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Weyl Formula for Quasi-Elliptic Pseudo-Differential Operators.

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1. Introduction.

This paper is devoted to the asymptotic behaviour for large λ of the counting function $N(\lambda)$ associated with quasi-elliptic anisotropic pseudo-differential operators on compact manifolds.

Anisotropic operators are defined locally as standard, by imposing different weight to derivatives with respect to different groups of variables. Precisely, in an open subset Ω of $\mathbb{R}^n$, the symbols classes $S^{\mu, q}(\Omega \times \times \mathbb{R}^n)$ (where μ is a real number and q is a given n-tuple of rational numbers $\geqslant 1$) are defined by the inequalities:

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leqslant C_{\alpha, \beta, K}(1 + |\xi|_q)^{\mu - \langle \alpha, q \rangle}$$

for x in a fixed compact subset $K \subset \Omega$ and ξ in $\mathbb{R}^n$. Here the weight function $|\xi|_q$ is defined by $|\xi|_q = \sum_{i=1}^{n} |\xi_i|^{1/q_i}$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$. The related calculus is due originally to Hunt, Piriou [10], [11], whereas the invariance of these operators with respect to certain classes of diffeomorphisms has been investigated in Parenti [15], where it is also shown that the definition of anisotropic operators can be transferred to a foliated manifold M, provided the foliation preserves the previous groups of variables. Our work looks as a continuation of [15] in the new direction of the spectral theory.

We begin by seeing that the quasi-homogeneous principal symbol of an anisotropic operator takes invariant meaning on a new vector bundle

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$T_q^* M$ which replaces the cotangent bundle $T^* M$ of the homogeneous case. Attention is here fixed on a self-adjoint quasi-elliptic operator A on a compact Riemannian manifold M. In order to reach the so-called Weyl's formula, we shall use the *heat method* (cf. Gilkey [3], Maniccia, Panarese [12], Grubb [4], Grubb, Seeley [5], Melrose [13], Gil [2], Taylor [18]).

In general one can associate several interesting operator-functions F with a quasi-elliptic operator A, such that $F(A)$ is trace class. In particular, the following have been studied for elliptic operators: the resolvent $(A-\lambda)^{-1}$ (when A has order > dim M), the power operator A^{-s} for Re s sufficiently large, and the heat operator e^{-tA} for $t>0$ (see for example Grubb [4] for relations between these three functions). It is well-known that in any case the asymptotic behaviour of $N(\lambda)$ can be investigated studying the trace Tr $F(A)$ of $F(A)$ and therefore studying its kernel K_F:

$$\int_M K_F(x, x)\, dx = \operatorname{Tr} F(A) = \sum_{j \geq 1} F(\lambda_j) = \int F(\lambda)\, dN(\lambda)$$

where $\{\lambda_j\}_{j\geq 1}$ is the sequence of the eigenvalues of A, and then applying Tauberian theorems to deduce the asymptotic behaviour of $N(\lambda)$.

Now, usually one is able to obtain an explicit enough expression for $K_F(x, x)$ (for instance, using pseudo-differential techniques), and therefore direct estimates on K_F yield the sought informations.

The heat method develops this program for the heat operator e^{-tA}; in this way we shall find a formula with remainder for the trace Tr e^{-tA} as $t\to 0^+$, from which, by Karamata's Tauberian Theorem, the Weyl's formula will follow. Namely

$$N(\lambda) = C\lambda^{|q|/\mu} + o(\lambda^{|q|/\mu}) \qquad \text{as } \lambda \to +\infty,$$

with $|q| = \sum_{j=1}^{n} q_j$ and

$$C = (2\pi)^{-n} \int_{a_\mu(x, \xi) \leq 1} dx\, d\xi$$

where a_μ is the quasi-homogeneous principal symbol of A and $dx\, d\xi$ is the canonical volume density in $T_q^* M$.

As an example, we shall consider the differential operator $P = \partial_{\theta_1}^4 - - \partial_{\theta_2}^2$ on the flat foliated torus $\mathbb{T}^2 = \mathbb{R}^2 / 2\pi\mathbb{Z} \oplus 2\pi\mathbb{Z}$.

Let us observe that the spectrum of quasi-elliptic operators in $\mathbb{R}^n$ has been studied in Helffer, Robert [7], Mohamed [14], Boggiatto, Buzano, Rodino [1], using other methods, cf. Hörmander [9], and obtaining precise estimates of the remainder in the Weyl's formula. In our case, similar sharp remainders will be, hopefully, obtained by using the anisotropic Fourier integral operators of Parenti, Segala [16]; we leave to the future the full development of these ideas.

Finally, I thank Professor L. Rodino who suggested the argument of the research and guided the work.

