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Functorial Approximation to the Lateral Completion in Archimedean Lattice-Ordered Groups with Weak Unit.

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ABSTRACT - In the category of the title, we evaluate quite explicity the maximum monoreflection beneath the lateral completion operator.

1. Introduction.

The problem addressed in this paper is a multiplication-free version of the problem of describing, in archimedean f-rings, the maximum monoreflection beneath the complete ring of quotients operator, which we have called «the maximum functorial ring of quotients.» Thus this paper has a position, perhaps penultimate, in a series beginning with [HM1,2], which we hope to terminate in a sequel [HM ∞]. But we will not multiply here. The present paper is constituted as follows (with the considerable technicalities explained in the body of the text).

A lattice-ordered group is called laterally complete if each pairwise disjoint family of elements has the supremum. For a lattice-ordered group A, $A \leqslant lA$ denotes the lateral completion of A, that is, the unique minimum essential extension of A to a laterally complete group lA. It is no mean feat that this exists; see the discussion in [AF].

W is the category of archimedean lattice-ordered groups with distinguished weak order unit (a positive element e for which $e \wedge |a| = 0$ im-

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plies $a = 0$), with morphisms the group and lattice homomorphisms preserving units. This is a natural multiplication-free generalization of the category of archimedean f-rings with identity (see the discussion in [HR2]), and offers the considerable technical advantage (over archimedean groups without unit) of the canonical Yosida representation (see § 3 here).

$\mathcal{L}$ will denote the class of W-objects which are laterally complete. It is well known (and visible in [AF], and made explicit in § 3 here) that for $A \in W$, the extension $A \leqslant lA$ lies in W also. If is well known that l is not a functor in W (there are W-morphisms $A \to B$ with no extension $lA \to lB$, and morphisms with extensions which are not unique). But, as a consequence of general principles, there is a functor $\mu(l)$ in W maximum beneath the operator l (meaning $A \leqslant \mu(l)\, A \leqslant lA$ canonically for each A), and $\mu(l)$ is a monoreflection in W [HM3,4,8]. We view $\mu(l)$ as «the best functorial approximation to l».

(Entirely analogously, there are «$\mu(l)$ in all lattice-ordered groups» and «$\mu(l)$ in abelian lattice-ordered groups». It is shown in [HM4; 6.4] that the latter is the identity functor, and it follows that the former is also. Needless to say, the «$\mu(l)$ in W» is *not* the identity functor.)

This paper answers the question «What is $\mu(l)$?» as follows.

W has (among others) the two extreme monoreflections: the epicomplete monoreflection β, which is the maximum monoreflection [BH2]; the maximum essential monoreflection ε (also denoted c^3) [BH4], which is the maximum monoreflection beneath Conrad's essential hull e [C2] (and might be denoted also $\mu(e)$, as the notation $\mu(l)$). The operators β and ε are somewhat complicated but more-or-less understood.

As a consequence of more general principles (developed exactly to address our present problems), there is the formula (reviewed in § 5 here)

$$(1.1) \qquad \mu(l)A = \varepsilon A \cap \bar{l}A \leqslant \beta A \qquad (A \in W)$$

where $\bar{l}$ is the monoreflection onto $R(\mathcal{L})$, the epireflective hull of $\mathcal{L}$ in W. See [HM8]. Our description of $\mu(l)$ is then accomplished in two steps:

(1) We show in § 4 that $\mathcal{R}(\mathcal{L}) = \Sigma$, the class of W-objects which are laterally σ-complete (each *countable* pairwise disjoint family has the supremum). Thus $\bar{l} = \sigma$, the laterally σ-complete monoreflection. We have

studied σ earlier [HM6,9]; it is more-or-less understood. (By the way: First, Σ is distinguished among similar properties by being monoreflective [HM6; § 7]. In particular, $\mathcal{L}$ is not monoreflective (of course, since if it were, $\mu(l)$ would be l, and this paper would not exist). Second, $A \leqslant \sigma A$ is rarely an essential extension, while $A \leqslant lA$ always is. The operators l and σ are incomparable. We say this to avoid a common confusion.)

This theorem «$\mathcal{R}(\mathcal{L}) = \Sigma$» is the main new technical accomplishment of this paper. The proof uses (a) a method of «perfect homomorphisms» borrowed from General Topology, described here in § 2, which seems to be novel in lattice-ordered groups, and (b) considerable information about topological spaces which are compact and basically disconnected (the Stone spaces of σ-complete Boolean algebras), and (c) a mildly novel description of $A \leqslant lA$ as functions locally-A on the Yosida space of A (3.2 below).

(2) Since $\bar{l} = \sigma$, the equation (1.1) becomes «$\mu(l)A = \varepsilon A \cap \sigma A$.» We then in § 5 pick carefully at the known representations of εA and σA, and variants thereon, to achieve the description of $\mu(l)\,A$, various specific examples of which are presented.

2. Reflections and perfect morphisms.

This section is quite abstract, though we shall, on occasion, concretize notation. It all will be applied to W in the sequel.

Let $\mathcal{C}$ be a category. A «class of objects in $\mathcal{C}$» or a «subcategory of $\mathcal{C}$», always will be assumed isomorphism-closed, and subcategories will be assumed full.

Elaboration on the following can be found in [HS], especially Chapter X.

A morphism e is an *epimorphism* (or just epic) if it is right-cancellable ($fe = ge$ implies $f = g$), and a *monomorphism* (or monic) if left-cancellable.

The subcategory $\mathcal{R}$ is *reflective* if for each $A \in \mathcal{C}$ there is $rA \in \mathcal{R}$ (the reflection of A) and $r_A\colon A \to rA$ (the reflection map) such that whenever $R \in \mathcal{R}$ and $\varphi \in \mathrm{Hom}(A, R)$ there is unique $\overline{\varphi} \in \mathrm{Hom}(rA, R)$ with $\overline{\varphi} r_A = \varphi$. The functor $r : \mathcal{C} \to \mathcal{R}$ is called the reflection.

If each r_A is epic, we say «$\mathcal{R}$ is epireflective» and «r is an epireflection», and likewise, «monoreflective» and «monoreflection». By [HS, 36.3], each monoreflective subcategory is epireflective. For epireflective $\mathcal{R}$ to be monoreflective, it suffices that each object have a monic to an $\mathcal{R}$-object.

A monic $A \xrightarrow{m} B$ is *extremal monic* if ($m = fe$ with e epic implies e an isomorphism); this means A is a subobject of B with no epic enlargement within B. Then A is said to be an *extremal subobject* of B.

The following is the cornerstone of the theory of epireflections.

2.1 THEOREM. (SEE [HS; CH. 9]). *Suppose that $\mathcal{C}$ is co-well-powered, has products, each morphism f has an essentially unique factorization $f = me$ with e epic and m extremal monic, and that the composition of extremal monics is extremal monic. Then*

(a) *the subcategory $\mathcal{R}$ is epireflective if and only if $\mathcal{R}$ is closed under formation of products and extremal subobjects;*

(b) *each object class $\mathcal{A}$ has an epireflective hull, i.e., is contained in a smallest epireflective subcategory $\mathcal{R}(\mathcal{A})$, and $B \in \mathcal{R}(A)$ if and only if B is an extremal subobject of a product of objects in $\mathcal{A}$.*

It is noted in [BH1] and [HK] that W satisfies the hypothesis of 2.1. But it is not so clear what are extremal subobjects in W. That is roughly the *raison d'être* for the thoughts on perfect maps which follow.

