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## The Stochastic Logistic Equation: Stationary Solutions and their Stability.

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ABSTRACT - Starting from the logistic equation we introduce uncertainty on the parameters and then we look for stochastic stationary solutions and conditions for their stability. In the stochastic case we do not obtain all the deterministic stationary solutions, but we can find invariant distributions which can be considered the stochastic analogue of the deterministic stationary solutions. In particular, in this work we give conditions on the parameters of the equation under which these invariant distributions exist and conditions under which the stationary solutions are stable.

### 1. Introduction.

We consider the deterministic logistic equation

$$(1) \quad dx_t = x_t(\alpha - \beta x_t) dt$$

where  $\alpha$  and  $\beta$  are two positive constants. It is well known that the solutions of (1) are of the form

$$(2) \quad x(t) = \frac{\alpha x_0}{\beta x_0 + (\alpha - \beta x_0) e^{-\alpha t}}.$$

In particular, for  $x_0=0$  and  $x_0=\frac{\alpha}{\beta}$ , we have the stationary solu-

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tions

$$x_t \equiv 0 \quad \text{and} \quad x_t \equiv \frac{\alpha}{\beta}.$$

It is also well known that the stationary solution  $x_t \equiv 0$  is unstable since the eigenvalue of the Jacobian is greater than zero (it is equal to  $\alpha$ ) while the stationary solution  $x_t \equiv \frac{\alpha}{\beta}$  is asymptotically stable, in fact the eigenvalue of the Jacobian is  $-\alpha$ .

In this work we suppose that the parameters  $\alpha$  and  $\beta$  are not exactly known, so we introduce uncertainty on them and transform the deterministic problem into a corresponding stochastic problem. The aim is to study, for the latter, the stability of the stochastic equilibria.

Introducing the auxiliary processes  $\gamma_t^{(1)}$  and  $\gamma_t^{(2)}$ , equation (1) can be written in the following form

$$(3) \quad \begin{cases} dx_t = x_t d\gamma_t^{(1)} - x_t^2 d\gamma_t^{(2)} \\ d\gamma_t^{(1)} = \alpha dt \\ d\gamma_t^{(2)} = \beta dt \end{cases}$$

Now, we suppose that the processes  $\gamma_t^{(1)}$  and  $\gamma_t^{(2)}$  are subject to uncertainty, so we consider the system

$$(4) \quad \begin{cases} dx_t = x_t d\gamma_t^{(1)} - x_t^2 d\gamma_t^{(2)} \\ d\gamma_t^{(1)} = \alpha dt + \sigma_1 dw_t^{(1)} \\ d\gamma_t^{(2)} = \beta dt + \sigma_2 dw_t^{(2)} \end{cases}$$

where  $w_t^{(1)}$  and  $w_t^{(2)}$  are two independent Wiener processes and  $\sigma_1$  and  $\sigma_2$  two positive constants. In this case we obtain the stochastic differential equation

$$(5) \quad dx_t = x_t(\alpha - \beta x_t) dt + \sigma_1 x_t dw_t^{(1)} - \sigma_2 x_t^2 dw_t^{(2)}.$$

In what follows we will discuss the stability of the stationary solutions of equation (5) and of two particular equations obtained from it taking  $\sigma_2 = 0$  and  $\sigma_1 = 0$  respectively, that is

$$(6) \quad dx_t = x_t(\alpha - \beta x_t) dt + \sigma x_t dw_t$$

and

$$(7) \quad dx_t = x_t(\alpha - \beta x_t) dt - \sigma x_t^2 dw_t.$$

In equation (6) we do not know exactly the linear term, while we have a perfect knowledge of the quadratic term; conversely, in equation (7) we do not know exactly the quadratic term.

In equations (6) and (7) we have put, for simplicity,  $\sigma_1 \equiv \sigma$  and  $\sigma_2 \equiv \sigma$  respectively.

In section 2 we recall some useful results concerning stochastic stability of the equilibrium solution of a stochastic differential equation and in section 4 we apply these concepts to model (5). In section 3 we determine the stationary solutions and the conditions for the existence of invariant distributions for the three models given before.

## 2. Stochastic stability.

Consider a one-dimensional stochastic differential equation

$$(8) \quad dx_t = f(t, x_t) dt + \sum_{i=1}^m g_i(t, x_t) dw_t^{(i)}, \quad t \geq t_0, \quad x(t_0) = x_0$$

where  $W_t = (w_t^{(1)}, \dots, w_t^{(m)})$  is a standard  $m$ -dimensional Wiener process and the coefficients  $f$  and  $g_i$  ( $i = 1, \dots, m$ ) satisfy the assumptions for the existence and uniqueness of the solution of a stochastic differential equation.

**DEFINITION 1.** *The stochastic process  $x_t \equiv \bar{x}$  is a stationary solution of equation (8) with initial condition  $x(t_0) = \bar{x}$  if*

$$f(t, \bar{x}) = 0 \quad g_i(t, \bar{x}) = 0 \quad (i = 1, \dots, m).$$

**DEFINITION 2.** *The trivial solution  $x_t \equiv 0$  is said to be*

1. *stable in probability if for every  $\varepsilon > 0$  and  $s \geq t_0$*

$$(9) \quad \lim_{y \rightarrow 0} P \left( \sup_{t \in [s, \infty)} |x(t; s, y)| \geq \varepsilon \right) = 0$$

*where  $x(t; s, y)$  denotes the solution of (8) satisfying the constant initial condition  $x(s) = y$ .*

2. *asymptotically stable if it is stable in probability and moreover*

$$(10) \quad \lim_{y \rightarrow 0} P \left( \lim_{t \rightarrow +\infty} |x(t; s, y)| = 0 \right) = 1 \quad s > t_0.$$

Denoting by  $L$  the differential operator defined for a function  $\xi(t, x) \in C^{1,2}$  by

$$L\xi = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} f(t, x) + \frac{1}{2} \frac{\partial^2 \xi}{\partial x^2} \sum_{i=0}^m g_i^2(t, x)$$

the following theorem gives sufficient conditions for the stability of the equilibrium solution of the stochastic differential equation (8) in terms of Lyapunov functions.