2. Anisotropic operators.

In this section we recall in short the definition and the main properties of the anisotropic operators; for more details we refer to Parenti [15].

Let Ω, X be open subsets of $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively. Let $(m_1, \dots, m_n)$ be a given n-tuple of integer numbers; we set $m = \max_j \{m_j\}$, $q = (q_1, \dots, q_n) = \left(\frac{m}{m_1}, \dots, \frac{m}{m_n}\right)$. If $\xi \in \mathbb{R}^n$, we define $|\xi|_q = \sum_{j=1}^{n} |\xi_j|^{1/q_j}$.

DEFINITION 2.1. *Let μ be a real number; we shall denote by $S^{\mu, q}(X \times \mathbb{R}^n)$ the space of all functions $p(x, \xi) \in C^\infty(X \times \mathbb{R}^n)$ satisfying the following condition: for every multi-index α, β and for every compact subset $K \subset X$ there exists $C_{\alpha, \beta, K} > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{\alpha, \beta, K}(1 + |\xi|_q)^{\mu - \langle \alpha, q \rangle} \tag{2.1}$$

for every $x \in K$, $\xi \in \mathbb{R}^n$, where $\langle \alpha, q \rangle = \sum_{j=1}^{n} \alpha_j q_j$.

As usual we put $S^{-\infty, q} = \bigcap_\mu S^{\mu, q}$ and $S^{\infty, q} = \bigcup_\mu S^{\mu, q}$.

DEFINITION 2.2. *If $a \in S^{\mu, q}(\Omega \times \Omega \times \mathbb{R}^n)$, the pseudo-differential operator $A = \mathrm{Op}(a)$ with amplitude a is defined for $f \in C_0^\infty(\Omega)$ by*

$$Af(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} a(x, y, \xi) f(y)\, dy\, d\xi\,, \tag{2.2}$$

where the integral is understood as an oscillatory integral. We shall denote by $L^{\mu, q}(\Omega)$ the space of all operators of this form.

If $A \in L^{\mu, q}(\Omega)$ is defined by an amplitude $a(x, \xi)$ independent of y (i.e. a symbol), we write $A = a(x, D)$. Moreover one easily sees that

$L^{-\infty, q}(\Omega) := \bigcap_{\mu} L^{\mu, q}(\Omega)$ is exactly the space of all regularizing operators, namely having kernel in $C^{\infty}(\Omega \times \Omega)$.

We remark that the image of the function $\mathbb{Z}_+^n \ni \alpha \mapsto \langle \alpha, q \rangle$ consists of the (non-negative) integer multiplies of the rational number

$$\theta = \theta(q) = m \frac{m''}{m'} \tag{2.3}$$

where $m' = \text{m.c.m.}\ \{m_1, \ldots, m_n\}$ and $m'' = \text{M.C.D.}\left\{\frac{m'}{m_1}, \ldots, \frac{m'}{m_n}\right\}$. Observe again that $0 < \theta \leqslant 1$ and $\theta = 1$ if and only if m_j divide m for every $j = 1, \ldots, n$. This number θ arises in the asymptotic formulas of the symbolic calculus; for instance, the product of two operators $p(x, D) \in L^{\mu, q}(\Omega)$ and $q(x, D) \in L^{\mu', q}(\Omega)$, one of the them properly supported, with symbols $p(x, \xi)$ and $q(x, \xi)$ respectively, is in $L^{\mu + \mu', q}(\Omega)$ and, up to regularizing operators, it has a symbol

$$p \circ q \sim \sum_{j \geqslant 0} \sum_{\langle \alpha, q \rangle = \theta j} (\alpha!)^{-1} \partial_{\xi}^{\alpha} p D_x^{\alpha} q\,. \tag{2.4}$$

DEFINITION 2.3. *We say that $P = p(x, D) \in L^{\mu, q}(\Omega)$ has quasi-homogeneous principal symbol if there exists $p_{\mu, q}(x, \xi) \in C^{\infty}(\Omega \times (\mathbb{R}^n \setminus \{0\}))$ quasi-homogeneous of degree μ with respect to ξ (i.e. $p_{\mu, q}(x, t^{q_1}\xi_1, \ldots, t^{q_n}\xi_n) = t^{\mu} p_{q, \mu}(x, \xi)$ for every $t > 0$, $x \in \Omega$, $\xi \in \mathbb{R}^n \setminus \{0\}$), such that $p(x, \xi) - p_{\mu, q}(x, \xi) = O(|\xi|_q^{\mu - \varepsilon})$ as $|\xi|_q \to +\infty$, uniformly for x in compact subsets of Ω and for some $\varepsilon > 0$.*

DEFINITION 2.4. *An operator $P = p(x, D) \in L^{\mu, q}(\Omega)$ with quasi-homogeneous principal symbol $p_{\mu, q}(x, \xi)$ is called quasi-elliptic if $p_{\mu, q}(x, \xi) \neq 0$ for all $x \in \Omega$, $\xi \in \mathbb{R}^n$.*

Let us come now to study the behaviour of these operators as it concerns the changes of variables.