The object E is called *epicomplete* (*EC*) if ($E \xrightarrow{\varphi} A$ epic and monic implies φ is an isomorphism). In the event that all monics are embeddings, E is EC if and only if whenever E is contained epically in A, then $E = A$. For many categories $\mathcal{C}$, the class EC is monoreflective. When that occurs, EC is the smallest monoreflective subcategory, and its reflection functor — which we denote β — is the maximum monoreflection (meaning, for any other monoreflection r, there is for each A a unique monic m with $mr_A = \beta_A$; or, concretizing, $rA \leq \beta A$ canonically for each A). This *is* the situation in W, discussed in § 3 and § 5 below, and also, for example, in Tychonoff spaces where β is Stone-Čech compactification. See [HM3] for further discussion.

We make some suppositions on our ambient category $\mathcal{C}$ and its EC. These are all features of W.

2.2 HYPOTHESES. *$\mathcal{C}$ has products and essentially unique (extremal monic) $\circ$ epic factorizations.*

$\mathcal{C}$ is concrete, with the injective $\circ$ surjective factorization of each morphism lying in $\mathcal{C}$, with all monics injective, with products set-theoretic.

EC is monoreflective (with functor denoted β), and EC is closed under surjections.

(Technically, a concrete category is a category equipped with a faithful functor to Sets. It is safe to think then of «structured sets» with structure-preserving functions.)

In the setting of 2.2, all injective $\mathcal{C}$-morphisms are monic, and we are assuming the converse. Likewise, all surjective $\mathcal{C}$-morphisms are epic, but we are emphatically *not* assuming the converse. (Were that true in W, this paper would be void.)

Assuming 2.2, we take the liberty of concretizing some of the notation below, in particular, writing $A \leqslant B$ to mean A is a subobject of B *via* an inclusion or obvious monic, and writing $A \leqslant \beta A$ for the EC-monoreflection of A, $\beta A - A$ for the set-theoretic remainder, etc..
The material below constitutes generalization from Topology. See § 3.7 of [E], [HI] and [H1] (among many items in the literature).

We assume that the ambient category satisfies 2.2.

2.3 Definition. *The morphism $\varphi : A \to L$ is perfect* if its extension $\beta\varphi : \beta A \to \beta L$ satisfies:

$$(\beta\varphi)(\beta A - A) \subseteq \beta L - L\,.$$

2.4 Theorem. *If $\varphi : A \to L$ is perfect, then the image $\varphi(A)$ is an extremal subobject of L.*

Proof. First, note

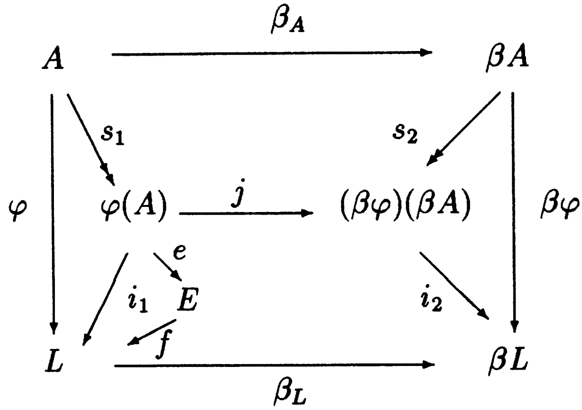
(*i*) φ is perfect if and only if $(\beta\varphi)(\beta A) \cap L = \varphi(A)$;

(*ii*) the monic m is extremal monic if and only if ($m = fe$, with e epic and f *monic* implies e an isomorphism);

(*iii*) if $B \leqslant D \leqslant C$ is the (extremal monic) $\circ$ epic factorization of $B \leqslant C$, and if $B \leqslant E \leqslant C$ has $B \leqslant E$ epic, then $E \leqslant D$.

(Here, (*i*) is immediate, and (*ii*) and (*iii*) are shown easily using information in [HS;§ 33].)

Now, consider the diagram



in which: $i_1 s_1$ and $i_2 s_2$ are the injective $\circ$ surjective factorizations of φ and $\beta\varphi$ respectively, j is the obvious inclusion, and, seeking to show i_1 extremal monic using (*ii*), $i_1 = fe$ is a monic $\circ$ epic factorization. We want e to be an isomorphism, or, viewing β_A, β_L, i_1 and i_2 as inclusions, we want $E = \varphi(A)$. Note that $s_2\beta_A = js_1$, and $i_2 j = \beta_L i_1 = \beta_L fe$.

Since s_1 is a surjection, it is epic and so is $s_2\beta_A$. Since $js_1 = s_2\beta_A$ is epic, so is its second factor j. Since EC is closed under surjections, $(\beta\varphi)(\beta A) \in EC$ and is therefore an extremal subobject wherever a subobject; so i_2 is extremal monic.

We have $(\beta_L f)\, e = \beta_L i_1 = i_2 j$, and the last expresses the (extremal monic) $\circ$ epic factorization of the first. From (*iii*) then, $E \leqslant (\beta\varphi)(\beta A)$. Inserting this into (*i*) yields $E = \varphi(A)$, as desired.

The following makes possible the construction of perfect morphisms one element (of $\beta A - A$) at a time.

2.5 Definition. *The set* $\Phi = \{\varphi_i \colon A \to L_i\}_i$ *is* jointly perfect *if*

$$\forall b \in \beta A - A \;\; \exists i \quad \text{with} \quad (\beta\varphi_i)(b) \in \beta L_i - L_i .$$

2.6 Proposition. *If* $\Phi = \{\varphi_i \colon A \to L_i\}_i$ *is jointly perfect, then the diagonal* $\Delta : A \to \Pi L_i$ *is perfect. So* $\Delta(A)$ *is an extremal subobject of* ΠL_i.

Proof. First, note: Suppose $\psi : A \to L$, $L \leqslant M$ with $M \in EC$, and let $\overline{\psi} \colon \beta A \to M$ denote the extension of ψ. If $\overline{\psi}(\beta A - A) \subseteq M - L$, then

ψ is perfect. (Proof. Let $\gamma : \beta L \to M$ extend the inclusion $L \leq M$. Then, $\overline{\psi} = \gamma \circ (\beta\psi)$, and the hypothesis on $\overline{\psi}$ ensures φ is perfect.)

The diagonal $\Delta : A \to \Pi L_i \equiv L$ is defined by the universal property of the product, by $\pi_i \circ \Delta = \varphi_i \forall i$, and will be the ψ of the previous paragraph. Let $M \equiv \Pi\beta L_i$. Then $M \in EC$ because EC is reflective, thus closed under product formations. (See [HS; § 36]. This does not use the hypotheses in 2.1.) Since Φ is jointly perfect, $\overline{\Delta} : \beta A \to M$ has $\overline{\Delta}(\beta A - A) \subseteq \subseteq M - L$. By the first paragraph, Δ is perfect.

By 2.4 now, $\Delta(A)$ is an extremal subobject of L.

3. The Yosida Representation; $\mathcal{L}$ and Σ.

The rest of the paper deals in quite specific calculations using the representation of W-objects as groups of continuous functions, and in those terms, some detailed information about laterally complete and σ-complete groups. We sketch this material.

Let X be a Tychonoff topological space, R the reals and $R \cup \{\pm\infty\}$ the extended reals (two-point compactification of R). $C(X) \equiv \{f | f : X \to R$ is continuous$\}$, with pointwise order and addition, and designated weak unit the constant function $1 : C(X) \in W$. $D(X) \equiv \{f | f : X \to R \cup \{\pm\infty\}$ continuous, $f^{-1}R$ dense in $X\}$ with pointwise order and partial addition: $f + g = h$ means $f(x) + g(x) = h(x)$ for $x \in f^{-1}R \cap g^{-1}R \cap h^{-1}R$. (This partially defined addition is fully defined if and only if X is a quasi-F space, meaning each dense cozeroset is C^*-embedded; see [HJ] and [DHH]). A W-object in $D(X)$ is a subset of $D(X)$ which contains 1, is a sublattice, is closed under + and contains $-f$ whenever it contains f, with 1 as designated weak unit.