**THEOREM 1.** *If there exists a Lyapunov function  $V(t, x)$ , defined on a bounded open neighborhood  $D$  of the origin (i.e. a function  $V \in C^{1,2}$  such that  $V(t, 0) = 0$  and  $V(t, x) > 0$  for any  $x \in D \setminus \{0\}$ ), such that*

$$(11) \quad LV(t, x) \leq 0 \quad (LV(t, x) < 0)$$

*for any  $x \in D \setminus \{0\}$ , then the equilibrium solution  $x_t \equiv 0$  of the stochastic differential equation (8) is stable (respectively, asymptotically stable) in probability.*

### 3. Stationary solutions and invariant distributions.

In order to find the stationary solutions to equation (5), we equal to zero both drift and diffusion coefficient of (5) (see definition 1), then we obtain only the stationary solution  $x_t \equiv 0$ , which is the same as in the deterministic case of equation (1). In the stochastic model we look at  $x_t \equiv 0$  as degenerate process with mass concentrated in 0. All the solutions obtained equaling to zero both drift and diffusion coefficients are degenerate processes. We remark that, in this way, we do not obtain the second stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ , that is  $x_t \equiv \frac{\alpha}{\beta}$  is not a degenerate solution of (5).

Numerically, the trajectories of the process solution of (5), obtained by an Euler discretization, are given in Figure 1 (for some value of  $\sigma_1$  and  $\sigma_2$ ).

As we can see from Figure 1, for  $\sigma_1$  and  $\sigma_2$  sufficiently small, the trajectories of  $x_t$ , satisfying equation (5), fluctuate around the deterministic trajectories. So we can think that there is something analogous to the deterministic equilibrium  $\frac{\alpha}{\beta}$ .

Associated to the equilibrium solution  $x_t \equiv 0$  there is an invariant Dirac delta distribution, so we can ask if there exists another invariant

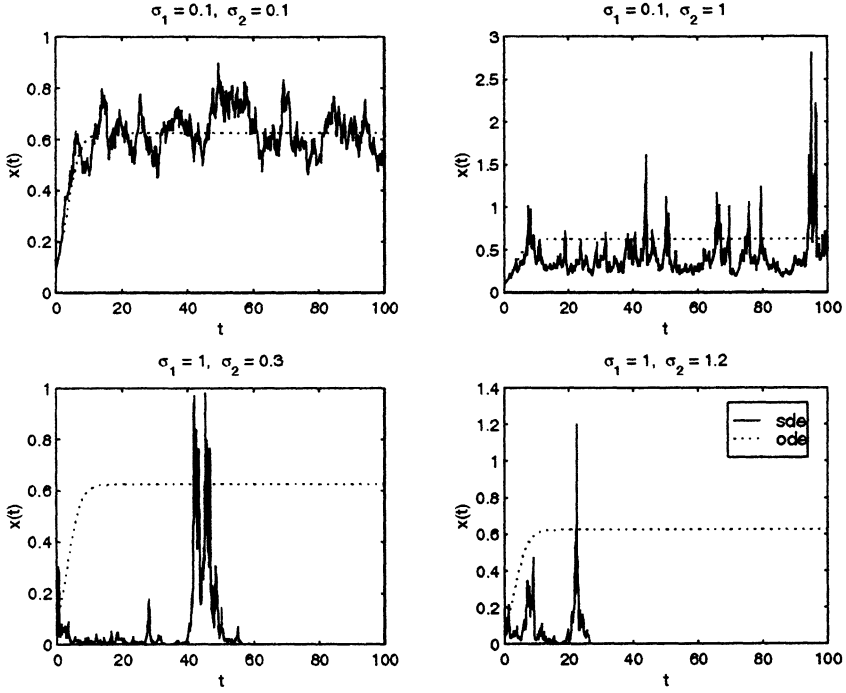


Fig. 1. – Deterministic and stochastic trajectories for some value of  $\sigma_1$  and  $\sigma_2$  for equation (5). We have denoted by *ode* and *sde* the deterministic and stochastic trajectories respectively. In these graphics  $\alpha = 0.5$ ,  $\beta = 0.8$  and  $x_0 = 0.1$ .

distribution which plays the role of the deterministic stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ . To this end we look for the solution of the stationary forward Fokker-Plank equation (see [5]).

In the case of equation (5), the stationary Fokker-Plank equation is

$$(12) \quad \frac{d}{dx}(x(\alpha - \beta x) p(x)) - \frac{1}{2} \frac{d^2}{dx^2}[(\sigma_1^2 x^2 + \sigma_2^2 x^4) p(x)] = 0$$

with  $p(x)$  a  $C^2$  function.

The following result holds:

**THEOREM 2.** *Equation (12) has a non-trivial solution if and only if*

$$(13) \quad \sigma_1^2 < 2\alpha .$$

The non-degenerate solution is a density function of the form

$$(14) \quad p(x) = Kx^{\frac{2\alpha}{\sigma_1^2} - 2} \left(1 + \frac{\sigma_2^2}{\sigma_1^2} x^2\right)^{-\frac{\alpha}{\sigma_1^2} - 1} e^{-\frac{2\beta}{\sigma_1\sigma_2} \arctan\left(\frac{\sigma_2}{\sigma_1} x\right)} \quad x > 0$$

where

$$(15) \quad K^{-1} = \int_0^{\infty} x^{\frac{2\alpha}{\sigma_1^2} - 2} \left(1 + \frac{\sigma_2^2}{\sigma_1^2} x^2\right)^{-\frac{\alpha}{\sigma_1^2} - 1} e^{-\frac{2\beta}{\sigma_1\sigma_2} \arctan\left(\frac{\sigma_2}{\sigma_1} x\right)} dx$$

is the normalizing constant.

PROOF. Write equation (12) in the following simpler form

$$(16) \quad y' - a(x)y = -c$$

where  $y = (\sigma_1^2 x^2 + \sigma_2^2 x^4) p(x)$ ;  $a(x) = \frac{2x(\alpha - \beta x)}{\sigma_1^2 x^2 + \sigma_2^2 x^4}$  and  $c$  is a constant.