DEFINITION 2.5. *Suppose that the n-tuple $(m_1, \ldots, m_n)$ satisfies the following condition: there exist positive integer numbers $r_1, \ldots, r_{\nu}$ such that*

$$\begin{cases} r_1 + \ldots + r_{\nu} = n \\ m_1 = \ldots = m_{r_1} < m_{r_1+1} = \ldots = m_{r_1+r_2} < \ldots < m_{r_1+\ldots+r_{\nu-1}+1} = \ldots = m_n . \end{cases}$$

Then, denoted by $GL(n, \mathbb{R})$ the Lie group of the invertible real $n \times n$ matrices, we write $GL_q(n, \mathbb{R})$ for the subgroup of all matrices of the form (in block matrix notation) $A = (A_{ij})_{i, j=1, 2, \dots, \nu}$, where A_{ij} is a $r_i \times$ $\times r_j$ matrix, $A_{jj} \in GL(r_j, \mathbb{R})$ for $1 \leqslant j \leqslant \nu$, and $A_{ij} = 0$ if $i > j$:

$$A = \begin{bmatrix} A_{11} & & & * \\ 0 & A_{22} & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & A_{\nu\nu} \end{bmatrix}. \tag{2.5}$$

In other words, the matrix $A = (a_{jk})_{j, k=1, \dots, n} \in GL(n, \mathbb{R})$ is in $GL_q(n, \mathbb{R})$ if and only if $a_{jk} = 0$ when $m_j > m_k$.

From now on we shall assume $(m_1, \dots, m_n)$ and $q = (q_1, \dots, q_n) =$ $= \left(\frac{m}{m_1}, \dots, \frac{m}{m_n}\right)$, $m = \max_j \{m_j\}$, satisfy the assumptions of Definition (2.5).

THEOREM 2.6. *Let Ω, Ω' be open subsets of $\mathbb{R}^n$; let $\phi : \Omega \to \Omega'$ be a diffeomorphism, and ϕ' its Jacobian matrix. Then, if ${}^t\phi' \in$ $\in C^\infty(\Omega, GL_q(n, \mathbb{R}))$, for every $A \in L^{\mu, q}(\Omega)$, the operator A^ϕ defined for $u \in C_0^\infty(\Omega')$ by*

$$A^\phi u = (A(u \circ \phi)) \circ \phi^{-1} \tag{2.6}$$

is in $L^{\mu, q}(\Omega')$.

Definition 2.5 and Theorem 2.6 were in Parenti [15]; one can easily establish the following result, concerning the corresponding quasi-homogeneous principal symbols.

THEOREM 2.7. *Let $\phi : \Omega \to \Omega'$ be a diffeomorphism with ${}^t\phi' =$ $= (A_{ij})_{i, j=1, \dots, \nu} \in C^\infty(\Omega, GL_q(n, \mathbb{R}))$ (in block matrix notation). Let $P \in$ $\in L^{\mu, q}(\Omega)$ with quasi-homogeneous principal symbol $p_\mu(x, \xi)$. Then the operator A^ϕ, defined in (2.6), has quasi-homogeneous principal symbol $p'_\mu(y, \eta)$ given by*

$$p'_\mu(y, \eta) = p_\mu(\phi^{-1}(y), ({}^t\phi')_d(\phi^{-1}(y))\eta) \tag{2.7}$$

for $y \in \Omega'$, $\eta \in \mathbb{R}^n \setminus \{0\}$, where $({}^t\phi')_d = [A_{11}, \dots, A_{\nu\nu}]$ is the diagonal part of ${}^t\phi'$.

Theorem 2.6 allows us to define our operators on those foliated manifolds (cf. Haefliger [6]) with the following property: each leaf must again have a structure of foliated manifold and so on, until the νth foliation on which any condition is not given.