The basic theorem below is reviewed in [HR1], [BH2] and [HM5].

3.1 THEOREM. (the Yosida functor) (a) *For each $A \in W$, there is a compact Hausdorff space YA and a W-isomorphism $A \to \widehat{A}$ onto a W-object $\widehat{A}$ in $D(YA)$, with $\widehat{A}$ separating the points of YA. The representation is essentially unique.*

(b) *For each $A \overset{\varphi}{\to} B \in W$, there is a unique continuous $YA \overset{Y\varphi}{\leftarrow} YB$ for which $\widehat{\varphi a} = (Y\varphi) \circ \widehat{a}$ $(a \in A)$. φ is one-to-one if and only if $Y\varphi$ is onto.*

We shall commonly identify A with $\widehat{A}$, thus viewing A as contained in $D(YA)$.

In a category, a monic m is called *essential* if φm monic implies φ monic. In W, monics are embeddings , and the embedding $A \leqslant B$ is essential if and only if A is *large* in B (for an ideal I of B, $I \neq (0)$ implies $I \cap A \neq \neq (0)$, equivalently, if $0 < b \in B$, there are $0 < a \in A$ and n with $a \leqslant nb$). (Here «$\Longleftarrow$» is clear, while «$\Longrightarrow$» requires noting that each nonzero ideal of B contains a nonzero W-kernel, for example a «polar«; see [BKW].)

We now describe the lateral completion $A \leqslant lA$ and the maximum essential extension $A \leqslant eA$ of an $A \in W$. The literature does not seem to contain exactly what we want to say, but we do not claim much novelty. See especially [C1,2], [FGL], [VG], [B], [J], [HM2], [AF] and references therein; some of the references concern the complete rings of quotients of archimedean f-rings, which is a very similar construct (see the first paragraph of § 1). We apologize to authors neglected or ill-used.

First, just suppose $A \subseteq D(X)$: for $G \subseteq X$ and $f \in C(G)$, f is *locally in A on G* if for each $x \in G$ there is open U in G and $a \in A$ such that $a | U = f | U$. The collection of all such f is denoted $A_{\mathrm{loc}}(G)$ Let $d.o.X$ denote the filter base of all *dense open* subsets of X. $\varinjlim\{A_{\mathrm{loc}}(G) \mid G \in d.o.X\}$ denotes the union modulo the equivalence: for $f_i \in A_{\mathrm{loc}}(G_i)$, $f_1 \sim f_2 \equiv (f_1 | G_1 \cap G_2 = = f_2 | G_1 \cap G_2)$. It is easy to see that, if A is a W-object in $D(X)$ then this construct $\varinjlim A_{\mathrm{loc}}(G)$ is in W, and is in fact the direct limit in W (with bonding homomorphisms: for $U \supseteq V$, $A_{\mathrm{loc}}(U) \ni f \mapsto f | V \in A_{\mathrm{loc}}(V)$), and $A \leqslant \varinjlim A_{\mathrm{loc}}(G)$ (*via* $A \ni f \mapsto f | f^{-1}R \in A_{\mathrm{loc}}(f^{-1}R)$).

3.2 THEOREM. *For $A \in W$, viewed as $A \subseteq D(YA)$, $A \leqslant \varinjlim\{A_{\mathrm{loc}}(G) \mid G \in d.o.YA\}$ is a model of $A \leqslant lA$.*

Regarding the proof of 3.2: Let B denote our direct limit. If is easy to see that $A \leqslant B$ is essential and that $B \in \mathcal{L}$. We need also that ($A \leqslant C \leqslant B$ with $C \in \mathcal{L}$ implies $C = B$). A direct proof of this is lengthy and we omit the details. It can also be derived *via* material in [AF; Ch. 8] and 3.8 below. Additionally, 3.2 strongly resembles a description of the complete ring of quotients of an archimedean f-ring given in [HM2; 3.2].

Note that 3.2 describes lA proceeding from the Yosida representation of A. We shall need also the Yosida representation of lA; we now describe it.

A continuous surjection $X \overset{\tau}{\leftarrow} X'$ between compact Hausdorff spaces (say) is called *irreducible* if $\tau(F) \neq X$ for each closed proper subset F of

X', equivalently, for each U open in X' there is V open in X with $\tau^{-1}V$ dense in U. This implies $\tau^{-1}D$ dense in X' for each D dense in X.

The Tychonoff space X is called *extremally disconnected* if $\overline{U}$ is open for each open U. This implies that all open sets, and all dense sets, are C^*-embedded [GJ].

The crux of the following is due to Gleason [G]. Expositions, with additional details, appear in [H3] and [PW], among other places.

3.3 THEOREM. (For compact Hausdorff spaces) *for each* X

(a) *there is an irreducible* $X \overset{\pi}{\longleftarrow} EX$ *with* EX *extremally disconnected*

(b) *if* $X \overset{\tau}{\longleftarrow} X'$ *is irreducible, then there is continuous* $X' \overset{\varrho}{\longleftarrow} EX$ *with* $\tau\varrho = \pi$; ϱ *is unique for that, is irreducible, and is a homeomorphism if and only if* X' *is extremally disconnected.*

EX (or the pair (EX, π)) is called the *absolute* of X, or Gleason's projective cover. It is (by 3.3) the unique extremally disconnected irreducible preimage of X, and the maximum irreducible preimage of X.

For X extremally disconnected, we have $D(X) \in W$ since dense sets, thus dense cozerosets, are C^*-embedded (as commented at the begining of this section). For $A \in W$, viewed as $A \subseteq D(YA)$, consider the absolute $YA \overset{\pi}{\longleftarrow} EYA$. For $a \in A$, $\pi \circ a \in D(EYA)$ since $(\pi \circ a)^{-1}R = a^{-1}\pi^{-1}R$, and preimages of dense sets are dense. This defines a W-embedding $\hat{\pi}: A \to D(EYA)$ with $Y\hat{\pi} = \pi$.

3.4 PROPOSITION. [HR2] *The W-extension* $A \overset{\varphi}{\leqslant} B$ *(φ is just a label) is essential if and only if* $YA \overset{Y\varphi}{\longleftarrow} YB$ *is irreducible.*

3.5 COROLLARY. *For* $A \in W$, $A \overset{\hat{\pi}}{\leqslant} D(EYA)$ *is a maximum essential extension of* A, *i.e., a model of the essential closure* $A \leqslant eA$. *So* $YeA = EYA$, *and the Yosida representation of the embedding* $A \leqslant eA$ *is* $YA \overset{\pi}{\longleftarrow} EYA$.

3.5 is not so different from the description in [C2].

We extend $\hat{\pi}$ over the construct of 3.2: If $f \in A_{\mathrm{loc}}(G)$, with equivalence class $[f]$, define $\tilde{\pi}([f]) \in D(EYA)$ as the extension over EYA of $\pi \circ f \in C(\pi^{-1}G)$ (the dense set $\pi^{-1}G$ is C^*-embedded in EYA). It is easy to see that this is well-defined and a homomorphism which is one-to-one. Let $\tilde{l}A$ be the image $\tilde{\pi}(\varinjlim\{A_{\mathrm{loc}}(G) \mid G \in d.o.YA)$ in $D(EYA)$.

3.6 COROLLARY. *For $A \in W$, $\widehat{\pi}(A) \leqslant \tilde{l}(A)$ is a model of $A \leqslant lA$.*

$\tilde{l}A$ separates the points of EYA, so $Y\tilde{l}A = EYA$. Thus $\tilde{l}A$ is the Yosida representation of lA, and the representation of the embedding $A \leqslant lA$ is $YA \overset{\pi}{\leftarrow} EYA$.