The solution of (16) is

$$y(x) = A(x) \left( k - c \int_1^x \frac{1}{A(t)} dt \right)$$

with  $A(t) = \exp\left(\int_1^t a(u) du\right)$ .

It follows that

$$(17) \quad p(x) = \frac{1}{\sigma_1^2 x^2 + \sigma_2^2 x^4} A(x) \left( k - c \int_1^x \frac{1}{A(t)} dt \right) \quad x > 0.$$

We prove that  $p(x)$  is a density function if and only if  $c = 0$ .

Function  $p(x)$  is a density if its integral between 0 and infinity is bounded.

We show that, in a neighborhood of 0,  $p(x)$  has bounded integral only when  $c = 0$  and condition (13) holds.

For  $0 < x < 1$

$$p(x) = \frac{A(x)}{\sigma_1^2 x^2 + \sigma_2^2 x^4} \left( k + c \int_x^1 \frac{1}{A(t)} dt \right).$$

We take  $k > 0$  and  $c \geq 0$  because  $p(x)$  is a density.

Observing that

$$(18) \quad \int_1^x a(u) du = \frac{2\alpha}{\sigma_1^2} \ln x - \frac{\alpha}{\sigma_1^2} \ln \left( 1 + \frac{\sigma_2^2}{\sigma_1^2} x^2 \right) - \frac{2\beta}{\sigma_1 \sigma_2} \arctan \left( \frac{\sigma_2}{\sigma_1} x \right) + c_1$$

with  $c_1$  constant and therefore

$$A(x) = c_2 \frac{x^{\frac{2\alpha}{\sigma_1^2}}}{\left( 1 + \frac{\sigma_2^2}{\sigma_1^2} x^2 \right)^{\frac{\alpha}{\sigma_1^2}}} e^{-\frac{2\beta}{\sigma_1 \sigma_2} \arctan \left( \frac{\sigma_2}{\sigma_1} x \right)}$$

where  $c_2$  is a positive constant, in a neighborhood of 0 we have ( $0 < x < 1$ )

$$K_1 x^{\frac{2\alpha}{\sigma_1^2}} \leq A(x) \leq K_2 x^{\frac{2\alpha}{\sigma_1^2}}$$

with  $K_1$  and  $K_2$  positive constants. Consequently

$$\frac{K_2^{-1}}{1 - \frac{2\alpha}{\sigma_1^2}} \left( 1 - x^{1 - \frac{2\alpha}{\sigma_1^2}} \right) \leq \int_x^1 \frac{1}{A(t)} dt \leq \frac{K_1^{-1}}{1 - \frac{2\alpha}{\sigma_1^2}} \left( 1 - x^{1 - \frac{2\alpha}{\sigma_1^2}} \right).$$

It follows that, for  $\sigma_1^2 < 2\alpha$

$$(19) \quad p(x) \geq kK_3 x^{\frac{2\alpha}{\sigma_1^2} - 2} - cK_4 x^{\frac{2\alpha}{\sigma_1^2} - 2} + cK_5 \frac{1}{x}$$

with  $K_3, K_4$  and  $K_5$  positive constants. The integral of the right hand side of (19) between 0 and 1 is infinite for  $c \neq 0$ . Then, for  $\sigma_1^2 < 2\alpha$ , it must be  $c = 0$  otherwise  $p(x)$  is not a density.

Analogously, for  $\sigma_1^2 > 2\alpha$ , the integral of  $p(x)$  is greater than

$$kK_6 x^{\frac{2\alpha}{\sigma_1^2} - 2} + cK_7 x^{\frac{2\alpha}{\sigma_1^2} - 2} - cK_8 \frac{1}{x}$$

with  $K_6, K_7$  and  $K_8$  positive constants, and the integral of this quantity between 0 and 1 diverges positively for any positive value of  $c$  and  $k$ .

Finally, for  $\sigma_1^2 = 2\alpha$  we have

$$(20) \quad p(x) \geq kK_9 \frac{1}{x} - cK_{10} \frac{\log(x)}{x}$$



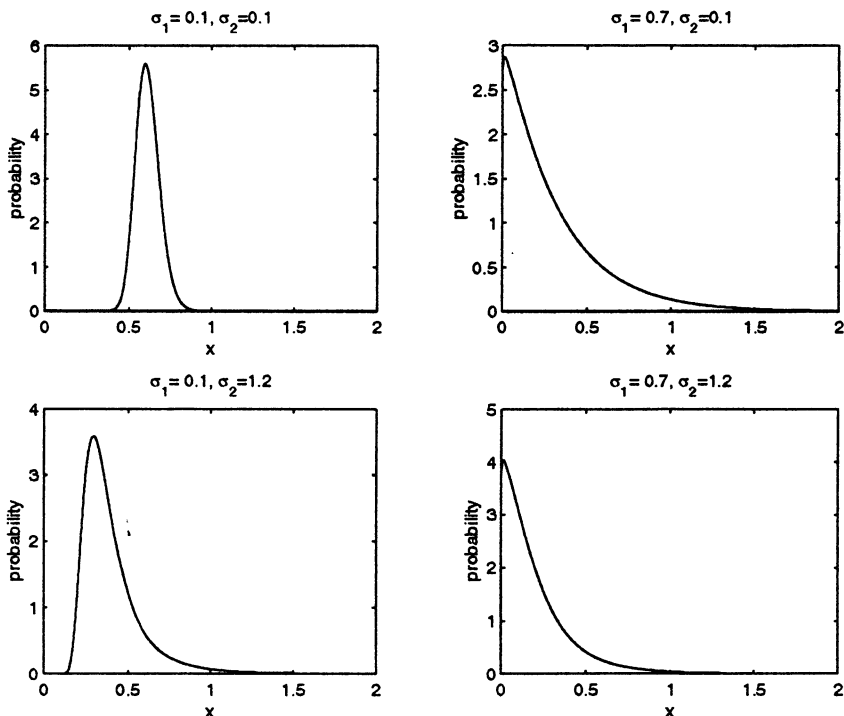


Fig. 2. – Density (14) for some value of  $\sigma_1$  and  $\sigma_2$ . The distributions are plotted for  $\alpha = 0.5$  and  $\beta = 0.8$ .

with  $K_9$  and  $K_{10}$  positive constants. Again, the right hand side of (20) has infinite integral between 0 and 1.