DEFINITION 2.8. (i) *Let M be a n-dimensional C^∞ manifold. We say that M is a q-manifold if it has a (maximal) atlas $\mathcal{A} = (\mathcal{O}_i, \phi_i)$ satisfying the following condition: for every i, j, ${}^t(\phi_i \circ \phi_j^{-1})' \in C^\infty(\phi_j(\mathcal{O}_i \cap \cap \mathcal{O}_j), GL_q(n, \mathbb{R}))$.*

(ii) *Let M be a q-manifold, and let A be a continuous linear map $A : C_0^\infty(M) \to C^\infty(M)$. We say that A is a pseudo-differential operator in $L^{\mu, q}(M)$ if, given any local chart $(\mathcal{O}, \chi)$, the transfer $A_\mathcal{O}^\chi := = (\chi^{-1})^* \circ A_\mathcal{O} \circ \chi^* : C_0^\infty(\chi(\mathcal{O})) \to C^\infty(\chi(\mathcal{O}))$ is a pseudo-differential operator in $L^{\mu, q}(\chi(\mathcal{O}))$, where $A_\mathcal{O}$ denotes the composition of the extension $C_0^\infty(\mathcal{O}) \to C_0^\infty(M)$, of the operator A and of the restriction $C^\infty(M) \to \to C^\infty(\mathcal{O})$.*

(iii) *We say that $A \in L^{\mu, q}(M)$ has quasi-homogeneous principal symbol if, given any local chart $(\mathcal{O}, \chi)$, the transfer $A_\mathcal{O}^\chi$ has quasi-homogeneous principal symbol according to Definition 2.3.*

Observe that by Theorem 2.7 the quasi-homogeneous principal symbol cannot have invariant meaning on the cotangent bundle T^*M. This justifies the following construction of a new vector bundle on a q-manifold M.

PROPOSITION 2.9. *On any q-manifold M there is a particular C^∞ vector bundle T_q^*M, see the next proof for the precise definition, such that the quasi-homogeneous principal symbol of an operator $P \in \in L^{\mu, q}(M)$ can be interpreted as a function in $C^\infty(T_q^*M \backslash 0)$.*

PROOF. Let us argue in the case of two groups of variables, i.e. $\nu = 2$ with the notations of Definition 2.5. As a consequence of the given foliation two particular vector bundles are defined on M. In fact, this foliation defines a (completely integrable) field $\mathcal{F}$ of $(n - r_1)$-planes (that are the tangent spaces to leaves). Then we consider the vector bundle $N^*\mathcal{F}$ on M, that is a sub-bundle of T^*M, with fiber at $x \in M$ given by the polar, or orthogonal, in T_xM of the tangent space at x to the leaf containing x; similarly we can consider the vector bundle $T^*\mathcal{F}$ on M with fiber at x given by the cotangent space at x to leaf containing x. By Theorem 2.7, if we assume on N^*M and T^*M fiber coordinates induced by the ones in

T^*M, it is straightforward to convince ourselves that in the complement of the null section of $T_q^*M := N^*M \oplus T^*M$ (Withney sum) the quasi-homogeneous principal symbol can be invariantly defined. In the general case, using inductive arguments, we shall set with obvious notations $T_q^*M = \left(\bigoplus_{i=1}^{\nu} N^*\mathcal{F}_i\right) \oplus T^*\mathcal{F}_\nu$. ■

In the following, to integrate functions defined in T_q^*M, we shall refer to *canonical* volume density defined (invariantly and locally) by $|dx_1 \wedge \ldots \wedge dx_n \wedge d\xi_1 \wedge \ldots \wedge d\xi_n|$.

DEFINITION 2.10. *Let M be a q-manifold and $P \in L^{\mu, q}(M)$, with quasi-homogeneous principal symbol p_μ. We say that P is quasi-elliptic if $p_\mu \neq 0$ in $T_q^*M \backslash 0$.*

Finally, we need introduce a scale of weighted Sobolev spaces.

DEFINITION 2.11. *Let $s \in \mathbb{R}$; we denote by $H^{s, q}(\mathbb{R}^n)$ the space of all distributions $u \in S'(\mathbb{R}^n)$ such that $(1 + |\xi|_q^2)^{s/2}\widehat{u}(\xi) \in L^2(\mathbb{R}^n)$.*

As usual, one defines the spaces $H^{s,q}_{\mathrm{loc}}(\Omega)$, $H_0^{s,q}(\Omega)$ and the Hilbert spaces $H^{s,q}(M)$ if M is a compact q-manifold. Then an operator $A \in \in L^{\mu, q}(M)$ can be regarded as continuous map $A : H^{s,q}(M) \to \to H^{s-\mu, q}(M)$.

