compare representations $\mathcal{R}_G$ and $\mathcal{R}_H$ where $T_G = T_H$. Notice, however, that when $H = G(\sigma)$, the map Φ in the proof of Theorem 2.1 can be construed as an imbedding of $\mathcal{R}_H$ into $\mathcal{R}_G$.

We shall now adopt further notational conventions that will remain in force throughout the remainder of this paper: $G = G[A_1, \ldots, A_n]$ and $H = G[B_1, \ldots, B_n]$ with $\tau_i =$ type (A_i) and $\sigma_i =$ type (B_i) for all $i \in \bar{n}$. The subrings $\bar{n}_G(\mu)$ are defined as in Theorem 2.1, while $\bar{n}_H(\mu) = \{E \in 2^{\bar{n}}: \sigma_E \vee \sigma_{E'} \geq \mu\}$ where, of course, $\sigma_E = \bigwedge_{i \in E} \sigma_i$ for any subset E of $\bar{n}$. We say that the representations $\mathcal{R}_H = (2^{\bar{n}}, \bar{n}_H(\mu): \mu \in T_H)$ and $\mathcal{R}_G = (2^{\bar{n}}, \bar{n}_G(\mu): \mu \in T_G)$ are *equivalent* provided $T_H = T_G$ and there exists a nonsingular linear transformation $\Omega: 2^{\bar{n}} \to 2^{\bar{n}}$ such that $\Omega(\bar{n}_H(\mu)) = \bar{n}_G(\mu)$ for all $\mu \in T_H$.

Our next theorem contains a fundamental criterion for establishing the equivalence of $\mathcal{R}_H$ and $\mathcal{R}_G$

THEOREM 2.2. *Let $G = G[A_1, \ldots, A_n]$ and $H = G[B_1, \ldots, B_n]$ where $T_G = T_H$ and* rank $H(\tau_k) \geq$ rank $G(\tau_k)$ *for all $k \in \bar{n}$. Suppose that there exist subsets $E_1, \ldots, E_n$ of $\bar{n}$ that satisfy the following two conditions:*

(1) $$E_k \in \bar{n}_G(\sigma_k) \text{ for all } k \in \bar{n}$$

(2) $$\text{For } F \subseteq \bar{n}, \sum_{k \in F} E_k = \bar{n} \text{ if and only if } F = \bar{n}.$$

Then the groups H and G are quasi-isomorphic and the representations $\mathcal{R}_H$ and $\mathcal{R}_G$ are equivalent.

PROOF. First observe that condition (2) is logically equivalent to the conjunction of the following two conditions:

$$\sum_{k\in\overline{n}} E_k = \overline{n}. \tag{3}$$

$$\sum_{k\in F} E_k \neq \emptyset \text{ if } F\neq\emptyset. \tag{4}$$

Define a map $\Omega : 2^{\overline{n}} \to 2^{\overline{n}}$ by $\Omega(F) = \sum_{k\in F} E_k$. Recalling that $I+J = (I\cap J')\cup(I'\cap J)$ where $(I\cap J')\cap(I'\cap J)=\emptyset$, we see that Ω is linear from the computation

$$\Omega(I+J) = \sum_{k\in I\cap J'} E_k + \sum_{k\in J\cap I'} E_k$$

$$= \Big(\sum_{k\in I\cap J} E_k + \sum_{k\in I\cap J'} E_k\Big) + \Big(\sum_{k\in I\cap J} E_k + \sum_{k\in I'\cap J} E_k\Big)$$

$$= \sum_{k\in I} E_k + \sum_{k\in J} E_k = \Omega(I) + \Omega(J).$$

Then (3) and (4) imply, respectively, that $\Omega(F') = \Omega(F)'$ and that Ω is nonsingular. It then follows that $\Omega(\overline{n}_H(\mu)) \subseteq \overline{n}_G(\mu)$, for all types μ. Indeed by (1) and Lemma 1.1 (a), $\tau_{\Omega(F)} \vee \tau_{\Omega(F)'} \geq \bigcup_{k\in F}(\tau_{E_k}\vee\tau_{E_k'}) \geq \bigcup_{k\in F}\sigma_k = \sigma_F$, and similarly $\tau_{\Omega(F)}\vee\tau_{\Omega(F)'} = \tau_{\Omega(F')}\vee\tau'_{\Omega(F')'} \geq \bigcup_{k\in F'}\sigma_k = \sigma_{F'}$; that is to say, $\sigma_F \vee \sigma'_F \geq \mu$ implies $\tau_{\Omega(F)}\vee\tau_{\Omega(F)'}\geq\mu$.

In order to prove the reverse inclusion $\overline{n}_G(\mu)\subseteq\Omega(\overline{n}_H(\mu))$, we shall show that Ω^{-1} has a description similar to the definition of Ω. Towards this end, observe, Ω is necessarily onto and there exist subsets $F_1, \dots, F_n$ of $\overline{n}$ such that $\Omega(F_i) = \{i\}$ for all $i\in\overline{n}$. We claim then that

$$(1') \quad F_i \in \overline{n}_H(\tau_i) \text{ for all } i\in\overline{n}.$$

Indeed the hypotheses $T_G = T_H$ and rank $G(\tau_i) \leq$ rank $H(\tau_i)$ imply that dim $\overline{n}_G(\tau_i)) \leq$ dim $\overline{n}_H(\tau_i)$, and consequently $\Omega(\overline{n}_H(\tau_i)) = \overline{n}_G(\tau_i)$. Then, since Ω is one-to-one and $\{i\}\in\overline{n}_G(\tau_i)$ condition (1′) follows. We furthermore maintain that

$$(3') \quad \sum_{i\in\overline{n}} F_i = \overline{n},$$

and

$$(4') \quad \sum_{i \in E} F_i \neq \emptyset \text{ if } E \neq \emptyset .$$

We shall establish the validity of these latter two conditions via some linear algebra over $\mathbb{Z}_{(2)}$. Let $\mathcal{E} = [e_{ik}]$ be the $n \times n$, $\{0, 1\}$-matrix where $e_{ik} = 1$ if and only if $i \in E_k$. Thus the column vectors of $\mathcal{E}$ can be identified with the characteristic functions of the corresponding E_k's or alternatively, $\mathcal{E}$ is the matrix associated with the linear transformation Ω relative to the canonical basis $\{1\}, \ldots, \{n\}$ of the vector space $2^{\overline{n}}$. Similarly, we let $\mathcal{F} = [f_{ji}]$ be the $n \times n$, $\{0, 1\}$-matrix where $f_{ji} = 1$ if and only if $j \in F_i$. From the familiar fact that the pointwise sum (mod 2) of characteristic functions yields the characteristic function of the corresponding sum of the subsets of $2^{\overline{n}}$, we see that (3) implies that (mod 2) all row sums of $\mathcal{E}$ equal 1, and (4) tells us that the column vectors of $\mathcal{E}$ are linearly independent in $\mathbb{Z}^n_{(2)}$. Similarly, the equations $\{i\} = \Omega(F_i) = \sum_{k \in F_i} E_k$ reflects the fact that the matrix product $\mathcal{E}\mathcal{F}$, as computed in $\mathbb{Z}_{(2)}$, is the $n \times n$ identity matrix $\mathfrak{I}_n$. Therefore $\det_2 \mathcal{E} = 1 = \det_2 \mathcal{F}$, and consequently the column vectors of $\mathcal{F}$ are also linearly independent in $\mathbb{Z}^n_{(2)}$; that is, condition (4′) holds.