It is easy to see, from the previous discussion, that the integral of  $p(x)$  is finite only in the case  $\sigma_1^2 < 2\alpha$  and  $c = 0$ .

We can conclude that, under hypothesis (13),  $p(x)$  is a density and it has the form (14). ■

Density (14) is plotted, for some values of  $\sigma_1$  and  $\sigma_2$ , in Figure 2.

The graphics in Figure 2 suggest that in this case the density function converges, as  $\sigma_1$  and  $\sigma_2$  go to zero, to a Dirac delta distribution with mass concentrated in  $\frac{\alpha}{\beta}$ , while for  $\sigma_1$  sufficiently large, density (14) is concentrated in a right neighborhood of zero.

Now we consider the particular case  $\sigma_2 = 0$ , that is equation (6).

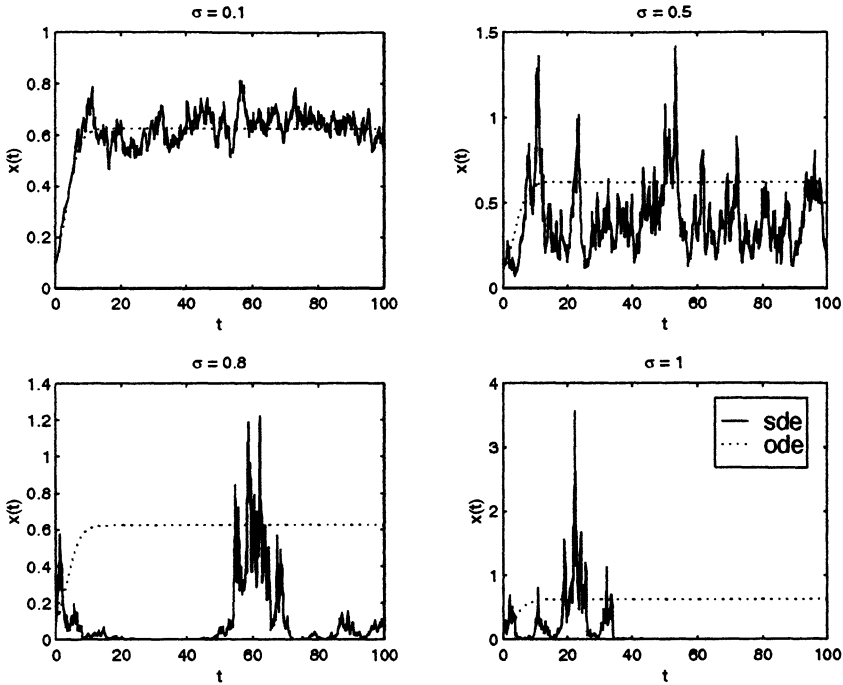


Fig. 3. – Deterministic and stochastic trajectories for  $\sigma = 0.1$ ,  $\sigma = 0.5$ ,  $\sigma = 0.8$  and  $\sigma = 1$ , respectively, for equation (6). We have denoted by *ode* and *sde* the deterministic and stochastic trajectories respectively. In these graphics  $\alpha = 0.5$ ,  $\beta = 0.8$  and  $x_0 = 0.1$ .

As we can see from the numerical simulation of the trajectories given in Figure 3, for small values of  $\sigma$  the trajectories fluctuate (for sufficiently large  $t$ ) around the deterministic stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ . In fact, following the proof of theorem 2 one can prove

**COROLLARY 1.** *Equation (12), for  $\sigma_2 = 0$  and  $\sigma_1 \equiv \sigma$ , has a non-trivial solution if and only if*

$$(21) \quad \sigma^2 < 2\alpha$$

*The non-trivial solution is a Gamma density with parameters  $\frac{2\beta}{\sigma^2}$  and  $\frac{2\alpha}{\sigma^2} - 1$ .*

Since  $p(x)$  is a Gamma density, it has the form

$$(22) \quad p(x) = \frac{\left(\frac{2\beta}{\sigma^2}\right)^{\frac{2\alpha}{\sigma^2}-1}}{\Gamma\left(\frac{2\alpha}{\sigma^2}-1\right)} x^{\frac{2\alpha}{\sigma^2}-2} e^{-\frac{2\beta}{\sigma^2}x}$$

for  $\sigma^2 < 2\alpha$ .

The mean of the density (22) is

$$(23) \quad E(x) = \frac{\alpha}{\beta} - \frac{\sigma^2}{2\beta}$$

and the variance

$$(24) \quad \text{var}(x) = \frac{\alpha\sigma^2}{2\beta^2} - \frac{\sigma^4}{4\beta^2}.$$

From corollary 1 we have that, when condition (21) holds, we have two invariant distribution for equation (6): the Dirac delta distribution relative to the null solution and a distribution with Gamma density (22).

**REMARK 1.** *The Gamma density (22), for  $\sigma^2 < \alpha$ , has a maximum in  $x = \frac{\alpha - \sigma^2}{\beta}$ , while for  $\sigma^2 \in [\alpha, 2\alpha)$  the maximum is in zero. So, for each  $t$ , the density (22) is concentrated around  $\frac{\alpha - \sigma^2}{\beta}$  and for  $\sigma^2 \in [\alpha, 2\alpha)$  in a right neighborhood of zero.*

Remark 1 tells us that, in the case  $\sigma^2 < 2\alpha$ , the process, for  $t$  sufficiently large, fluctuates around the maximum of the density (22) but remaining always positive.