3. The heat semigroup e^{-tA}.

Let M be a *compact* Riemannian q-manifold. A quasi-elliptic operator $A \in L^{\mu,q}(M)$, $\mu > 0$, can be regarded as a closed unbounded operator on $L^2(M)$ with dense domain $H^{\mu,q}(M)$. Standard arguments (cf. Shubin [17]) show that its resolvent is compact, and therefore, if A is formally self-adjoint, it has a spectrum made of real eigenvalues $\{\lambda_j\}_{j \geq 1}$ of finite multiplicity, clustering at infinity. Studying the asymptotic behaviour of the spectrum, we assume A satisfies the following properties:

(3.1) $A^* = A \in L^{\mu,q}(M), \mu > 0$, with quasi-homogeneous principal symbol

$$a_\mu(x, \xi) > 0 \quad \text{in } T_q^*M \backslash 0 .$$

In particular, for any chart $(\mathcal{O}, \chi)$, called $a(x, \xi)$ a symbol for the transfer

$A^{\chi}_{\mathcal{O}}$, this means that

(3.2) $$a(x,\xi)-a_\mu(x,\xi)=O(|\xi|_q^{\mu-\varepsilon}) \quad \text{as } |\xi|_q\to+\infty$$

uniformly for x in compact subsets of $\chi(\mathcal{O})$ and for some $\varepsilon>0$. This number ε, which might depend on the local chart, *will be assumed constant.*

By the positivity of the principal symbol, A is semi-bounded from below and so the sequence of eigenvalues is definitively positive. Moreover, denoted by $\{A_\lambda\}$ the spectral resolution of A, for $t\geq 0$ is well defined $e^{-tA}=\int e^{-t\lambda}dA_\lambda$ as one-parameter semigroup of bounded operators $L^2(M)\to L^2(M)$ or, more generally, $H^{s,q}(M)\to H^{s,q}(M)$ for any integer s (this follows from the fact that if $A\geq c$, then e^{-tA} commutes with $(A-c+1)^s$ for any s and the H^s norm of u is equivalent to the L^2 norm of $(A-c+1)^s u$).

Our purpose is to determine the singularities of the heat kernel $H(t,x,y)$ of e^{-tA} for small t.

THEOREM 3.1. *For $t\in[0,+\infty)$ one can represent e^{-tA} in the form of a sum of an operator with kernel smooth in t, x, y and a pseudo-differential operator $U(t)$. In any local chart $\mathcal{O}$, $U(t)$ has a symbol $u(t,x,\xi)$ smooth in t and satisfying the following condition: for any compact subset $K\subset\mathcal{O}$, $\alpha,\beta\in\mathbb{Z}^n_+$, $l\in\mathbb{Z}_+$, $T\in(0,+\infty)$ there exists $C>0$ such that*

$$|t^N\partial_t^l\partial_\xi^\alpha\partial_x^\beta u(t,x,\xi)|\leq C(1+|\xi|_q)^{(l-N)\mu-\langle\alpha,q\rangle}$$

for all $t\in[0,T]$, $\xi\in\mathbb{R}^n$ and $x\in K$. In particular, $U(t)$ and therefore e^{-tA} are regularizing for $t>0$.

PROOF. Following a standard pattern, see for example Taylor [18], we limit ourselves here to the main lines. Let us construct the pseudo-differential operator $U(t)$, which approximates e^{-tA}, as solution of the problem

(3.3) $$\begin{cases}\left(\dfrac{\mathrm{d}}{\mathrm{d}t}+A\right)U(t)\in L^{-\infty}(M)\\ U(0)=\mathrm{Id}_{C^\infty(M)}.\end{cases}$$

More precisely, the left side of the first equation in (3.3) will also be a smooth function of t for $t\geq 0$, with values in $L^{-\infty}(M)$. The linearity of the problem allows a reduction, using a partition of unity, to the case of constructing the symbol $u(t, x, \xi)$ of $U(t)$ in a local chart $\mathcal{O}$. We assume the given volume density in $\mathcal{O}$ agrees with the Lebesgue measure and we choose a properly supported representative $a(x, D)$ of A in $\mathcal{O}$. It is natural to look for $u(t, x, \xi)$ with asymptotic expansion $u(t, x, \xi) \sim \sum_{j \geq 0} u_j(t, x, \xi)$, $u_j \in S^{-\theta j}(\mathcal{O} \times \mathbb{R}^n)$ (where θ is defined in (2.3)). By (2.4) we can write the problem (3.3) in terms of symbols; regrouping the terms of the same order, we deduce the following transport equation, coupled with the respective initial conditions:

$$\text{(3.4)} \qquad \begin{cases} \partial_t u_j(t, x, \xi) + a(x, \xi)\, u_j(t, x, \xi) + \\ \qquad\qquad + \sum\limits_{\substack{\theta l + \langle \alpha, q\rangle = \theta j \\ 0 \leq l \leq j-1}} (\alpha!)^{-1} \partial_\xi^\alpha a(x, \xi)\, D_x^\alpha u_l(t, x, \xi) = 0 \\ u_0(t, x, \xi) = 1 \\ u_j(0, x, \xi) = 0 \qquad \text{for } j \geq 1\,. \end{cases}$$

By inductive arguments it is easily proved that

$$\text{(3.5)} \qquad \begin{cases} u_0 = e^{-ta} \\ u_j = e^{-ta} \sum\limits_{h=2}^{2j} t^h S^{h\mu - \theta j}, \qquad j \geq 1\,, \end{cases}$$

where S^k stands for a function in $S^{k,q}(\mathcal{O} \times \mathbb{R}^n)$.