To show finally that condition (3′) is satisfied, we need only verify that (mod 2) all row sums of $\mathcal{F}$ equal 1. This, however, follows from the matrix equation $\mathcal{F}\mathcal{E} = \mathfrak{I}_n$ and the fact that $\mathcal{E}$ enjoys this same property. Perhaps the most convenient way to substantiate this last remark is to introduce the linear functional $\varrho : \mathbb{Z}^n_{(2)} \to \mathbb{Z}_{(2)}$ defined by $\varrho(x_1, \ldots, x_n) = x_1 + \ldots + x_n$. Then if $\mathcal{E}^k$ and $\mathcal{F}^i$ denote the k-th row and i-th row vectors of $\mathcal{E}$ and $\mathcal{F}$, respectively, $\varrho(\mathcal{F}^i) = \sum_{k \in \overline{n}} f_{ik} = \sum_{k \in \overline{n}} f_{ik} \varrho(\mathcal{E}^k) = \varrho\left(\sum_{k \in \overline{n}} f_{ik} \mathcal{E}^k\right) = 1$ because $\varrho(\mathcal{E}^k) = 1$ for all $k \in \overline{n}$ and $\sum_{k \in \overline{n}} f_{ik} \mathcal{E}^k$ is the i-th row vector of $\mathcal{F}\mathcal{E} = \mathfrak{I}_n$.

Because of (1′), (3′) and (4′), the linear map $\Omega^* : 2^{\overline{n}} \to 2^{\overline{n}}$ defined by $\Omega^*(E) = \sum_{i \in E} F_i$ is nonsingular with $\Omega^*(\overline{n}_G(\mu)) \subseteq \overline{n}_H(\mu)$ for all types μ. By either a direct computation or by the observation that $\mathcal{F} = \mathcal{E}^{-1}$ (computed over $\mathbb{Z}_{(2)}$), we see that $\overline{n}_G(\mu) = \Omega(\Omega^*(\overline{n}_G(\mu))) \subseteq \Omega(\overline{n}_H(\mu))$ for all types μ. Since $T_H = T_G$ by hypothesis, we conclude that the representations $\mathcal{R}_H$ and $\mathcal{R}_G$ are equivalent. Finally, we note that H and G are quasi-isomorphic. Indeed, since all the requisite conditions of Corollary 1.1 are satisfied, we have monomorphisms $\Psi_{\mathcal{E}} : H \to G$ and $\Psi_{\mathcal{F}} : G \to H$. This completes the proof. ■

COROLLARY 2.1. *If the representations $\mathcal{R}_H$ and $\mathcal{R}_G$ are equivalent, then the groups $H = G[B_1, \dots, B_n]$ and $G = G[A_1, \dots, A_n]$ are quasi-isomorphic.*

PROOF. Let $\Omega : 2^{\overline{n}} \to 2^{\overline{n}}$ be a nonsingular linear transformation such that $\Omega(\overline{n}_H(\mu)) = \overline{n}_G(\mu)$ for all μ in $T_H = T_G$. It then follows that rank $G(\tau_i) = \text{rank } H(\tau_i)$ for each $i \in \overline{n}$ because $T_G = T_H$ and hence dim $\overline{n}_G(\tau_i) = \dim \overline{n}_H(\tau_i)$ for each $i \in \overline{n}$. It suffices to show that the other hypotheses of Theorem 2.2 are satisfied. Naturally then we introduce subsets $E_1, \dots, E_n$ of $\overline{n}$ where $E_k = \Omega(\{k\})$ for each k. Since $\{k\} \in \overline{n}_H(\sigma_k)$, $E_k \in \overline{n}_G(\sigma_k)$ for each $k \in \overline{n}$. Also we clearly have $\Omega(F) = \sum_{k \in F} E_k$ because $F = \sum_{k \in F} \{k\}$, and therefore condition (4) in the proof of Theorem 2.2 is satisfied since Ω is a nonsingular linear transformation. If we have $\Omega(\overline{n}) = \overline{n}$, then condition (3) in that proof will be satisfied and it will follow that H and G are quasi-isomorphic.

It is possible for $\Omega(\overline{n}) \neq \overline{n}$, but this anomaly can only occur in the extreme situation where T_G contains a unique maximal type. In fact, if σ and μ are distinct maximal types in T_G, then $\overline{n}_G(\sigma) \cap \overline{n}_G(\mu) = \{\emptyset, \overline{n}\} = \overline{n}_H(\mu) \cap \overline{n}_H(\mu)$ which will force $\Omega(\overline{n}) = \overline{n}$. The point is that if $E \in \overline{n}_G(\sigma) \cap \overline{n}_G(\mu)$ with $E \notin \{\emptyset, \overline{n}\}$, then G will contain a nonzero $x = \langle x_1, \dots, x_n \rangle$ that is constant on both E and E'. This, however, is absurd since Theorem 1.1 would imply type $x = \tau_E \vee \tau_{E'} \geqslant \sigma \vee \mu$. So if $\Omega(\overline{n}) \neq \overline{n}$, we may assume that $T_H = T_G$ contains a unique maximal type σ, in which case $\overline{n}_H(\sigma) \subseteq \overline{n}_H(\mu)$ and $\overline{n}_G(\sigma) \subseteq \overline{n}(\mu)$ for all μ in $T_H = T_G$. There certainly exists a nonsingular transformation $\Omega_0 : \overline{n}_G(\sigma) \to \overline{n}_H(\sigma)$ with $\Omega_0(\overline{n}) = \overline{n}$. Then if $\Omega_1 : 2^{\overline{n}} \to 2^{\overline{n}}$ is the linear transformation that restricts to Ω_0 on $\overline{n}_H(\sigma)$ and agrees with Ω on some fixed complement, it is routine to check that $\Omega_1(\overline{n}_H(\mu)) = \overline{n}_G(\mu)$ for all $\mu \in T_H$. Finally, redefining the E_k's in terms of Ω_1, we see that Theorem 2.2 implies that H and G are quasi-isomorphic since $\Omega_1(\overline{n}) = \overline{n}$. ∎

The proof of the converse of Corollary 2.1 is more subtle. Exploiting Theorem 2.1 and a fairly intricate induction on rank, we shall establish this converse result for strongly indecomposable $B^{(1)}$-groups in § 3. Finally in the last section, we take advantage of the fact that each $G[\mathcal{A}]$ is quasi-isomorphic to a direct sum of strongly indecomposable $\mathcal{B}^{(1)}$-groups in order to show that when $H = G[B_1, \dots, B_n]$ is quasi-isomorphic to $G = G[A_1, \dots, A_n]$, then the representations $\mathcal{R}_H$ and

$\mathcal{R}_G$ are necessarily equivalent. In both instances, Theorem 2.2 will be an indispensable tool.