**REMARK 2.** *It is important to observe that, for  $\sigma$  going to zero, this non-trivial invariant distribution approaches the Dirac delta distribution with mass concentrated in  $\frac{\alpha}{\beta}$ . In fact, as one can see from equations (23) and (24),  $\text{var}(x)$  goes to zero and  $E(x)$  goes to  $\frac{\alpha}{\beta}$  as  $\sigma$  goes to zero. This means that, for  $\sigma$  sufficiently small, the distribution is concentrated around the deterministic stationary solution  $\frac{\alpha}{\beta}$ .*

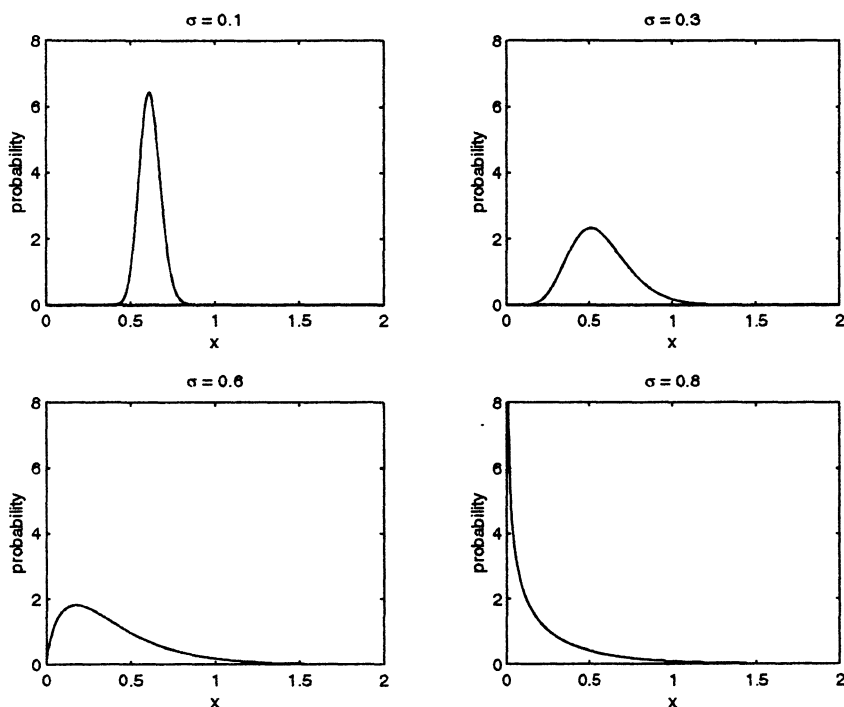


Fig. 4. – Gamma distribution for  $\sigma = 0.1$ ,  $\sigma = 0.3$ ,  $\sigma = 0.6$  and  $\sigma = 0.8$ , respectively. The distributions are plotted for  $\alpha = 0.5$  and  $\beta = 0.8$ .

The previous remarks are confirmed by the numerical results given in Figure 4.

Figure 5 shows that also in the particular case of equation (7), namely when  $\sigma_1 = 0$  in equation (5), the trajectories fluctuate around the deterministic stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ . In fact, following again the proof of theorem (2), one can prove

**COROLLARY 2.** *Equation (12), for  $\sigma_1 = 0$  and  $\sigma_2 \equiv \sigma$ , has a non-trivial solution, that is a density function of the form*

$$(25) \quad p(x) = \frac{K}{\sigma^2 x^4} e^{-\frac{\alpha}{\sigma^2 x^2} + \frac{2\beta}{\sigma^2 x}} \quad x > 0$$

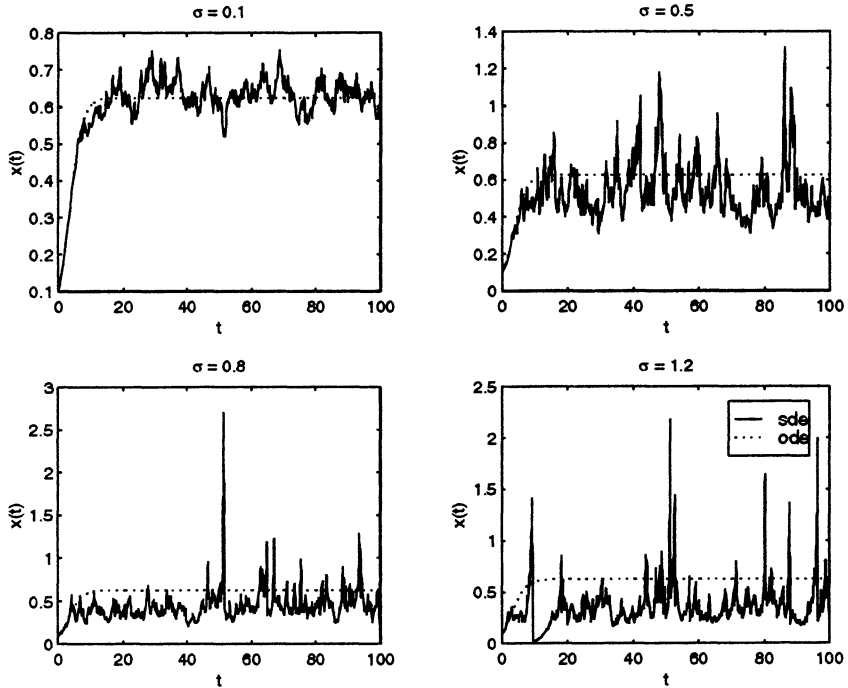


Fig. 5. – Deterministic and stochastic trajectories for  $\sigma = 0.1$ ,  $\sigma = 0.5$ ,  $\sigma = 0.8$  and  $\sigma = 1.2$ , respectively, for equation (7). We have denoted by *ode* and *sde* the deterministic and stochastic trajectories respectively. In these graphics  $\alpha = 0.5$ ,  $\beta = 0.8$  and  $x_0 = 1$ .

where  $K$  is the following constant

$$(26) \quad K = \frac{2\alpha\sqrt{2\alpha}}{\sigma} \left[ \frac{3\sqrt{2}\beta}{\sqrt{\alpha\sigma}} + \sqrt{2\pi}e^{-\frac{\beta^2}{\alpha\sigma^2}} \left( 1 - \Phi\left(-\frac{\sqrt{2}\beta}{\sqrt{\alpha\sigma}}\right) \right) \left( 1 + \frac{2\beta^2}{\alpha\sigma^2} \right) \right]^{-1}$$

and  $\Phi(\cdot)$  represents the normal cumulative distribution function.