In particular, if $A \in \mathcal{L}$, then YA is extremally disconnected.

PROOF. The first assertion is clear. The rest follows from this, 3.1, and the point-separation, so we verify the last. For $p \neq q$ in EYA, choose clopen U containing p, not q. Choose V open in YA with $\pi^{-1}V$ dense in U (by irreducibility). Then $V^{\circ} = V \cup (YA - \overline{V}) \in d.\ o.\ YA$ and the characteristic function $\psi_V \in A_{\text{loc}}(V^{\circ})$. Then we have $\overline{\pi}([\psi_V]) = \psi_U \in D(EYA)$ (one sees easily), and $\psi_U(p) = 1$, $\psi_U(q) = 0$.

3.6 is similar to a result in [J] about complete rings of quotients.

We shall not need the next result. We record it now because of inevitable comparison of essential closedness and epicompleteness (3.8 below). In terms of our present exposition, the result follows easily from 3.8 and 3.9. Recall that a lattice is called conditionally ($\sigma-$) complete if each family of elements which is bounded above (and countable) has the supremum.

3.7 THEOREM. (Conrad [C2]). *For $A \in W$, these are equivalent.*

(a) *A is essentially closed.*

(b) *A is divisible and both conditionally and laterally complete.*

(c) *YA is extremally disconnected and $A = D(YA)$.*

The Tychonoff space X is called *basically disconnected* if $\overline{U}$ is open for each cozeroset U. Each basically disconnected space is quasi-F, whence $D(X) \in W$. (See [GJ] and [DHH].) These spaces are central to our considerations.

3.8 THEOREM. [BH 1,2] *For $A \in W$, these are equivalent.*

(a) *A is epicomplete in W (or, $A \in EC$).*

(b) *A is divisible and both conditionally and laterally σ-complete.*

(c) *YA is basically disconnected and $A = D(YA)$.*

Let X be a compact Hausdorff space. A closed subset T is called a P-set in X if any G_δ-set which contains T is actually a neighborhood of T.

3.9 THEOREM. (Tzeng [T], Veksler [V].) *Let X be compact basically disconnected.*

(a) *T is a P-set of X if and only if $T = \cap \mathcal{V}$ for some family $\mathcal{V}$ of clopen sets of X for which $(V_1, V_2, \ldots \in \mathcal{V} \Rightarrow \bigcap_n V_n \supseteq V$ for some $V \in \mathcal{V})$.*

(b) *If T is a P-set of X, then T is basically disconnected.*

(c) *If T is a P-set of X, then $D(X) \ni f \mapsto f|T \in D(T)$ is a W-homomorphism onto $D(T)$.*

(d) *Any W-surjection of $D(X)$ is (isomorphic to one) of the form* (c).

3.10 Corollary. [BH2]. *If $A \in EC$ and $A \twoheadrightarrow B$ is a W-surjection, then $B \in EC$.*

More on P-sets and the connections to W can be found in [V], [BHM], and [BH3].

We now turn to Σ.

3.11 Theorem. [HM6]. *Let $A \in \Sigma$.*

(a) *YA is basically disconnected.*

(b) *If U is a clopen subset of YA and $a \in A$, then the function $a(U) \equiv (a$ on U; 0 off $U)$ is in A.*

(c) *$A \leqslant D(YA)$ is a model of the EC monoreflection $A \leqslant \beta A$.*

Of course, (c) is a very special result about β. β will be discussed further, in general, in § 5 below.

4. $\mathcal{R}(\mathcal{L}) = \Sigma$.

We prove this now. The heart of the proof is the following.

4.1 Theorem. *Let $A \in \Sigma$. For each $f \in D(YA) - A$, there is a nonvoid P-set $T(f)$ of YA for which the restriction $f|T(f)$ is «nowhere locally-A on $T(f)$», i.e., there is no pair (G, a), G a nonvoid open set in $T(f)$ and $a \in A$, for which $a|G = f|G$.*

The point of 4.1, probably visible to the reader, is that the composite homomorphism $\varphi_{T(f)}: A \to A \mid T(f) \leqslant l(A \mid T(f))$ will have $(\beta\varphi_{T(f)})(f) \notin l(A|T(f))$, so the family $\{\varphi_T \mid T \in P(YA)\}$ will be jointly perfect. We explain this carefully below.

Proof of 4.1. Let $A \in \Sigma$ and let $f \in D(YA) - A$. We may suppose $f \geqslant 0$. (For if the Theorem is true for each such f, then given general f, write $f = f^+ - f^-$ in the usual way, and we have $T(f^+)$ and $T(f^-)$. Then $T(f^+) \cup T(f^-)$ is a P-set on which f is nowhere locally-A.)

Let $\mathcal{U}(f)$ be the family of all U which are nonvoid cloper sets in YA, for which there is $a \in A$ with $a|U = f|U$.

If $\mathcal{U}(f) = \emptyset$, let $T(f) = YA$.

If $\mathcal{U}(f) \neq \emptyset$, let $\mathcal{V}(f) \equiv \{YA - \overline{\bigcup_n U_n} \mid U_1, U_2, \ldots \in \mathcal{U}(f)\}$, and then $T(f) \equiv \cap\, \mathcal{V}(f)$. The family $\mathcal{V}(f)$ satisfies the condition in 3.3 (a): first, the members are clopen because the sets $\bigcup_n U_n$ are cozero (being open F_σ [E]), and YA is basically disconnected (by 3.4); second, $\bigcap_\kappa (Y - \overline{\bigcup_n U_n^\kappa}) = Y - \overline{\bigcup_\kappa \bigcup_n U_n^\kappa} \supseteq Y - \overline{\bigcup_{n\kappa} U_n^\kappa}$. Thus $\dot{T}(f)$ is a P-set.

$T(f)$ will be nonvoid if each member of $\mathcal{V}(f)$ is nonvoid (for then clearly, $\mathcal{V}(f)$ will have the finite intersection property, and YA is compact). Seeking a contradiction, suppose U_n's $\in \mathcal{U}(f)$ have $Y - \overline{\cup U_n} = \emptyset$. Then $\cup U_n$ is dense in YA. For each n, choose $a_n \in A$ with $a_n | U_n = f| U_n$. Let $W_n = U_n - (U_{n-1} \cup \ldots \cup U_1)$. The W_n's are clopen and parwise disjoint, and $\cup W_n = \cup U_n$ is dense in YA. By 3.4 (b), each $a_n(W_n) \in A$ (see 3.11 (b)). Since the W_n's are pairwise disjoint, the $a_n(W_n)$'s are pairwise disjoint in the sense of lattice-ordered groups. Since $A \in \Sigma$, $g \equiv \bigvee_n a_n(W_n)$ exists in A. Note the easily-proved general lemma: If $G \in W$ and $\{g_\alpha\}$ is a pairwise disjoint family in G, with each $g_\alpha \geqslant 0$, which has the supremum $g = \bigvee g_\alpha$ in G, then in the Yosida representation, $g | coz g_\alpha = g_\alpha | \mathrm{coz}\, g_\alpha$ for each α. Applying this to $g = \bigvee a_n(W_n)$ we have $g | \mathrm{coz}\, a_n(W_n) = a_n(W_n) | W_n = a_n | W_n = f| W_n$. Thus, we have f, $g \in D(YA)$ with $g(x) = f(x)$ for each x in the dense set $\cup W_n$. Therefore $f = g$ (see [E]). This says $f \in A$, a contradiction.