3. – Strongly indecomposable $\mathcal{B}^{(1)}$-groups..

We now turn our attention to the quasi-isomorphism problem for strongly indecomposable groups of the form $G[A_1, \ldots, A_n]$. As noted in § 1, independent solutions of this problem have already been given in [6] and in [11]. Our approach to this problem will, of course, apply techniques developed in the preceding sections and will lead to refinements of results obtained in these earlier treatments. There are various characterizations of when $G = G[A_1, \ldots, A_n]$ is strongly indecomposable (see [5], [6], [11], and [14]), but the one we find most convenient is due to Goeters and Ullery.

THEOREM 3.1. [12] *The group* $G = G[A_1, \ldots, A_n]$ *is strongly indecomposable if and only if, for every* $k \in \overline{n}$ *and every nontrivial partition* I, J *of* $\{k\}' = \overline{n} \setminus \{k\}$, $\tau_I \vee \tau_J \not\geq \tau_k$.

The influence of this particular characterization is evident in the proof of Theorem 1.2 above. As a consequence of Theorem 3.1, we can remove the redundant «cotrimmed» hypothesis from the characterization of strongly indecomposable groups of the form $G[A_1, \ldots, A_n]$ given in [5] and [6].

COROLLARY 3.1. *The group* $G = G[A_1, \ldots, A_n]$ *is strongly indecomposable if and only if* rank $G(\tau_k) = 1$ *for all* $k \in \overline{n}$.

PROOF. For each $k \in \overline{n}$, let $\tau_k' = \tau_k \vee \bigwedge_{i \neq k} \tau_i$, and note that $0 \neq G(\tau_k') \subseteq G(\tau_k)$ because $\tau_k' \in T_G$ and $\tau_k' \geq \tau_k$. First assume that rank $G(\tau_k) = 1$ for all $k \in \overline{n}$. Then for all $k \in \overline{n}$, $G(\tau_k') = G(\tau_k)$, from which it follows both that τ_k' is maximal in T_G and that τ_k' is the least element of T_G which is $\geq \tau_k$. Now suppose by way of contradiction that G is not strongly indecomposable; that is, there exists some k such that $\{k\}'$ has a partition I, J with $\tau_I \vee \tau_J \geq \tau_k$. But then $\tau_I \vee \tau_{I'} = \tau_I \vee (\tau_J \wedge \tau_k) = (\tau_I \vee \tau_J) \wedge (\tau_I \wedge \tau_k) \geq \tau_k$ and hence $\tau_I \vee \tau_{I'} = \tau_k'$. By Corollary 1.4, $I = \{k\}'$; that is, $J = \emptyset$ and I, J is not a partition of $\{k\}'$.

Conversely, assume that $G = G[A_1, \ldots, A_n]$ is strongly indecomposable. Then, by Theorem 3.1 and Corollary 1.4, rank $G(\tau_k') = 1$ for all $k \in \overline{n}$.

It remains to show, for each k, that $G(\tau_k) = G(\tau'_k)$, or equivalently, that τ'_k is the least element of T_G which is $\geqslant \tau_k$. Suppose then that σ is the least element of T_G with $\sigma \geqslant \tau_k$ and let $I_1, \ldots, I_m$ be the canonical partition of $\overline{n}$ associated with σ. Since $\tau'_k \geqslant \sigma$, we may assume, by Theorem 1.2 number 2, that $I_m = \{k\}$ and $\{k\}' = \bigcup_{1 \leqslant j < m} I_j$. The desired conclusion that $\tau'_k = \sigma$ will be obtained once we establish that $m = 2$. But if $m \neq 2$, then $I = \bigcup_{1 \leqslant j < m-1} I_j$, $J = I_{m-1}$ yields a partition of $\{k\}'$ such that

$$\tau_I \vee \tau_J = \Big(\bigwedge_{1 \leqslant j < m-1} I_j\Big) \vee \tau_{I_{m-1}} = \bigwedge_{1 \leqslant j < m-1} (\tau_{I_j} \vee \tau_{I_{m-1}}) \geqslant$$

$$\geqslant \bigwedge_{1 \leqslant j < m-1} (\tau_{I_j} \vee \tau_{I'_j}) \geqslant \sigma \geqslant \tau_k,$$

contradicting the fact that $G = G[A_1, \ldots, A_n]$ is strongly indecomposable. ■

COROLLARY 3.2. *Let* $G = G[A_1, \ldots, A_n]$ *be strongly indecomposable and suppose* $\sigma \in \text{typeset}(G)$ *with rank* $G(\sigma) = 1$. *If* $\sigma = \tau_E \vee \tau_{E'}$ *where* $E = \{i_1, \ldots, i_s\}$ *and if* $B = \bigcap_{i \in E} A_i + \bigcap_{i \in E'} A_i$, *then* $G(\tau_E) \simeq$ $\simeq G[A_{i_1} \ldots, A_{i_s}, B]$ *is also strongly indecomposable.*

PROOF. For each $k \in \overline{n}$, let $A'_k = A_k + \bigcap_{i \neq k} A_i$ and $\tau'_k = \tau_k \vee \bigwedge_{i \neq k} \tau_i =$ $= \text{type}(A'_k) \geqslant \tau_k$. By Corollary 1.2, we can identify $K = G[A_{i_1}, \ldots A_{i_s}, B]$ with $G[A'_{i_{11}}, \ldots, A'_{i_s}, B]$, which in turn can be viewed, via Corollary 1.1, as a pure subgroup of $G = G[A_1, \ldots, A_n]$. Then $1 \leqslant \text{rank } K(\tau'_{i_j}) \leqslant$ $\leqslant \text{rank } K(\tau_{i_j}) \leqslant \text{rank } G(\tau_{i_j}) = 1$ for all $j \in \overline{s}$ by Corollary 3.1. Similarly, rank $K(\sigma) = 1$ and consequently K is strongly indecomposable by another application of Corollary 3.1.