Corollary 2 tells us that, for equation (7), we have two invariant distributions: the Dirac delta distribution associated to the null solution and a distribution with density (25).

**REMARK 3.** *In the case « $\beta$  uncertain» ( $\sigma_1 = 0$ ), the non-trivial invariant distribution always exists without conditions on the parameter  $\sigma$ .*

The mean of the density (25) is

$$(27) \quad E(x) = \frac{\sqrt{2\alpha} \left[ \sqrt{\frac{\alpha}{2}} \frac{\sigma}{\beta} e^{-\frac{\beta^2}{\alpha\sigma^2}} + \sqrt{2\pi} \left( 1 - \Phi \left( -\frac{\sqrt{2}\beta}{\sqrt{\alpha\sigma}} \right) \right) \right]}{3\sigma e^{-\frac{\beta^2}{\alpha\sigma^2}} + \sqrt{2\pi} \left( \sqrt{\frac{\alpha}{2}} \frac{\sigma^2}{\beta} + \sqrt{\frac{2}{\alpha}} \beta \right) \left( 1 - \Phi \left( -\frac{\sqrt{2}\beta}{\sqrt{\alpha\sigma}} \right) \right)}$$

and the variance

$$(28) \quad \text{var}(x) = \frac{2\alpha\sqrt{2\pi} \left( 1 - \Phi \left( -\frac{\sqrt{2}\beta}{\sqrt{\alpha\sigma}} \right) \right)}{\frac{3\sqrt{2}\beta\sigma}{\sqrt{\alpha}} e^{-\frac{\beta^2}{\alpha\sigma^2}} + \sqrt{2\pi} \left( \sigma^2 + \frac{2\beta}{\alpha} \right) \left( 1 - \Phi \left( -\frac{\sqrt{2}\beta}{\sqrt{\alpha\sigma}} \right) \right)}.$$

REMARK 4. *Observe that, as in the previous case ( $\alpha$  uncertain), the non-degenerate distribution converges, as  $\sigma$  goes to zero, to a Dirac*

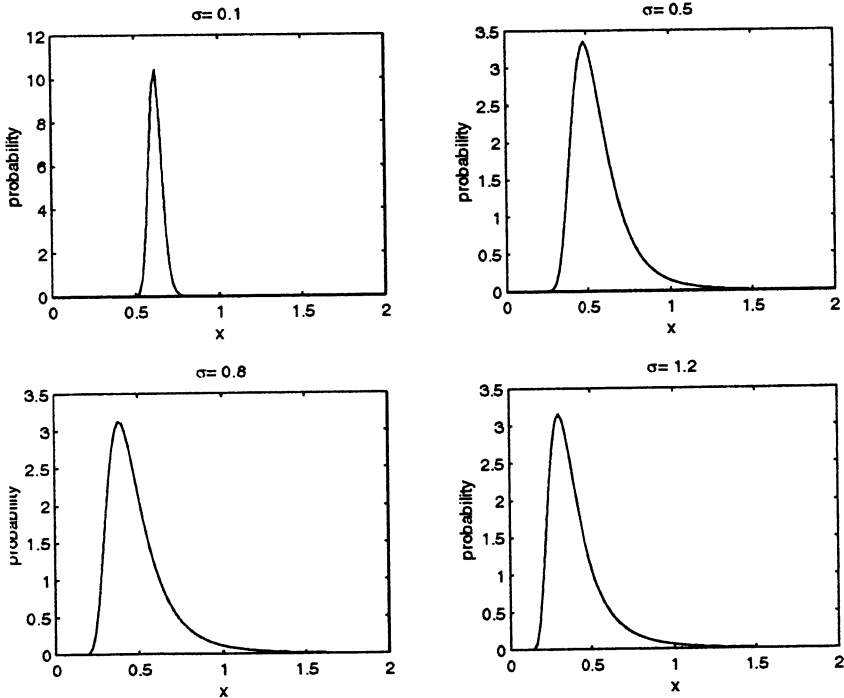


Fig. 6. – Density (25) for  $\sigma = 0.1$ ,  $\sigma = 0.5$ ,  $\sigma = 0.8$  and  $\sigma = 1.2$ , respectively. The densities are plotted for  $\alpha = 0.5$  and  $\beta = 0.8$ .

delta distribution with mass concentrated in  $\frac{\alpha}{\beta}$  which is the second stationary solution of the deterministic logistic equation (1). In fact, one can see that, as  $\sigma$  goes to zero,  $E(x)$  goes to  $\frac{\alpha}{\beta}$  and  $\text{var}(x)$  goes to zero.

These results are confirmed by Figure 6.

#### 4. Stability of the stationary solutions.

The invariant distribution with density (14) exists only under condition (13). So we want to analyze the behavior of the solution for  $\sigma_1^2 > 2\alpha$ . To this end we study the stability of the process  $x_t \equiv 0$ .

Figures 1 and 2 suggest that the null process is stable for  $\sigma_1$  large and unstable for  $\sigma_1$  small.

By choosing an appropriate Lyapunov function and applying theorem 1, we show the stability of  $x_t \equiv 0$  for  $\sigma_1$  large.