In order to make this formal construction meaningful and prove the regularity properties with respect to the variable t, we observe that from (3.5) we have for every $j \geq 0$, $\alpha, \beta \in \mathbb{Z}_+^n$, $l \in \mathbb{Z}_+$,

$$\partial_\xi^\alpha \partial_x^\beta \partial_t^l u_j(t, x, \xi) = e^{-ta(x, \xi)} \sum_{s = s^*}^{2j + |\alpha + \beta|} (ta(x, \xi))^s S^{l\mu - \theta j - \langle \alpha, q\rangle},$$

where $s^*(\alpha, \beta, l, j) \geq 0$. It readily follows from the positivity of the principal symbol that for every compact subset $K \subset \mathcal{O}$, $T \in (0, +\infty)$, $N \in \mathbb{Z}_+$, there exists a constant $C > 0$ such that

$$|t^N \partial_t^l \partial_\xi^\alpha \partial_x^\beta u_j(t, x, \xi)| \leq C(1 + |\xi|_q)^{(l-N)\mu - \theta j - \langle \alpha, q\rangle}$$

for all $t \in [0, T]$, $x \in K$, $\xi \in \mathbb{R}^n$. Now, it is a standard procedure to construct $u(t, x, \xi) \sim \sum_{j \geq 0} u_j(t, x, \xi)$ satisfying the estimates

$$\text{(3.6)} \qquad \left| t^N \partial_t^l \partial_\xi^\alpha \partial_x^\beta \Big(u - \sum_{j \leq J-1} u_j \Big) \right| \leq C(1 + |\xi|_q)^{(l-N)\mu - \theta J - \langle \alpha, q\rangle}$$

for all $t \in [0, T]$, $x \in K$, $\xi \in \mathbb{R}^n$ with $u(0, x, \xi) \equiv 1$.

The regularity properties of $K(t) := \left(\frac{\mathrm{d}}{\mathrm{d}t} + A\right) U(t)$ with respect to t follow from the ones of $u(t, x, \xi)$.

Then it suffices to observe that by (3.3) it follows that

$$e^{-tA} - U(t) = \int_0^t e^{-(t-s)A} K(s)\, ds \tag{3.7}$$

as an equality between operators from $C^\infty(M)$ in itself. Since e^{-tA} defines a continuous map $C^\infty(M) \to C^\infty([0, +\infty) \times M)$, we conclude that $e^{-tA} - U(t)$ has kernel in $C^\infty([0, +\infty) \times M \times M)$. ■

4. Heat trace asymptotics.

Let M be a compact Riemannian q-manifold and let A be an operator satisfying (3.1). This section is directed to study the asymptotic behaviour of the counting function $N(\lambda) := \sum_{\lambda_j \leq \lambda} 1$ associated with A. By Theorem 3.1, e^{-tA} is regularizing for $t > 0$, therefore trace class. The study of $\mathrm{Tr}\ e^{-tA}$, which we are going to carry out, will therefore provide, by means of Karamata's Tauberian Theorem, the Weyl's estimate for $N(\lambda)$ we are seeking.

Denote by Γ the Euler's Gamma Function.

THEOREM 4.1. *Let M be a compact Riemannian q-manifold and let A be an operator satisfying* (3.1). *Then the trace of the corresponding heat semigroup e^{-tA} can be estimate by*

$$\mathrm{Tr}\ e^{-tA} = \Gamma\left(1 + \frac{|q|}{\mu}\right) C t^{-\frac{|q|}{\mu}} + O\left(t^{-\frac{|q|}{\mu} + \frac{\min\{\varepsilon, \theta\}}{\mu}}\right) \quad \text{as } t \to 0^+ \tag{4.1}$$

where θ and ε are defined in (2.3) *and* (3.2) *respectively,* $|q| = \sum_{j=1}^n q_j$ *and*

$$C = (2\pi)^{-n} \int_{a_\mu(x,\xi) \leq 1} dx\, d\xi\,, \tag{4.2}$$

where $dx\, d\xi$ is the canonical volume density in $T_q^* M$.