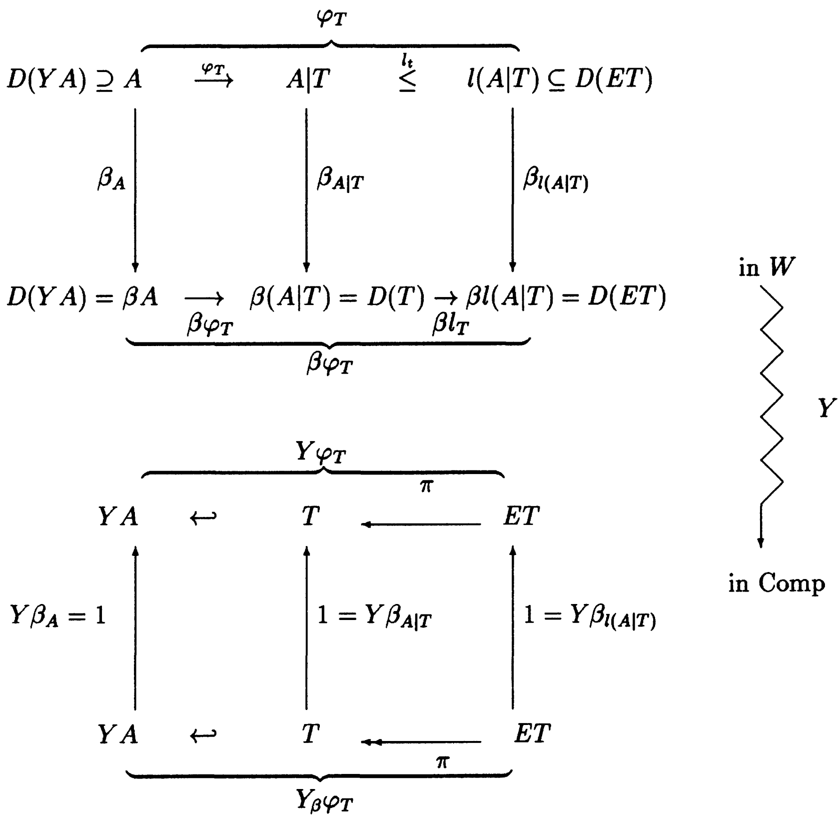
So $T(f)$ is nonvoid P-set. Now (seeking a contradiction) suppose there is nonvoid open G in $T(f)$ and $a \in A$ with $a|G = f|G$. There is open H in YA with $H \cap T(f) = G$, and then there is clopen $W \subseteq H$ with $W \cap T(f) \neq \emptyset$ (since YA is basically disconnected, and thus the family of clopen sets is a basis [GJ]). We have $a| W \cap T(f) = f| W \cap T(f)$. Since $T(f)$ is a P-set and W is clopen, $W \cap T(f)$ is a P-set. (This is easy, but it's [BHM; 2.4].) Now, $Z \equiv \{y \in YA \mid a(y) = f(y)\}$ is a G_δ-set containing the P-set $W \cap T(f)$, and thus Z contains a neighborhood of $W \cap T(f)$, and thus $Z \supseteq U \supseteq W \cap T(f)$ for some clopen U (since $W \cap T(f)$ is compact and $clop YA$ is a basis). So we have $U \neq \emptyset$ and $a|U = f|U$. This means $U \in \mathcal{U}(f)$, which implies $U \cap T(f) = \emptyset$, a contradiction.

The proof of 4.1 is complete.

4.2 COROLLARY. *Let $A \in \Sigma$. For each P-set T of YA, let $\varrho_T \colon A \to A|T$ be the restriction homomorphism (for $a \in A \subseteq D(YA)$, $\varrho_T(a) = a|T$), let $l_T \colon A|T \to l(A|T)$ be the lateral completion of $A|T$, and let $\varphi_T \equiv$*

$\equiv l_T \varrho_T$: $A \to l(A\,|\,T)$. *The family* $\{\varphi_T\,|\,T$ *is a* P*-set of* $YA)\}$ *is jointly perfect.*

PROOF. By 3.4, $\beta A = D(YA)$. For a P-set T of YA, $A\,|\,T$ as presented is in its Yosida representation, by 3.3 (c) and the uniqueness in 3.1. So by 3.2, $l(A\,|\,T)$ may be viewed as all locally-A functions on dense open subsets of T, and the Yosida representation is described in 3.2 also. We display our homomorphisms and the dual continuous maps:



(In the above, $l(A\,|\,T) \in \Sigma$ and $Yl(A\,|\,T) = ET$, so $\beta l(A\,|\,T) = D(ET)$ by 2.4. And note that the action of $\beta\varphi_T$ is just $\beta\varphi_T(f) = (f\,|\,T) \circ \pi$.)

Now let $g \in D(ET) = \beta l(A\,|\,T)$. Of course by 3.2, $g \in l(A\,|\,T)$ means $g = \widehat{h}$ for some h in some $(A\,|\,T)_{\text{loc}}(G)$, so $g\,|\,\pi^{-1}G = h \circ (\pi\,|\,\pi^{-1}G)$.

After those preliminaries, we let $f \in \beta A - A$, then choose $T = T(f)$ by 4.1, on which f is nowhere locally-A. We claim $\beta\varphi_T(f) \in \beta l(A|T) - l(A|T)$. If not, by the previous paragraph $f \circ \pi = \beta\varphi_T(f)$ has $f \circ \pi | \pi^{-1} G =$ $= h \circ (\pi | \pi^{-1} G)$ for some $G \in d.o.T$ and $h \in (A|T)_{\text{loc}}(G)$. Take any $p \in G$, then U clopen in T and $p \in U \subseteq G$, with $a \in A$ for which $h|U = a|U$. We then find $f|U = a|U$, which contradicts the feature of $T = T(f)$.

This proves 4.2.

4.3 Corollary. (*a*) *$A \in \Sigma$ if and only if A is an extremal subobject of a laterally complete object.*

(*b*) $\mathcal{R}(\mathcal{L}) = \Sigma$.

Proof. It is seen easily that $\mathcal{L}$ is closed under formation of products in W. Since Σ is monoreflective, thus epireflective, Σ is closed under formation of products and extremal subobjects (2.1). Of course, $\mathcal{L} \subseteq \Sigma$.

a) Thus, if A is an extremal subobject of $B \in \mathcal{L}$, then $A \in \Sigma$. Conversely, if $A \in \Sigma$, then the diagonal Δ for the jointly perfect family in 4.2 has image $\Delta(A)$ extremal in $\Pi\{l(A|T) \,|\, T \in P(YA)\}$ by 2.6. This product is in $\mathcal{L}$ as noted above, and Δ is monic since its single component φ_{YA} is monic.

b) Now, using 2.1, $\mathcal{L} \subseteq \Sigma$ implies $\mathcal{R}(\mathcal{L}) \subseteq \mathcal{R}(\Sigma) = \Sigma$, and $\Sigma \subseteq \mathcal{R}(\mathcal{L})$ by a).

4.4 Remarks. a) For $A \in \Sigma$, the embedding $A \leqslant lA$ need *not* be extremal monic (so 4.3 b) can not be proved like this), indeed, it is frequently epic. An example of this is visible in 5.10, using $A = l(\omega_1)S(X)$ for X extremally disconnected and failing the countable chain condition. Here, $A \leqslant lA = lS(X)$ is uniformly dense, hence epic.