The following result holds

**THEOREM 3.** *For the model (5), the stationary solution  $x_t \equiv 0$  is stable in probability if  $\sigma_1^2 > 2\alpha$  and asymptotically stable in probability if  $\sigma_1^2 > \frac{2\alpha}{1-\varepsilon}$  (with  $\varepsilon$  a positive constant less than 1).*

**PROOF.** Consider the function

$$(29) \quad V(x) = \left| \int_0^x |y|^{-A} \left( 1 + \frac{\sigma_2^2}{\sigma_1^2} y^2 \right)^{\frac{\alpha}{\sigma_1^2}} e^{\frac{2\beta}{\sigma_1\sigma_2} \arctan\left(\frac{\sigma_2}{\sigma_1} y\right)} dy \right|$$

where  $A < 1$  is a positive constant.  $V(x)$  is a Lyapunov function and

$$(30) \quad \begin{aligned} LV(x) &= \\ &= \text{sgn}(x) x e^{-A \ln|x| + \frac{\alpha}{\sigma_1^2} \ln\left(1 + \frac{\sigma_2^2}{\sigma_1^2} x^2\right) + \frac{2\beta}{\sigma_1\sigma_2} \arctan\left(\frac{\sigma_2}{\sigma_1} x\right)} \left( 1 + \frac{\sigma_2^2}{\sigma_1^2} x^2 \right) \left( \alpha - \frac{A\sigma_1^2}{2} \right) \end{aligned}$$

$LV \leq 0$  for  $\sigma_1^2 \geq \frac{2\alpha}{A}$ , that is the stationary solution  $x_t \equiv 0$  is stable for  $\sigma_1^2 \geq \frac{2\alpha}{A}$  (in particular for  $A = \frac{2\alpha}{\sigma_1^2}$ ). More precisely, applying theorem 1 we obtain that

$$x_t \equiv 0 \text{ is stable in probability for } \sigma_1^2 > 2\alpha$$

$x_t \equiv 0$  is asymptotically stable in probability if  $\sigma_1^2 > \frac{2\alpha}{1-\varepsilon}$

where  $\varepsilon$  is a positive constant less than 1. ■

Theorem 3 gives the stability of the stationary solution  $x_t \equiv 0$  in the case  $\sigma_1^2 > 2\alpha$  without conditions on the coefficient  $\sigma_2$ .

In fact, in the particular case  $\sigma_1 = 0$ , theorem 2 asserts that there always exists the non-degenerate distribution with density (25), so the process fluctuates in a neighborhood of the deterministic solution  $\frac{\alpha}{\beta}$  and the stationary solution  $x_t \equiv 0$  is always unstable (see Figure 5).

The particular case  $\sigma_2 = 0$  is analogous to that of equation (5). In fact we have

**COROLLARY 3.** *For the model (6), the stationary solution  $x_t \equiv 0$  is stable in probability if  $\sigma^2 > 2\alpha$  and asymptotically stable in probability for  $\sigma^2 > \frac{2\alpha}{1-\varepsilon}$  (with  $\varepsilon$  a positive constant less than one).*

**PROOF.** We consider the Lyapunov function

$$(31) \quad V(x) = \left| \int_0^x e^{-A \ln|y| + \frac{2\beta}{\sigma^2} y} dy \right|$$

where  $A < 1$  is a positive constant (for  $A > 1$  the integral in (31) does not exist).

Following the proof of theorem 3 we obtain the thesis. ■

Corollary 3 gives the stability of the stationary solution  $x_t \equiv 0$  in the case  $\sigma^2 > 2\alpha$ , that is when the non-degenerate invariant distribution does not exist. In the case  $\sigma^2 < 2\alpha$ , instead, the process fluctuates, for  $t$  sufficiently large, around the maximum of the density function (22), as stated in remark 1. It is also possible to determine a band within which, for  $t$  sufficiently large, the process is with probability approximately one. Denoting by  $\sigma_x$  the standard deviation of the process (that is  $\sigma_x = \sqrt{\text{var}(x)}$  with  $\text{var}(x)$  defined in (24)), the probability that, for  $t$  sufficiently large, the process differs from its mean by less than  $K$  times the



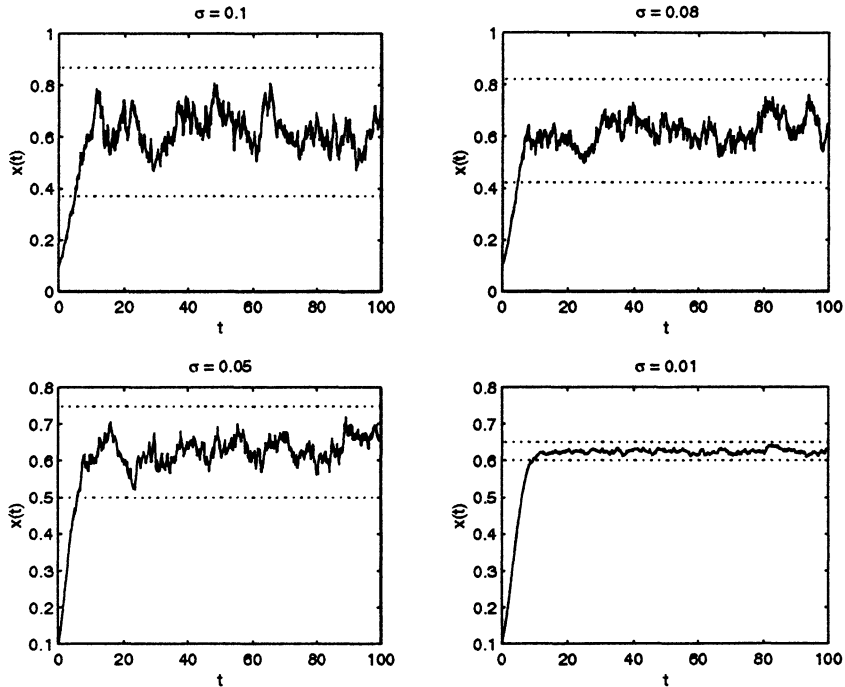


Fig. 7. – The trajectories of the process (case of equation (6)), for  $t$  sufficiently large, remains in the region between the dotted lines. This region is the same indicated in formula (34).

standard deviation is

$$(32) \quad P(|x - E(x)| \leq K\sigma_x) = \int_{E(x) - K\sigma_x}^{E(x) + K\sigma_x} p(x) dx$$

Since  $p(x)$  is a Gamma distribution, the integral cannot be computed in a simple way, but it can be approximated numerically.