Now let $\tau'_I = \wedge_{k \in I} \tau'_k$ for each $I \subseteq \overline{n}$, and observe that Theorem 2.1 (1) implies $\tau'_E \vee \tau'_{E'} = \tau_E \vee \tau_{E'}$. Thus $(\tau'_{i_1} \wedge \ldots \wedge \tau'_{i_s}) \wedge (\tau_E \vee \tau_{E'}) = \tau'_E \wedge$ $\wedge (\tau'_E \vee \tau'_{E'}) = \tau'_E$. Notice that if I, J is a partition of E', then $\tau_I \vee \tau_J \geqslant \tau_{E'}$ and consequently $\tau_I \vee \tau_J \ngeqslant \tau_E$ by Corollary 1.4. Therefore $\tau_I \vee \tau_J \ngeqslant \tau'_E$ for I, J a partition of E', and it follows (see condition $(**)$ in the proof of Theorem 1.2) that $\{i_1,\}, \ldots, \{i_s\}, E'$ is the canonical partition of $\overline{n}$ associated with τ'_E. Then, by Theorem 1.2 (3), $K = G(\tau'_E)$. Finally, a slight modification of the second half of the proof of Corollary 3.1 leads to the conclusion that τ'_E is the least element in T_G with $\tau'_E \geqslant \tau_E$ and hence $G(\tau_E) =$ $= G(\tau'_E) \simeq G[A_{i_1}, \ldots, A_{i_s}, B]$, as desired. ■

As will be seen in proof of our next theorem, the importance of Corollary 3.2 is that it provides a method for establishing certain results about strongly indecomposable $\mathcal{B}^{(1)}$-groups via induction on rank. (For other illustrations of this technique in the dual context, see [14] and [17].) We are now in position to prove the main theorem of this section, in which we adopt the convention that $\sum_{\sigma \in M} \overline{n}_G(\sigma) = \{\emptyset, \overline{n}\}$ when $M = \emptyset$. In the sequel, we shall frequently write $G \sim H$ to indicate that G and H are quasi-isomorphic.

THEOREM 3.2. *Let* $G = G[A_1, \ldots, A_n]$ *and* $H = G[B_1, \ldots, B_n]$ *be strongly indecomposable groups with equal typesets. Then the following three conditions are equivalent:*

(a) *G and H are quasi-isomorphic.*

(b) *The $\mathbb{Z}_{(2)}$-representations $\mathcal{R}_G$ and R_H are equivalent.*

(c) $\dim\left(\sum_{\sigma \in M} \overline{n}_G(\sigma)\right) = |M| + 1$ for every proper subset M of $\{\sigma_1, \ldots, \sigma_n\}$.

PROOF. We shall first prove that (a) implies (c). Suppose then that $G \sim H$ and the desired conclusion (c) holds for strongly indecomposable groups of rank $< n-1$. By Corollary 3.1 and Theorem 2.1, $\overline{n}_G(\tau_k)$ is 2-dimensional and is hence spanned by the vectors $\{k\}$ and $\overline{n}$. It then follows readily that $\dim\left(\sum_{k \in E} \overline{n}_G(\tau_k)\right) = |E| + 1$ whenever E is a proper subset $\overline{n}$. Thus condition (c) is certainly satisfied if the σ_i's are merely a permutation of the τ_i's. So without loss of generality, we may assume that τ_1 is distinct from all the σ_j's. Then $G \sim H$ implies that $G(\tau_1) \sim H(\tau_1)$ and, by Corollary 3.1, rank $H(\tau_1) =$ rank $G(\tau_1) = 1$. Therefore, by Corollary 1.4, there is a nonempty proper subset F of $\overline{n}$ such that $\sigma_F \vee \sigma_{F'} = \tau_1' = \tau_1 \vee \bigwedge_{i \neq 1} \tau_i$. Similarly, rank $G(\sigma_k) =$ rank $H(\sigma_k) = 1$ for all $k \in \overline{n}$ and consequently, by Theorem 2.1 and the proof of Corollary 3.2, $\dim \overline{n}_G(\sigma_k) = \dim \overline{n}_G(\sigma_k') = 2$ where $\sigma_k' = \sigma_k \vee \bigwedge_{i \neq k} \sigma_i$ is an element of $T_H = T_G$ for all $k \in \overline{n}$. In other words, condition (c) is satisfied in the special case where $|M| = 1$.

We next observe that $\overline{n}_G(\sigma_F) \cap \overline{n}_G(\sigma_{F'}) = \overline{n}_G(\tau_1)$. Indeed if E is in $\overline{n}_G(\sigma_F) \cap \overline{n}(\sigma_{F'})$ then $\tau_E \vee \tau_{E'} \geqslant \sigma_F \vee \sigma_{F'} = \tau_1'$; that is, $\overline{n}_G(\sigma_F) \cap \overline{n}_G(\sigma_{F'}) \subseteq \overline{n}_G(\tau_1') = \overline{n}_G(\tau_1)$. On the other hand, this inclusion cannot be proper since $\dim \overline{n}_G(\tau_1') = 2$ and $\{1\}$ is in $\overline{n}_G(\sigma_F) \cap \overline{n}_G(\sigma_{F'})$.

Now suppose $F = \{i_1, \ldots, i_{s-1}\}$ and $F' = \{j_1, \ldots, j_{t-1}\}$. Then, by

Corollary 3.2, there is a rank 1 group A_1' of type τ_1' such that $H(\sigma_F) \simeq \simeq G[B_{i_1}, \ldots, B_{i_{s-1}}, A_1']$ and $H(\sigma_{F'}) \simeq G[B_{j_1}, \ldots, B_{j_{t-1}}, A_1']$ are strongly indecomposable groups of ranks $s-1$ that $t-1$, respectively, where $s<n$, $t<n$ and $s+t=n+2$. Since $G \sim H$, $G_1 = G(\sigma_F) \sim H(\sigma_F)$ and $G_2 = G(\sigma_{F'}) \sim H(\sigma_{F'})$ are also strongly indecomposable groups of ranks $s-1$ and $t-1$, respectively.

Write $M = \{\sigma_i : i \in I\}$ where I is a proper subset of $\overline{n}$, and then take $S = I \cap F$ and $T = I \cap F'$. By Theorem 2.1 we have ring isomorphisms $\Theta_F : \overline{n}_G(\sigma_F) \to 2^{\overline{s}}$ and $\Theta_{F'} : \overline{n}_G(\sigma_{F'}) \to 2^{\overline{t}}$, where Θ_F maps $V_1 = \sum_{i \in S} \overline{n}_G(\sigma_i)$ onto $W_1 = \sum_{i \in S} \overline{s}_{G_1}(\sigma_i)$ and $\Theta_{F'}$ maps $V_2 = \sum_{j \in T} \overline{n}_G(\sigma_j)$ onto $W_2 = \sum_{j \in T} \overline{t}_{G_2}(\sigma_j)$. Thus, by the induction hypothesis, $\dim W_1 = |S| + 1$ and $\dim V_2 = = \dim W_2 = |T| + 1$. Because $|S| + |T| = |M| < n$, either $|S| < s-1$ or $|T| < t-1$. If $|S| < s-1$, then another application of the induction hypothesis yields $\dim (W_1 + \overline{s}_{G_1}(\tau_1')) = |S| + 2$; that is, $\overline{s}_{G_1}(\tau_1') \not\subseteq W_1$. Considering inverse images, we see that $\overline{n}_G(\tau_1') \not\subseteq V_1$. Similarly, if $|T| < t- -1$, then $\overline{n}_G(\tau_1') \not\subseteq V_2$. Consequently, one or the other of V_1 and V_2 fails to contain $\overline{n}_G(\tau_1')$, and therefore $V_1 \cap V_2$ is a proper subspace of $\overline{n}_G(\sigma_F) \cap \cap \overline{n}_G(\sigma_{F'}) = \overline{n}_G(\tau_1')$. It follows, since both V_1 and V_2 contain $\{\emptyset, \overline{n}\}$, that $\dim (V_1 \cap V_2) = 1$. Finally observe that