(It is perhaps an interesting problem to describe those A for which $A \leqslant lA$ is extremal monic.)

b) Nontrivial theorems of the form «$\mathcal{R}(\mathcal{A}) = \mathcal{B}$» are not very common in lattice-ordered algebra, but for comparison, here are two more in W:

(*i*) Let $\mathcal{E}x = \{D(X) \,|\, X$ compact extremally disconnected$\}$. Then $\mathcal{R}(\mathcal{E}x) = EC = \{D(X) \,|\, X$ compact basically disconnected$\}$ (see 3.3). ($\mathcal{E}x \subseteq$ $\subseteq EC$ so $\mathcal{R}(\mathcal{E}x) \subseteq \mathcal{R}(EC) = EC$. Conversely, any A embeds into $\mathcal{E}x$-objects, among others, Conrad's essential closure (see 3.5 here), and of course, if $A \in EC$, then A is an extremal subobject whenever it is a subobject.)

This more-or-less easy theorem «$\mathcal{R}(\mathcal{E}x) = EC$» would seem very close to «$\mathcal{R}(\mathcal{L}) = \Sigma$», because for $A \in \Sigma$ (respectively $\mathcal{L}$), $A \in EC$ (respectively, $\mathcal{E}x$) if and only if A is divisible and uniformly complete. (See [HM4].) But we see no further connection.

(*ii*) Let $\mathcal{F} = \{D(X)\,|\,X$ is compact quasi-$F\}$. Then [BH4] $\mathcal{R}(\mathcal{F}) = c^3$ (more on which in § 5 below). This theorem enables the proof that c^3 is the least essentially monoreflective subcategory, which is important to § 5 below. The proof is not so easy and seems to bear no resemblance to the proof above that $\mathcal{R}(\mathcal{L}) = \Sigma$.

5. The maximum monoreflection beneath l.

We explain and interpret the equation (1.1) from the introduction.

In a category, an *extension operator* (*EO*) is an assignment «d», to each object A a monic $d_A \colon A \to dA$. For two *EO*'s, $c \leq d$ means that for each A there is a monic f with $fc_A = d_A$. With some mild further hypotheses, it is shown in [HM3] that beneath an *EO* d there is a maximum monoreflection $\mu(d)$. With further data available in the category, $\mu(d)$ has the following description.

5.1 THEOREM. [HM8]. *Suppose the ambient category satisfies the hypotheses of* 2.1, *and has the maximum monoreflection* β *and the maximum essential monoreflection* ε. *Let* $\mathcal{E}$ *denote the range of* ε. *Let* d *be an idempotent EO whose monics* d_A *are restrictably essential* ($d_A = fg$ *with* f *monic implies* g *is essential monic*). *Let* $\mathcal{D}$ *denote the range of* d, *and let* $\bar{d}$ *be the monoreflection functor for* $\mathcal{R}(\mathcal{D})$. *Then*:

(*a*) $\mu(d)$ *exists*, $\mu(d) = \varepsilon \wedge \bar{d}$ (*the infimum of monoreflections in the partial order* $\leq$ *defined above*), *and this is the maximum essential monoreflection beneath* $\bar{d}$.

(*b*) *The range of* $\mu(d)$ *is* $\mathcal{R}(\mathcal{E} \cup \mathcal{D})$, *and this is the least essentially monoreflective category containing* $\mathcal{D}$.

(*c*) $\mu(d)A = \varepsilon A \cap \bar{d}A \leq \beta A$, for each A.

We focus on 5.1 (c), for l in W. In § 4, we proved that $\mathcal{R}(\mathcal{L}) = \Sigma$, so that $\bar{l} = \sigma$ (since reflection functors are unique). Also, l satisfies the hypothesis on d in 5.1 [C1]. Thus,

5.2 COROLLARY. $\mu(l)A = \varepsilon A \cap \sigma A \leq \beta A$ *for each* $A \in W$. *The range of the functor* $\mu(l)$ *is* $\mathcal{R}(\mathcal{E} \cup \mathcal{L})$.

(Two other plausible candidates for equations for $\mu(l)$ will be refuted in 5.11 below.)

We now interpret 5.2 using the known representations of β, σ, and ε, and some variations thereon.

For X a Tychonoff space, $B(X)$ denotes the W-object of real-valued Baire functions on X. (The Baire field $\mathcal{B}(X)$ is the σ-field generated by the cozerosets of X. $f \in B(X)$ means $f^{-1}(a, b) \in \mathcal{B}(X)$ for all a, b.)

For $A \in W$, we define $B_{\omega, A}(YA)$ by: $f \in B_{\omega, A}(YA)$ means $f \in R^{YA}$ and there are disjoint $Y_1, Y_2, \ldots \in \mathcal{B}(YA)$ with $\bigcup_n Y_n = YA$, and there are $a_1, a_2, \ldots \in A$, for which $a_n \mid Y_n = f \mid Y_n$ for each n. (Note that $B_{\omega, A}(YA) \subseteq \subseteq B(YA)$.)

We abbreviate $B = B(YA)$ and $B_{\omega, A} = B_{\omega, A}(YA)$.

A is inserted into $B_{\omega, A}$ by $a \mapsto \overline{a} = (a$ on $a^{-1}R$; 0 off $a^{-1}R)$. This is not an algebraic embedding, but will become so upon a factoring, as follows.

On B, an equivalence relation $\sim$ is defined: $f \sim g$ means $\{x \in \in YA \mid f(x) \neq g(x)\} \subseteq \bigcup_n a_n^{-1}\{\pm\infty\}$ for some $a_1, a_2, \ldots \in A$.

5.3 THEOREM. ([BH2] and [HM9].) *For each* $A \in W$, $A \leqslant B_{\omega A}/\sim \leqslant \leqslant B/\sim$ *is a model of* $A \leqslant \sigma A \leqslant \beta A$.

Next, for $A \in W$, let $(A^{-1}R)_\delta \equiv \{\bigcap_n a_n^{-1}R \mid a_1, a_2, \ldots \in A\}$; this is a filter base of dense subsets of YA. For $F \in (A^{-1}R)_\delta$, we let $A(F) \equiv \equiv \{a \mid F \mid a \in A,\ a^{-1}R \supseteq F\}$; $C(F)$ and $B(F)$ are the real-valued continuous, and Baire, functions respectively; and $B_{\omega, A}(F) \equiv \{f \in B(F) \mid$ there are $F_1, F_2, \ldots,$ disjoint Baire sets of YA,

with $F = \cup F_n$, and $a_1, a_2., \ldots \in A$ such

that $a \mid F_n = f \mid F_n$ for each $n\}$.

With arrows denoting inclusion as W-subobjects, we have

$$\text{(5.4)} \qquad \begin{array}{ccccc} & & C(F) & & \\ & \nearrow & \uparrow & \searrow & \\ A(F) & \longrightarrow & C(F) \cap B_{\omega,A}(F) & \rightarrow & B(F) \\ & \searrow & \downarrow & \nearrow & \\ & & B_{\omega,A}(F) & & \end{array}$$

For $F_2 \subseteq F_1$ in $(A^{-1}R)_\delta$, we have for the various l-groups in (5.4), restriction homomorphisms $A(F_1) \to A(F_2)$, ..., $B(F_2) \to B(F_1)$. We note that $C(F_1) \to C(F_2)$ is always an embedding, but $B_{\omega, A}(F_1) \to B_{\omega, A}(F_2)$ is not unless $F_2 = F_1$. These homomorphisms are constituents in direct systems of W-objects, which have limits, for example, $\varinjlim B(F)$. This is realized as $\cup B(F)/\approx$, where $\approx$ is the equivalence relation: for $f_i \in B(F_i)$ $(i = 1, 2)$, $f_1 \approx f_2$ means there is $F \in (A^{-1}R)_\delta$ with $F \subseteq F_1 \cap F_2$ and $f_1 | F = f_2 | F$; likewise for the other limits.