We need some technical lemmas. We begin by considering the function $\sigma : \mathbb{R}^n \to \mathbb{R}$, $\sigma(\xi) = \left(\sum_{i=1}^n \xi_i^{2m_i}\right)^{1/2m}$, where m_i, m are the integer numbers defining the n-tuple q (cf. the beginning of Section 2). Observe that σ is continuous and $\sigma|_{\mathbb{R}^n\setminus\{0\}} \in C^\infty(\mathbb{R}^n\setminus\{0\})$. Then we denote by Σ_q the manifold of equation $\sigma(\xi)=1$.

REMARK 4.2. It is readily seen that $n^{-1/2m}\sigma(\xi) \leqslant |\xi|_q \leqslant n\sigma(\xi)$ for all $\xi\in\mathbb{R}^n$. Hence, for $|\xi|\geqslant R>0$, the weight functions $\sigma(\xi)$, $|\xi|_q$ and $1+|\xi|_q$ are all equivalent.

LEMMA 4.3. *Let $\Omega\subset\mathbb{R}^n$ be open, $\phi\in C^\infty(\Omega\times(\mathbb{R}^n\setminus\{0\}))$ and let us suppose there exists $l>0$ such that $\phi(x, t^{q_1}\xi_1, \dots, t^{q_n}\xi_n)=t^l\phi(x,\xi)$ for all $(x,\xi)\in\Omega\times(\mathbb{R}^n\setminus\{0\})$ and $t>0$. For any fixed $k>-|q|$, define $k^*=\frac{k+|q|}{l}$. If $\phi>0$ in $\Omega\times(\mathbb{R}^n\setminus\{0\})$, then for any fixed $x\in\Omega$ we have*

$$\int_{\mathbb{R}^n} e^{-t\phi(x,\xi)}\sigma(\xi)^k\,d\xi=\frac{\Gamma(k^*)}{l}t^{-k^*}\int_{\Sigma_q}\phi(x,\theta)^{-k^*}\,d\theta\,, \tag{4.3}$$

where $d\theta$ is a suitable volume density in Σ_q, defined as in the subsequent proof.

PROOF. We begin by defining the anisotropic polar coordinates σ, $\theta=(\theta_1,\dots,\theta_{n-1})$ by

$$\begin{cases} \xi_1=\sigma^{q_1}(\cos\theta_1)_s^{1/m_1}\\ \xi_2=\sigma^{q_2}(\sin\theta_1\cos\theta_2)_s^{1/m_2}\\ \vdots\\ \xi_{n-1}=\sigma^{q_{n-1}}(\sin\theta_1\sin\theta_2\dots\sin\theta_{n-2}\cos\theta_{n-1})_s^{1/m_{n-1}}\\ \xi_n=\sigma^{q_n}(\sin\theta_1\sin\theta_2\dots\sin\theta_{n-2}\sin\theta_{n-1})_s^{1/m_n}, \end{cases} \tag{4.4}$$

where $(t)_s^{1/m_i}$ stands for $|t|^{1/m_i}\operatorname{sign}t$ (working with such coordinates we leave out the points of the coordinate planes). Denoted by $\mathfrak{J}(\sigma,\theta)$ the Jacobian determinant of this transformation, we have $\mathfrak{J}(\sigma,\theta)=$ $=\sigma^{|q|-1}\mathfrak{J}(1,\theta)$. Moreover, the manifold Σ_q has (4. 4) with $\sigma=1$ as parametric equations. In Σ_q we assume $|\mathfrak{J}(1,\theta)\,d\theta_1\wedge\dots\wedge d\theta_{n-1}|$ as volume density; in short it will be denoted by $d\theta$. Let us switch to such coordinates and let us put $\sigma=\left(\frac{\varrho}{t\phi(x,\theta)}\right)^{/l}$; writing $\phi(x,\theta)$ for $\phi(x,\xi(1,\theta))$, the left-hand side of (4.3) becomes

$$\int_{\Sigma_q} \int_0^{+\infty} e^{-t\sigma^l \phi(x,\,\theta)} \sigma^{k+n-1} d\sigma\, d\theta = \frac{1}{l} \int_{\Sigma_q} \left(\int_0^{+\infty} e^{-\varrho} \varrho^{k^\star - 1} (t\phi(x,\,\theta))^{-k^\star} d\varrho \right) d\theta\,.$$

This concludes the proof.

LEMMA 4.4. *For any fixed x, the following formula holds*:

$$\int_{a_\mu(x,\,\xi) \le 1} d\xi = \frac{1}{|q|} \int_{\Sigma_q} a_\mu(x,\,\theta)^{-|q|/\mu} d\theta\,. \tag{4.5}$$

PROOF. Switching to the anisotropic polar coordinates $(\sigma,\,\theta)$ given by (4.4), we have

$$\int_{a_\mu(x,\,\xi) \le 1} d\xi = \int_{\Sigma_q} \int_0^{d(\theta)} \sigma^{|q|-1} d\sigma\, d\theta = \frac{1}{|q|} \int_{\Sigma_q} d(\theta)^{|q|} d\theta\,, \tag{4.6}$$

where the positive real number $d(\theta)$ is defined by the condition $a_\mu(x,\, d(\theta)^{q_1} \xi_1(1,\,\theta),\, \ldots,\, d(\theta)^{q_n} \xi_n(1,\,\theta)) = 1$.