$$\dim \Big(\sum_{\sigma \in M} \overline{n}_G(\sigma) \Big) = \dim (V_1 + V_2) = \dim V_1 + \dim V_2 - \dim (V_1 \cap V_2)$$

$$= (|S| + 1) + (|T| + 1) - 1 = |S| + |T| + 1 = |M| + 1 .$$

Since (b) implies (a) by Corollary 2.1, it suffices to prove that condition (c), together with the other hypotheses on G and H, imply the representations $\mathcal{R}_G$ and $\mathcal{R}_H$ are equivalent. So we assume (c), and than show that the relevant conditions of Theorem 2.2 are all satisfied. To begin with, notice that $\tau_k' = \tau_k \vee \bigwedge_{i \neq k} \tau_i$ is in $T_G = T_H$ and, by the proof of Corollary 3.2, is in fact the least element of T_G is $\geqslant \tau_k$. Thus $0 \neq H(\tau_k') = = H(\tau_k)$; and, since rank $G(\tau_k) = 1$ by Corollary 3.1, we certainly have the requisite hypothesis; rank $H(\tau_k) \geqslant$ rank $G(\tau_k)$, satisfied for all $k \in \overline{n}$.

Next observe that condition (c) implies that $\dim \overline{n}_G(\sigma_k') = 2$ for all $k \in \overline{n}$, where $\sigma_k' = \sigma_k \vee \bigwedge_{i \neq k} \sigma_i$ is the least element of $T_H = T_G$ that is $\geqslant \sigma_k$. Then by Theorem 2.1, $G(\sigma_k') = G(\sigma_k)$ is a rank 1 pure subgroup of G for each $k \in \overline{n}$, and therefore an application of Corollary 1.4 yields nonempty proper subsets $E_1, \ldots, E_n$ of $\overline{n}$ such that $\tau_{E_k} \vee \tau_{E_k'} = \sigma_k'$ for all $k \in \in \overline{n}$. In particular, we have $E_k \in \overline{n}_G(\sigma_k)$ for all $k \in \overline{n}$. It remains then to argue that the E_k' s satisfy condition (2) in the statement of Theorem 2.2, or equivalently, that the E_k's satisfy both conditions (3) and (4) appearing in the proof of that theorem.

Towards this end, we begin by noting that the two vectors E_k and $\overline{n}$ form a basis for $\overline{n}_G(\sigma_k)$ since $\dim \overline{n}_G(\sigma_k) = 2$. But then another application of condition (c) implies that the n vectors $E_1, \ldots, E_{n-1}, \overline{n}$ span $2^{\overline{n}}$ and hence form a basis for $2^{\overline{n}}$. Therefore for some choice of scalars $a_k \in \in \mathbb{Z}_{(2)}$ and $a \in \mathbb{Z}_{(2)}$, we can write

$$E_n = a_1 E_1 + \ldots + a_{n-1} E_{n-1} + a\overline{n}.$$

Recalling the basic fact that $(I+J)' = I + J'$, we see that

$$E_n' = a_1 E_1 + \ldots + a_{n-1} E_{n-1} + b_n \overline{n}$$

where $b_n = 0$ if $a = 1$ and $b_n = 1$ when $a = 0$. Replacing E_n by E_n' if necessary, we may assume without loss of generality that $a = 1$. In fact it then follows that all the a_k's equal to 1. Indeed if, say, $a_1 = 0$, then the $n-1$ vectors $E_2, \ldots, E_{n-1}, \overline{n}$ would span the subspace $\sum_{k=2}^{n} \overline{n}_G(\sigma_k)$, contrary to the fact that this subspace has dimension n by condition (c). In summary, $E_n = E_1 + \ldots + E_{n-1} + \overline{n}$, or equivalently, $E_1 + E_2 + \ldots + E_n = \overline{n}$. Since $E_1, \ldots, E_{n-1}, \overline{n}$ form a basis for $2^{\overline{n}}$, the last equation above implies that the vectors $E_1, E_2, \ldots, E_n$ are also a basis for $2^{\overline{n}}$. Finally, by linear independence, $\sum_{k \in F} E_k \neq \emptyset$ if $F \neq \emptyset$; and this observation completes the proof of the theorem. ■

Recall that when given an $n \times n, \{0, 1\}$-matrix $\mathcal{E}$, the matrix $\mathcal{E}_k$ represents the matrix obtained by substituting the vector 1_n of 1's into the k^{th}-column of $\mathcal{E}$. In [11], $\mathcal{E}$ is called *admissible*, if the integral determinant of $\mathcal{E}_k$ is nonzero for each index k.

COROLLARY 3.3. *Let* $G = G[A_1, \ldots, A_n]$ *and* $H = G[B_1, \ldots, B_n]$ *be strongly indecomposable groups with equal typesets, and suppose* $\mathcal{E} = [e_{ik}]$ *is an* $n \times n, \{0, 1\}$*-matrix with the property that, for each* $k \in \overline{n}$, $E_k = \{i \in \overline{n} : e_{ik} = 1\}$ *is a nonempty proper subset of* $\overline{n}$ *such that* $\tau_{E_k} \vee \tau_{E_k'} \geqslant \text{type}\,(B_k)$. *Then the following conditions are equivalent:*

(1) *G and H are quasi-isomorphic.*

(2) *All row sums of* $\mathcal{E}$ *have the same parity and* det $\mathcal{E}$ *is an odd integer.*

(3) $\mathcal{E}$ *is admissible.*

PROOF. Notice first that the existence of such an $\mathcal{E}$ is not in doubt. By the proof of Corollary 3.2, $\sigma_k' = \sigma_k \vee \bigwedge_{i \in k} \sigma_i$ is a maximal element of $T_H = T_G$ and hence, by Theorem 1.2, there exists a nonempty proper subset E_k of $\overline{n}$ such that $\tau_{E_k} \vee \tau_{E_k'} = \sigma_k' \geqslant \sigma_k =$ type (B_k). As one further preliminary observation, we have from the linearity of the determinant function, for all $k \in \overline{n}$,

$$(\dagger) \quad \det \mathcal{E}_k = \det \mathcal{E} + \det \mathcal{E}^{(k)},$$

where $\mathcal{E}^{(k)}$ is obtained from $\mathcal{E}$ modifying the k-th column only by replacing each 0 by 1 and each 1 by 0. In other words $\mathcal{E}^{(k)}$ is formed in the same manner as $\mathcal{E}$ except that E_k is replaced by its complement E_k'.