Under passage to $\varinjlim$, (5.4) is transmuted as follows. (The information about εA is from [AH], [H2], and [BH 4].)

5.5 THEOREM. *Let $A \in W$.*

$$\begin{array}{ccccc} & \nearrow & \varinjlim C(F) & \searrow & \\ & & \uparrow & & \\ (a)\quad \varinjlim A(F) & \to & \varinjlim[C(F) \cap B_{\omega,A}(F)] & \to & \varinjlim B(F) \quad (\text{over } F \in A^{-1}R) \\ & & \downarrow & & \\ & \searrow & \varinjlim B_{\omega,A}(F) & \nearrow & \end{array}$$

(with arrows denoting limit embeddings) is an item-by-item model of

$$\begin{array}{ccccc} & \nearrow & \epsilon A & \searrow & \\ & & \uparrow & & \\ (b)\quad A & \longrightarrow & \epsilon A \cap \sigma A & \longrightarrow & \beta A \\ & & \downarrow & & \\ & \searrow & \sigma A & \nearrow & \end{array}$$

PROOF. (Sketch). One must check that the various arrows in (5.4) become embeddings in the limit. This is an easy exercise in the relation $\approx$.

The Yosida Representation 3.1 implies that $\varinjlim A(F) = A$. That $\varinjlim C(F) = \varepsilon A$ has been referenced above.

To see that $\varinjlim A(F) \leq \varinjlim B_{\omega, A}(F) \leq \varinjlim B(F)$ models $A \leq \sigma A \leq \beta A$, we refer to the model in 5.3 and perform the very easy comparison of $\sim$ with $\approx$: the homomorphism $\psi : B(YA)/\sim \to \cup B(F)/\approx$ defined by $\psi(f/\sim) = f/\approx$ fixes the copies of A elementwise, is a surjective embedding, and carries $B_{\omega, A}(YA)/\sim$ onto $\cup B_{\omega, A}(F)/\approx$. We omit the details.

Finally, on the face of it, $\varepsilon A \cap \sigma A = [\varinjlim C(F)] \cap [\varinjlim B_{\omega, A}(F)]$. We are asserting that this is, in fact, $\varinjlim [C(F) \cap B_{\omega, A}(F)]$. This too is an easy exercise in $\approx$.

Combining 5.2 and the appropriate part of 5.5, we have

5.6 COROLLARY. *For* $A \in W$, $\mu(l)\, A = \varinjlim [C(F) \cap B_{\omega, A}(F)]$ *(over* $F \in (A^{-1}R)_\delta$*).*

We illustrate this equation with some relatively simple special cases. In a lattice-ordered group A an element u is a strong unit if $u \geq 0$ and for each $a \in A$, $|a| \leq nu$ for some $n \in N$. For $A \in W$, the designated unit will be strong if and only if $A \subseteq C(YA)$ in the Yosida representation (i.e., every function in A is bounded). Then, $(A^{-1}R)_\delta = \{YA\}$, $\varepsilon A = C(YA)$, $\sigma A = B_{\omega, A}(YA)$, and

5.7 COROLLARY. *For* $A \in W$, *if the unit is strong, then* $\mu(l)A = C(YA) \cap B_{\omega, A}(YA)$.

(One may note here that this presentation of $\mu(l)A$ is in its Yosida representation, even though $\sigma A = B_{\omega, A}(YA)$ is not.)

Specializing further, let X be a compact zero-dimensional space, and $S(X) \equiv \{f \in C(X) \mid f(X) \text{ is finite}\}$ (called the Specker group of continuous step functions). Note that $YS(X) = X$, and 1 is a strong unit.

5.8 COROLLARY.

(*a*) $\sigma S(X) = \{f \in B(X) \mid f(X)$ *is countable*$\}$

(*b*) $\mu(l)S(X) = \{f \in C(X) \mid f(X)$ *is countable*$\}$.

(c) $\mu(l)\, S(X) = C(X)$ *if and only if* X *is scattered (each nonvoid closed subspace has an isolated point).*

PROOF.

a) See [HM9].

b) This follows from a) and 5.7.

c) W. Rudin has shown that compact X is scattered if and only if $f(X)$ is countable for each $f \in C(X)$; see [R] and [S]. So c) follows from this and b).

We now consider the possibility of more direct formulas for $\mu(l)$ in the vein of 5.2.

For $A \in W$, let eA be Conrad's essential closure, which is the maximum essential extension of A in W (or in all archimedean lattice-ordered groups) [C2]. Analogous to lA, let $l(\omega_1)A$ be the lateral σ-completion, the unique minimum *essential* extension of A to a Σ-object. We have $lA = \cap\{B \mid A \leqslant B \leqslant eA,\ B \in \mathcal{L}\}$ and $l(\omega_1)A = \cap\{B \mid A \leqslant B \leqslant eA,\ B \in \Sigma\}$ [HM7]. (The extension operator $l(\omega_1)$ is *not* σ; only rarely does $l(\omega_1)A = \sigma A$ occur.)

Among our earlier thoughts on the subject of this paper were: $\mu(l)A \overset{?}{=} \varepsilon A \cap lA$ for each A? Then, $\mu(l)A \overset{?}{=} \varepsilon A \cap l(\omega_1)A$ for each A? (These intersections are taken in eA.) Shortly, we shall see that this is not the case.

5.9 PROPOSITION. $A \leqslant \mu(l)A \leqslant \varepsilon A \cap l(\omega_1)A \leqslant \varepsilon A \cap lA \leqslant eA$ $(A \in W)$.

PROOF. Whenever r is an *essential* monoreflection, with range $\mathcal{R}$, rA can be achieved from *any* embedding $A \leqslant E$, $E \in \mathcal{R}$, as $rA = \cap\{R \mid A \leqslant R \leqslant E,\ R \in \mathcal{R}\}$ [HMad]. Apply this to $r = \mu(l)$ and $\mathcal{R} = \mathcal{R}(\mathcal{E} \cup \mathcal{L}) = \mathcal{R}(\mathcal{E} \cup \mathcal{R}(\mathcal{L})) = \mathcal{R}(\mathcal{E} \cup \Sigma)$ using 5.1 and 4.3. Since $\varepsilon A \in \mathcal{E} \subsetneq \mathcal{R}$ and $l(\omega_1)A \in \Sigma \subsetneq \mathcal{R}$, we find $\mu(l)A \leqslant \varepsilon A \cap l(\omega_1)A$.

We return to our favorite examples $S(X)$, X compact zero-dimensional. A *quasipartition* of X is a pairwise disjoint family of clopen sets with dense union. Recall the Stone-Nakano theorems [GJ; 3N]: X is extremally (resp., basically) disconnected if and only if $C(X)$ is conditionally complete (resp., conditionally σ-complete) (also if and only if laterally complete (resp., laterally σ-complete), it can be shown).

5.10 PROPOSITION. *If X is basically disconnected, then*

$$l(\omega_1)S(X) = \{f \in D(X) \mid X \textit{ has a countable quasipartition } \mathcal{U} \textit{ with } f|U \textit{ constant } \forall U \in \mathcal{U}\}.$$

If X is extremally disconnected, then

$$lS(X) = \{f \in D(X) \mid X \text{ has a quasipartition } \mathcal{U} \text{ with } f|U \text{ constant } \forall U \in \mathcal{U}\}.$$

PROOF. We treat the first case. The second is exactly similar, and also can be derived easily from 3.2 or 3.9. (No such derivation is available for the first case.)

Let B be the described collection of functions.