Nevertheless, we recall that the standardization of a random variable distributed Gamma, with parameter  $r$  and  $\lambda$ , converges to a standard normal random variable for  $r$  going to infinity. This allows us to make the following approximation, for  $\sigma$  sufficiently small,

$$(33) \quad P(|x - Ex| \leq K\sigma_x) = \Phi(K) - \Phi(-K)$$

where we have denoted by  $x$  a random variable having density (22) and by  $\Phi$  the cumulative normal distribution function.

Recalling (23) and (24) we obtain that the process is with probability approximately one (for  $\sigma$  sufficiently small and  $t$  sufficiently large) in the interval

$$(34) \quad \left[ \frac{2\alpha - \sigma^2}{2\beta} - 4 \sqrt{\frac{2\alpha\sigma^2 - \sigma^4}{4\beta^2}}, \frac{2\alpha - \sigma^2}{2\beta} + 4 \sqrt{\frac{2\alpha\sigma^2 - \sigma^4}{4\beta^2}} \right].$$

In Figure 7 we have plotted the «limiting region» for the trajectories for some value of  $\sigma$ .

## 5. Conclusions.

If we introduce uncertainty on the parameter  $\alpha$ , obtaining equation (6), we can have one or two invariant distributions, it depends on the ratio  $\frac{\sigma^2}{\alpha}$ . If  $\sigma^2 < 2\alpha$ , then there exists a non-degenerate invariant distribution (besides the Dirac delta distribution with mass concentrated in 0) with Gamma density given by (22). For  $\sigma^2 < \alpha$  the trajectories of the process solution of (6) fluctuate around the value  $\frac{\alpha - \sigma^2}{\beta}$ , while for  $\sigma^2 \in [\alpha, 2\alpha]$  the most probable values are those of a right neighborhood of zero (as stated in remark 1). These observations are confirmed by Figure 4.

For  $\sigma$  sufficiently small, the maximum value of the density function (22) approaches the deterministic equilibrium  $\frac{\alpha}{\beta}$  and, in particular, for  $\sigma$  going to zero, the non-trivial invariant distribution becomes a degenerate distribution with mass concentrated in  $\frac{\alpha}{\beta}$ . It follows that the non-degenerate invariant distribution plays the role of the deterministic stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ .

Moreover, for  $\sigma$  sufficiently small, we can say that the process, for  $t$  sufficiently large, is, with probability approximately one, in a «band» of width depending on  $\alpha$  and  $\beta$  and given by (34). This means that the trajectories, for  $t$  sufficiently large and  $\sigma$  sufficiently small, remain in a neighborhood of the mean value (23), with probability approximately one.

If  $\sigma^2 > 2\alpha$ , then the non-trivial invariant distribution does not exist

any more and the degenerate process  $x_t \equiv 0$  is stable in probability as stated in theorem 3.

We can conclude that, for  $\sigma^2 < 2\alpha$ , the process, for  $t$  sufficiently large, fluctuates around the maximum value of the stationary density (which exists only for  $\sigma^2 < 2\alpha$ ), otherwise the process goes to zero which becomes a stable stationary solution.

In the case of  $\beta$  uncertain we always have two invariant distributions: the Dirac delta distribution associated to the unstable stationary solution  $x_t \equiv 0$  and a non-degenerate invariant distribution with density given in (25). The trajectories fluctuate in a neighborhood of the deterministic equilibrium  $\frac{\alpha}{\beta}$ . Also in this case, for  $\sigma$  going to zero, the non-trivial invariant distribution becomes a degenerate distribution with mass concentrated in  $\frac{\alpha}{\beta}$ . It follows that this invariant distribution can be considered as the stochastic analogue of the deterministic stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ .

If we introduce uncertainty both on  $\alpha$  and  $\beta$  we have a situation similar to that of equation (6), that is we can have one or two invariant distributions depending on the ratio  $\frac{\sigma_1^2}{\alpha}$ ; there exist two invariant distributions only in the case  $\sigma_1^2 < 2\alpha$ : the Dirac delta distribution with mass concentrated in 0 and an invariant distribution with density given in (14).

When the non-trivial invariant distribution does not exist, namely if  $\sigma_1^2 > 2\alpha$ , the process  $x_t \equiv 0$  becomes stable in probability (see theorem 3).

We conclude that if we introduce uncertainty only on parameter  $\beta$  the situation is quite similar to the deterministic case, that is the null solution is always unstable and the process fluctuates around the deterministic stationary solution  $x_t \equiv \frac{\alpha}{\beta}$ .

Uncertainty on  $\alpha$ , on the contrary, causes a change in the stability properties of the null solution, depending on the value of the parameter  $\sigma_1$  in equation (5) (or  $\sigma$  in equation (6)): for small values of  $\sigma_1$  (or  $\sigma$ ), the null solution is unstable, while it becomes stable for large values of  $\sigma_1$  (or  $\sigma$ ) and in that case the Dirac delta distribution with mass concentrated in 0 is the only invariant distribution.

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## REFERENCES

- [1] H. DENG - M. KRSTIC, *Stochastic nonlinear stabilization*, Systems Control Lett., **32** (1997), pp. 143-159.
- [2] P. FLORCHINGER, *Lyapunov-like techniques for stochastic stability*, SIAM J. Control Optim., **33** (1995), pp. 1151-1169.
- [3] C. W. GARDINER, *Handbook of stochastic methods*, Springer-Verlag, Berlin, 1985.
- [4] R. KHASMINSKII, *Stochastic stability of differential equations*, Sijthoff & Noordhoff, Alphen aan den Rijn, 1980.
- [5] P. E. KLOEDEN - E. PLATEN, *Numerical solution of stochastic differential equations*, Applications of Mathematics Series Vol. 23, Springer-Verlag, Heidelberg, 1992.
- [6] H. J. KUSHNER, *Stochastic stability and control*, Academic Press, New York, 1967.
- [7] E. M. LUNGU - B. ØKSENDAL, *Optimal harvesting from a population in a stochastic crowded environment*, Math. Biosci., **145** (1997), pp. 47-75.
- [8] H. RISKEN, *The Fokker-Planck equation*, Springer-Verlag, Berlin, 1984.

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