To see that (1) implies (2), we refer to that portion of the preceding proof in which it is shown that condition (b) follows from condition (c). As is noted there, $E_1, \ldots E_{n-1}, \overline{n}$ span $2^{\overline{n}}$. Furthermore, forgoing the possible replacement of E_n by E_n' in that earlier argument, we see that either $E_1 + \ldots + E_{n-1} + E_n = \overline{n}$ or else $E_1 + \ldots + E_{n-1} + E_n' = \overline{n}$. Recalling the relation between sums of the E_k's and the sums (mod 2) of the column vectors of $\mathcal{E}$ (see proof of Theorem 2.2), we have in the first instance that all row sums of $\mathcal{E}$ are odd and in the second that all row sums of $\mathcal{E}$ are even. Thus when $E_1, \ldots, E_{n-1}, E_n$ span $2^{\overline{n}}$ and $E_1 + \ldots + E_{n-1} + E_n' = \emptyset$, we have $\det_2 \mathcal{E} = 1$ and $\det_2 \mathcal{E}^{(n)} = 0$; while, when $E_1, \ldots, E_{n-1}, E_n'$ span $2^{\overline{n}}$ and $E_1 + \ldots E_{n-1} + E_n = \emptyset$, we have $\det_2 \mathcal{E}^{(n)} = 1$ and $\det_2 \mathcal{E} = 0$. In either case, $\det \mathcal{E}_n$ is an odd integer by $(\dagger)$.

Observe, by linearity of the determinant function, that $\det \mathcal{E}_k + \det \mathcal{E}_k^{(n)} = 0$ for $k \neq n$, and that (2) and $(\dagger)$ imply that $\det \mathcal{E}$ and $\det \mathcal{E}^{(n)}$ have opposite parity. Thus, in view of the proof of Corollary 1.3, condition (2) implies that $\det \mathcal{E}_k \neq 0$ for all k. Finally, assume that condition (3) is satisfied and note then, the induced map $\Psi_{\mathcal{E}}: H \to G$ of Proposition 1.2 is a monomorphism. Then, because $\tau_k' = \tau_k \vee \bigwedge_{i \neq k} \tau_i$ is in $T_H =$ typeset (H) and rank $G(\tau_k') = 1$, $\Psi_{\mathcal{E}}(H(\tau_k'))$ necessarily has finite index in $G(\tau_k') = G(\tau_k)$. Consequently, $\Psi_{\mathcal{E}}(H)$ has finite index in $G = \sum_{k \in \overline{n}} G(\tau_k)$; that is, $H \sim G$, as desired. ■

The equivalence of conditions (1) and (3) in Corollary 3.4 is, of course, due to Fuchs and Metelli, and our proof that (3) implies (1) is the same as given in Proposition 4.5 of [11]. The advantage of our new condition (2) over condition (3) is that one can check more rapidly whether or not the former is satisfied. A proof that $G \sim H$ implies condition (2) of Corollary 3.4 can also be derived from Yom's «Vertex Switch» Theorem [17], but the proof of the latter itself involves a quite complicated induction on rank using the same general method needed to establish our Theorem 3.2.

On the other hand, the original Fuchs-Metelli characterization remains a potent tool with applications that do not follow directly from condition (2) of Corollary 3.4. As an illustration of this fact, we shall next show that when G and H are as in Theorem 3.2 then the analogue of condition (c) with the $\overline{n}_G(\sigma)$'s replaced by the corresponding $G(\sigma)$'s is also equivalent to $G \sim H$. It should be noted that this yields a slight refinement of the Arnold-Vinsonhaler characterization [6].

COROLLARY 3.4. *Let $G = G[A_1, \dots, A_n]$ and $H = G[B_1, \dots, B_n]$ be strongly indecomposable groups with equal typesets. Then G and H are quasi-isomorphic if and only if the following condition is satisfied:*

(‡) $\operatorname{rank}\left(\sum_{\sigma\in M} G(\sigma)\right) = |M|$ *for every proper subset* M *of* $\{\sigma_1, \dots, \sigma_n\}$.

PROOF. From Corollary 3.2 and Theorem 1.1 (3), it is a routine induction to show that $\operatorname{rank}\left(\sum_{\sigma\in M} H(\sigma)\right) = |M|$ whenever M is a proper subset of $\{\sigma_1, \dots, \sigma_n\}$. Indeed if F is a nonempty proper subset of $\overline{n}$, then each nonzero element of $\sum_{i\in F} H(\sigma_i)$ has a positive multiple $x = \langle x_1, \dots, x_n\rangle$ with $x_j = 0$ for all $j \in F'$. Thus, if $G \sim H$ then $\operatorname{rank}\left(\sum_{\sigma\in M} G(\sigma)\right) = \operatorname{rank}\left(\sum_{\sigma\in M} H(\sigma)\right)$ and (‡) follows.

Conversely, assume that condition (‡) is satisfied. In particular then, $\operatorname{rank} G(\sigma_i) = 1$ for each $i \in \overline{n}$. Let $\mathcal{E}$ be as in Corollary 3.4, and assume by way of contradiction that G and H fail to be quasi-isomorphic. Then by condition (3) of Corollary 3.4, there must exist some $k \in \overline{n}$ such that $\det \mathcal{E}_k = 0$; say, $\operatorname{rank} \mathcal{E}_k = m < n$. We shall let $E_1, \dots, E_n$ denote the column vectors of $\mathcal{E}$. By permuting the elements of $\overline{n}$, we may assume that $k = n$ and that the column space of $\mathcal{E}_n$ is spanned by 1_n and the column vectors $E_1, \dots, E_{m-1}$ of $\mathcal{E}$. Since then E_m is a rational linear combination of these vectors, there exist integers $a_1, \dots, a_{m-1}$ and a such that

$$a_m E_m = a_1 E_1 + \dots + a_{m-1} E_{m-1} + a 1_n$$

with $a_m \neq 0$. Clearly, we can take these integers to lie in $\bigcap_{i\in\overline{n}} A_i$ and hence we may view $a_1 E_1, \dots, a_m E_m$ as elements of the group $A_1 \oplus \dots \oplus A_n$.

Notice also by Theorem 1.1 (3) that $g_i = \pi_{\mathcal{a}}(a_i E_i)$ is in $G(\sigma_i)$ for $i = 1, \ldots, m$. Consequently, $g_m = g_1 + \ldots + g_{m-1}$ is a nonzero element of $G(\sigma_m) \cap (G(\sigma_1) + \ldots + G(\sigma_{m-1}))$. Since rank $G(\sigma_m) = 1$, $G(\sigma_m)$ is contained in the pure subgroup $(G(\sigma_1) + \ldots + G(\sigma_{m-1}))_*$ generated by $\sum_{i=1}^{m-1} G(\sigma_i)$. But then

$$\text{rank}\left(\sum_{i=1}^{m} G(\sigma_i)\right) \leqslant \text{rank}\left(\sum_{i=1}^{m-1} G(\sigma_i)\right)_* \leqslant m-1$$

which contradicts condition (‡). ■

4. – Quasi-isomorphism implies equivalence of representations..