$B \in W$: For example, if f_i has associated countable quasipartition $\mathcal{U}_i (i = 1, 2)$, let $\mathcal{U} = \{U_1 \cap U_2 \mid U_i \in \mathcal{U}_i\}$. This is a countable quasipartition with $f_1 + f_2$ defined on $\cup \mathcal{U}$ and constant on each member of $\mathcal{U}$. Since $\cup \mathcal{U}$ is cozero and X is basically disconnected, $f_1 + f_2 \in C(\cup \mathcal{U})$ extends to «$f_1 + f_2$» $\in D(X)$ which is the sum in B.

$A \leqslant B$ is essential: if $0 < f \in B$ and f has associated quasipartition $\mathcal{U}$, then $f|U = r > 0$ for some $U \in \mathcal{U}$, and $0 < r\chi(\mathcal{U}) \leqslant f$ with $r\chi(\mathcal{U}) \in A$.

$B \in \Sigma$: Let $\{f_n\}$ be a disjoint family in B, with $\mathcal{U}_n$ a quasipartition associated to f_n, for each n. Now, $\{\operatorname{coz} f_n\}$ is a disjoint family of subsets of X, and for each n, $\operatorname{coz} f_n = \bigcup_k V_{nk}$ for a disjoint family $\{V_{nk}\}_k$ of clopen sets. ($\operatorname{coz} f_n$ is σ-compact, thus Lindelöf. Cover it by clopen subsets, take a countable subcover, and disjointify.) Then,

$$\mathcal{U} \equiv \{U \cap V_{nk} \mid n, k \in N;\ U \in \mathcal{U}_n\} \cup \{X - \overline{\bigcup_n \operatorname{coz} f_n}\}$$

is a countable clopen quasipartition. ($\overline{\bigcup_n \operatorname{coz} f_n}$ is open since X is basically disconnected). Define $f \in D(X)$ by: for $W \in \mathcal{U}$, $f|W \equiv f_n|W$ for the unique n with $W \subseteq \operatorname{coz} f_n$; extend over X (possible since $\cup \mathcal{U}$ is cozero, thus C^*-embedded). Then $f = \vee f_n$ in B (for example, by Lemma 4.1 in [BH2], since $\{x \mid f(x) = \vee f_n(x)\} \supseteq \cup \mathcal{U}$, and is dense in X).

5.11 THEOREM. *a) For any infinite compact basically disconnected space X, $\mu(l)\, S(X) \neq \varepsilon S(X) \cap l(\omega_1)\, S(X)$.*

b) For any compact extremally disconnected X which fails the countable chain condition, $\varepsilon S(X) \cap l(\omega_1)\, S(X) \neq \varepsilon S(X) \cap lS(X)$.

PROOF. *a)* Since X is infinite Hausdorff and zero-dimensional, there is a disjoint countable family $U_1, U_2, \ldots$ of nonvoid clopen sets. Let $U_0 = X - \overline{\bigcup_{n \geqslant 1} U_n}$, which is clopen by basic disconnectivity. Let $\{r_n\}$ be an enumeration of the rationals in [0, 1], and let $f \in C(X)$ be $f | U_n = r_n$ for each n, extended continuously over X. (Each basically disconnected space is an F-space, meaning cozero sets are C^*-embedded [GJ]). Then $f \in \varepsilon S(X) \cap l(\omega_1) S(X)$ (since $\varepsilon S(X) = C(X)$, and 5.10) while $f(X) \supseteq [0, 1]$ so $f \notin \mu(l) S(X)$ by 5.8 (b). To see that, let $r \in [0, 1]$ and choose $r_{n_i} \to r$. There is $x \in \overline{\cup U_{n_i}} - \cup U_{n_i}$ by compactness, and $f(x) = r$ by continuity.

b) Take an uncountable pairwise disjoint family of clopen sets, and with Zorn's Lemma, enlarge to a maximal such family $\mathcal{U}$; this is a quasi-partition. Let $\varphi : \mathcal{U} \to [0, 1]$ have uncountable range. Let $f \in C(X)$ be $f | U = \varphi(U)$ for each $U \in \mathcal{U}$, then extended over X. (By extremal disconnectivity, dense sets are C^*-embedded [GJ].) So $f \in \varepsilon S(X) = C(X)$. By 5.10, $f \in lS(X)$ and $f \notin l(\omega_1) S(X)$.

6. Comments and questions.

6.1. *What are the elements of* $\mu(l) A$?

We saw in § 5 that $\mu(l) A < \varepsilon A \cap lA$ can occur. Still, one wonders if there is a general method of picking out from lA the elements of $\mu(l)A$ (the «functorial elements»). We note in archimedean f-rings (a) in [HM1] a successful effort in this vein for $\mu(q) \equiv$ the maximum monoreflection under $q \equiv$ the classical ring of quotients, and (b) in [HM2], an unsuccessful such effort for $\mu(Q)$, $Q \equiv$ the complete ring of quotients. (As mentioned earlier $\mu(Q)$ and $\mu(l)$ are very similar and closely related. Our paper [HM ∞] details that connection.)

The same question can be asked about $\mu(l)$ within $\varepsilon A \cap l(\omega_1)A$. This appears even less tractable, since we know no description of $l(\omega_1)A$ not involving transfinite iteration. (See [HM7].)

6.2.*What is the range of* $\mu(l)$?

We noted in 5.2 that $\mathcal{R}(\mathcal{E} \cup \mathcal{L})$ is the range of $\mu(l)$. Thus $A = \mu(l) A$ if and only if $A \in \mathcal{R}(\mathcal{E} \cup \mathcal{L})$ if and only if A is an extremal subobject of some $E \times L$, $E \in \mathcal{E}$ and $L \in \mathcal{L}$ (using 2.1 and the fact that $\mathcal{E}$ and $\mathcal{L}$ are closed under product formation). From a categorical perspective perhaps it's hard to say more, but this certainly is not a satisfacory *algebraic* description. What is an algebraic condition on A equivalent to the equality $A = \mu(l) A$? (One asks this leaving aside the debatable question, Is the study of the class W really algebra?)

6.3. *What happens without a unit*?

Consider the category *Arch* of archimedean lattice-ordered groups (without benefit of distinguished weak unit, without benefit of the Yosida representation). Here all the categorical machinery of § 2 and 5.1 is perfectly in place, and the relevant operators and functors exist. For the nonce, we use notation such as, for example: β° for the maximum monoreflection in *Arch*, whose range is EC°, whose objects are those *Arch*-objects which are divisible and both laterally and conditionally σ-complete [BH1] – but, it is not so clear what $\beta^\circ A$ «is» (as opposed to βA as described in 5.3; even for $A \in W$, $\beta^\circ A \neq \beta A$ can occur [K]); ε° for the maximum essential monoreflection in *Arch*, whose range ε° consists of the relatively uniformly complete *Arch* objects, with $\varepsilon^\circ A$ the relative uniform completion [BH 5]; l° the lateral completion, which is described in [AF]; Ch. 8.

From 5.1, we have $\mu(l^\circ)\, A = \varepsilon^\circ A \cap \bar{l}^\circ A \leqslant \beta^\circ A$ $(A \in Arch)$, where $\bar{l}^\circ$ is the monoreflection for the epireflective hull $\mathcal{R}(\mathcal{L}^\circ)$ in *Arch*, and $\mu(l^\circ)$ ranges on $\mathcal{R}(\mathcal{E}^\circ \cup \mathcal{L}^\circ)$.

It seems plausible enough that «4.2 is true in *Arch*», in particular, that $\mathcal{R}(\mathcal{L}^\circ) = \Sigma^\circ$, so $\bar{l}^\circ$ would be σ°, but we have no idea how to prove that. ([HM 6, 9] discuss σ°. As with β°, «what is it?»)

The following line from [BH 1; p. 475] seems in order: We are adrift in *Arch* without the anchor of the Yosida representation.

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