In this final section, we complete our proof that $G = G[A_1, \ldots, A_n] \sim H = G[B_1, \ldots, B_n]$ implies that the representations $\mathcal{R}_G$ and $\mathcal{R}_H$ are equivalent. The proof will be by induction on the rank of the groups and will, of course, rely heavily on the fact that the result holds in the case where G and H are strongly indecomposable (Theorem 3.2).

We begin with a basic fact about quasi-decompositions of groups of the form $G[\mathcal{a}]$. In the formulation of this preliminary result, we find it convenient to use the following *ad hoc* notation: If $I = \{i_1, \ldots, i_s\}$ is the subset of $\overline{n}$ with $|I| = s$, then we let $G[\mathcal{a}_I]$ denote the $\mathcal{B}^{(1)}$-group $G[A_{i_1}, \ldots, A_{i_s}]$.

LEMMA 4.1. *Let I and J be subsets of $\overline{n}$ such that $I \cup J = \overline{n}$ and $I \cap J = \{r\}$. If $\tau_I \vee \tau_J \geqslant \tau_r$, then*

(a) $G[A_1, \ldots, A_n] \sim G[\mathcal{a}_I] \oplus G[\mathcal{a}_J]$; *and*

(b) $\tau_E \vee \tau_{E'} = (\tau_E \vee \tau_{I \setminus E})$ *whenever E is a subset of I not containing r.*

PROOF. The assertion (1) is established in both [12] and [15]. Now suppose $\tau_I \vee \tau_J = \tau_r$ and let E be a subset of I with $r \notin E$. Then $\tau_E \vee \tau_{I \setminus E} \leqslant \tau_E \vee \tau_r = \tau_E \vee \tau_I \vee \tau_J = \tau_E \vee \tau_J$ because $E \subseteq I$. Therefore, $\tau_E \vee \tau_{E'} = \tau_E \vee (\tau_{I \setminus E} \wedge \tau_J) = (\tau_E \vee \tau_{I \setminus E}) \wedge (\tau_E \vee \tau_J) = \tau_E \vee \tau_{I \setminus E}$ as needed. ■

As noted in [12], repeated applications of the first part of Lemma 4.1 yields, via Theorem 3.1, the following fundamental fact:

$G[A_1, \ldots, A_n] \sim G[\mathcal{A}_{I_1}] \oplus \ldots \oplus G[\mathcal{A}_{I_s}]$, where $\overline{n} = \bigcup_{I \leq k \leq s} I_k$, $|I_J \cap I_k| \leq 1$ whenever $j \neq k$ and each $G[\mathcal{A}_{I_k}]$ is strongly indecomposable. This result also appears (with less formal notation) in [11], and was first proved in the dual setting in [4]. By Jónsson's celebrated version of the Krull-Schmidt theorem for quasi-decompositions of finite rank torsion-free groups [16], we conclude the following: If $H = G[B_1, \ldots, B_n]$ is quasi-isomorphic to $G[A_1, \ldots, A_n]$, then $H \sim G[B_{J_1}] \oplus \ldots \oplus G[\mathcal{B}_{J_s}]$ with $G[\mathcal{B}_{J_k}] \sim G[\mathcal{A}_{I_k}]$ for each k. These observations coupled with Theorem 3.4 serve as a basis for the proof of our main theorem, but the second assertion in Lemma 4.1 will play an equally important role.

Although the simultaneous decompositions of quasi-isomorphic groups $G[A_1, \ldots, A_n] \sim G[\mathcal{A}_{I_1}] \oplus \ldots \oplus G[\mathcal{A}_{I_s}]$ and $H \sim G[B_{J_1}] \oplus \ldots \oplus G[\mathcal{B}_{J_s}]$ with $G[\mathcal{B}_{J_k}] \sim G[\mathcal{A}_{I_k}]$, ostensibly resolve the issue of the equivalence of $\mathcal{R}_G$ and $\mathcal{R}_H$ by Theorem 3.2 and the fact that coproducts of representations match up quite nicely, some precautions must be adopted. Plainly put, when G quasi-decomposes into $G = G[\mathcal{A}_I] \oplus G[\mathcal{A}_J]$ arising from an $r \in I \cap J$ satisfying $\tau_r = \tau_I \vee \tau_J$, $\mathcal{R}_G$ fails to be the coproduct of $\mathcal{R}_{G[A_I]}$ and $\mathcal{R}_{G[A_J]}$. This is because the vector space dimensions are simply not right. This difficulty is overcome by passing from $\mathcal{R}_G$ to the *reduced representation* $\overline{\mathcal{R}}_G$ wherein $2^{\overline{n}}$ and each component subspace is replaced by the corresponding quotient space modulo $\{\emptyset, \overline{n}\}$.

Note that equivalence of representations is preserved under this reduction; that is, $\mathcal{R}_H \simeq \mathcal{R}_G$ if and only if $\overline{\mathcal{R}}_H \simeq \overline{\mathcal{R}}_G$. Indeed if $\mathcal{R}_H \simeq \mathcal{R}_G$ via a vector space isomorphism $\Omega : 2^{\overline{n}} \to 2^{\overline{n}}$, then as observed in the proof of Corollary 2.3 we may assume that $\Omega(\overline{n}) = \overline{n}$ and hence Ω induces an isomorphism on quotients modulo $\{\emptyset, \overline{n}\}$ which will yield $\overline{\mathcal{R}}_H \simeq \overline{\mathcal{R}}_G$. On the other hand, if $\overline{\Omega}: 2^{\overline{n}}/\{\emptyset, \overline{n}\} \to 2^{\overline{n}}/\{\emptyset, \overline{n}\}$ is a vector space isomorphism associated with an equivalence $\overline{\mathcal{R}}_H \simeq \overline{\mathcal{R}}_G$, then we can take $\Omega : 2^{\overline{n}} \to 2^{\overline{n}}$ to be any of the 2^{n-1} liftings of $\overline{\Omega}$ as follows: For each $1 \leq j < n$, select E_j such that $\overline{\Omega}(\{j\} + \{\emptyset, \overline{n}\}) = E_j + \{\emptyset, \overline{n}\}$. Then extend Ω linearly, in that $\Omega(F) = \sum_{j \in F} E_j$. Since $\overline{\Omega}(E + \{\emptyset, \overline{n}\}) = F + \{\emptyset, \overline{n}\}$ is an element of $\overline{n}_G(\sigma)/\{\emptyset, \overline{n}\}$, then either possibility $\Omega(E) = F$ or $\Omega(E) = F'$ yields a member of $\overline{n}_G(\sigma)$, showing that Ω is a representation isomorphism from $\mathcal{R}_H$ onto $\mathcal{R}_G$.

LEMMA 4.2. *Let I, J and r be as in the statement of Lemma 4.1 so that $G \sim G_1 \oplus G_2$ where $G_1 = G[\mathcal{A}_I]$ and $G_2 = G[\mathcal{A}_J]$. Then $\overline{R}_G \simeq \overline{\mathcal{R}}_{G_1} \oplus \oplus \overline{\mathcal{R}}_{G_2}$.*

PROOF. First note that, for a fixed subset I of $\overline{n}$, the map $E \to E \cap I$ is a ring epimorphism from $2^{\overline{n}}$ onto 2^I. Therefore the correspondence $E \to (E \cap I, E \cap J)$ induces a linear map $\Omega : 2^{\overline{n}}/\{\emptyset, \overline{n}\} \to 2^I/\{\emptyset, I\} \oplus 2^J/\{\emptyset, J\}$. Moreover, the hypothesis on I and J insures that Ω is one-to-one and hence a vector space isomorphism since $n - 1 = (|I| - 1) + (|J| - 1)$. If $I(\sigma)$ and $J(\sigma)$ are defined in the manner analogous to the definition of $\overline{n}_G(\sigma)$, then by Theorem 2.1, $\dim (I(\sigma)/\{\emptyset, I\}) + \dim (J(\sigma)/\{\emptyset, J\}) = \operatorname{rank} G_1(\sigma) + \operatorname{rank} G_2(\sigma) = \operatorname{rank} G(\sigma) = \dim (\overline{n}_G(\sigma)/\{\emptyset, \overline{n}\})$. Finally, Lemma 4.1 (b) and this latter observation imply that Ω defines an equivalence between $\overline{\mathcal{R}}_G$ and $\overline{\mathcal{R}}_{G_1} \oplus \overline{\mathcal{R}}_{G_2}$. ■

We now have all the ingredients needed to establish our main result.

THEOREM 4.1. *Suppose that* $G = G[A_1, \ldots, A_n]$ *and* $H = G[B_1, \ldots, B_n]$ *have the same typeset. Then* $G \sim H$ *if and only if the* $\mathbb{Z}_{(2)}$-*representations* $\mathcal{R}_G$ *and* $\mathcal{R}_H$ *are equivalent.*

PROOF. By Corollary 2.1, it remains only to show that $H \sim G$ implies $\mathcal{R}_H \simeq \mathcal{R}_G$; or equivalently, $\overline{\mathcal{R}}_H \simeq \overline{\mathcal{R}}_G$. From the discussion following Lemma 4.1, we have $G \sim G[\mathcal{A}_{I_1}] \oplus \ldots \oplus G[\mathcal{A}_{I_s}]$ and $H \sim G[\mathcal{B}_{J_1}] \oplus \ldots \oplus G[\mathcal{B}_{J_s}]$ where, for each $i = 1, 2, \ldots, s$, $G[\mathcal{A}_{I_i}]$ and $G[\mathcal{B}_{J_i}]$ are quasi-isomorphic strongly indecomposable groups.

For notational simplicity, let $G_i = G[\mathcal{A}_{I_i}]$ and $H_i = G[\mathcal{B}_{J_i}]$ for $i = 1, 2, \ldots, s$. Since the process of obtaining the quasi-decompositions for G is by successive applications of Lemma 4.1, repeated applications of Lemma 4.2 yields $\overline{\mathcal{R}}_G \simeq \overline{\mathcal{R}}_{G_1} \oplus \ldots \oplus \overline{\mathcal{R}}_{G_s} \simeq \overline{\mathcal{R}}_{H_1} \oplus \ldots \oplus \overline{\mathcal{R}}_{H_s} \simeq \overline{\mathcal{R}}_H$. We conclude that $\mathcal{R}_H \simeq \mathcal{R}_G$ as desired. ■

In closing, we note that it is possible to give an alternative proof of Theorem 4.3 using Theorem 2.2 together with Theorem 3.2 and an induction on rank. Such a proof is technically more complicated since it involves the detailed analysis carried out in [13] of quasi-decompositions $G[A_1, \ldots, A_n] \sim G[\mathcal{A}_I] \oplus G[\mathcal{A}_J]$ where I is a subset of $\overline{n}$ with $G[\mathcal{A}_I]$ a prescribed strongly indecomposable quasi-summand of $G[A_1, \ldots, A_n]$.

REFERENCES

[1] D. ARNOLD, *Representations of partially ordered sets and abelian groups*, Contemporary Math., 87 (1989), pp. 91-109.

[2] D. ARNOLD - M. DUGAS, *A survey of Butler groups and the role of representations*, in *Abelian Groups and Modules*, Lecture Notes in Pure and Applied Mathematics, **182**, 1996.

[3] D. ARNOLD - F. RICHMAN - C. VINSONHALER, *Representations of posets and finite valuated groups*, to appear in the Journal of Algebra.

[4] D. ARNOLD - C. VINSONHALER, *Quasi-isomorphism invariants for a class of torsion-free abelian groups*, Houston J. Math., **15** (1989), pp. 327-340.

[5] D. ARNOLD - C. VINSONHALER, *Invariants for a class of torsion-free abelian groups*, Proc. Amer. Math. Soc., **105** (1989), p. 327-340.

[6] D. ARNOLD - C. VINSONHALER, *Duality and invariants for Butler groups*, Pacific J. Math., **148** (1991), pp. 1-10.

[7] D. ARNOLD - C. VINSONHALER, *Isomorphism invariants for Butler groups*, Trans. Amer. Math. Soc., to appear in the Transactions of the AMS.

[8] D. ARNOLD - C. VINSONHALER, *Quasi-isomorphism invariants for two classes of finite rank Butler groups*, preprint.

[9] D. ARNOLD - C. VINSONHALER, *Finite rank Butler groups*, in *Abelian Groups*, Lecture Notes in Pure and Applied Mathematics, **146**, 1993.

[10] M. C. R. BUTLER, *Torsion-free modules and diagrams of vector spaces*, *Proc. London. Math. Soc.*, **18** (1968), pp. 635-652.

[11] L. FUCHS - C. METELLI, *On a class of Butler groups*, Manuscripta Math., **71** (1991), pp. 1-28.

[12] H. P. GOETERS - W. ULLERY, *Butler groups and lattices of types*, Comment. Math. Univ. Carolinae, **31** (1990), pp. 613-619.

[13] H. P. GOETERS - W. ULLERY, *Quasi-summands of a certain class of Butler groups*, in *Abelian Groups*, Lecture Notes in Pure and Applied Mathematics, **146** (1993), pp. 167-174.

[14] P. HILL - C. MEGIBBEN, *The classification of certain Butler groups*, to appear in the Journal of Algebra.

[15] W. Y. LEE, *Co-diagonal Butler groups*, Chinese J. Math. **17** (1989), pp. 259-271.

[16] B. JÓNSSON, *On direct decompositions of torsion-free abelian groups*, Math. Scand., **7** (1959), pp. 361-371.

[17] P. D. YOM, *A characterization of class of torsion-free abelian groups*, preprint.

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