By the quasi-homogeneity of a_μ, we get $d(\theta) = a_\mu(x,\,\theta)^{-1/\mu}$. Substituting this expression for $d(\theta)$ in (4.6), we obtain (4.5). ■

LEMMA 4.5. *Let $\mathcal{O} \subset M$ be a coordinate open subset, $\phi \in S^{k,\,q}(\mathcal{O} \times \mathbb{R}^n)$ and let a and a_μ be respectively the symbol and the quasi-homogeneous principal symbol of A in $\mathcal{O}$. Then*

$$\int_{\mathbb{R}^n} e^{-ta(x,\,\xi)} \phi(x,\,\xi)\, d\xi = \int_{\sigma(\xi) \ge 1} e^{-ta_\mu(x,\,\xi)} \phi(x,\,\xi)\, d\xi + O\left(t^{\min\left\{0,\, \frac{-|q| - k + \varepsilon}{\mu}\right\}}\right)$$

as $t \to 0^+$, uniformly for x in compact subsets of $\mathcal{O}$.

PROOF. Let K be an arbitrary compact subset of $\mathcal{O}$ and $x \in K$. In this proof each asymptotic formula is meant uniformly for x in K. Define

$$I(t) = \int_{\mathbb{R}^n} e^{-ta(x,\,\xi)} \phi(x,\,\xi)\, d\xi - \int_{\sigma(\xi) \ge 1} e^{-ta_\mu(x,\,\xi)} \phi(x,\,\xi)\, d\xi\,.$$

It is clear that

$$I(t) = O(1) + \int_{\sigma(\xi) \geq r} (e^{-ta(x,\,\xi)} - e^{-ta_\mu(x,\,\xi)})\, \phi(x,\, \xi)\, d\xi\,,$$

for any fixed $r > 0$. Now we distinguish two cases: $\varepsilon \leq |q| + k$ and $\varepsilon > |q| + k$.

In the first case, let us verify that

$$t^{\frac{|q|+k-\varepsilon}{\mu}} \int_{\sigma(\xi) \geq r} (e^{-ta(x,\,\xi)} - e^{-ta_\mu(x,\,\xi)})\, \phi(x,\, \xi)\, d\xi = O(1) \tag{4.7}$$

as $t \to 0^+$. Switching to the anisotropic polar coordinates $(\sigma,\, \theta)$ given by (4.4) and putting $\sigma = \left(\dfrac{\varrho}{ta_\mu(x,\, \theta)} \right)^{1/\mu}$, the expression in (4.7) turns into

$$t^{\frac{|q|+k-\varepsilon}{\mu}} \int_{\Sigma_q} \int_r^{+\infty} \sigma^{|q|-1} (e^{-ta(x,\xi(\sigma,\theta))} - e^{-ta_\mu(x,\xi(\sigma,\theta))})\, \phi(x,\, \xi(\sigma,\, \theta))\, d\sigma\, d\theta = \tag{4.8}$$

$$= \frac{1}{\mu} t^{\frac{k-\varepsilon}{\mu}} \int_{\Sigma_q} \int_{tr^\mu a_\mu(x,\theta)}^{+\infty} a_\mu(x,\theta)^{-\frac{|q|}{\mu}} e^{-\varrho} (e^{g(t,x,\varrho,\theta)} - 1)\, \varrho^{\frac{|q|}{\mu}-1} \overline{\phi}(t, x, \varrho, \theta)\, d\varrho\, d\theta\,,$$

with $g(t,\, x,\, \varrho,\, \theta) = t(a_\mu(x,\, \xi(\sigma,\, \theta)) - a(x,\, \xi(\sigma,\, \theta)))$ (we recall that σ is function of ϱ, t, x, θ) and $\overline{\phi}(t,\, x,\, \varrho,\, \theta) = \phi(x,\, \xi(\sigma,\, \theta))$. By (3.2) and Remark 4.2 there exists a constant $C > 0$ such that

$$|g(t,\, x,\, \varrho,\, \theta| \leq C t^{\frac{\varepsilon}{\mu}} \left(\frac{\varrho}{a_\mu(x,\, \theta)} \right)^{\frac{\mu-\varepsilon}{\mu}} \tag{4.9}$$

for r large enough. Hence in the integration domain we have $|g(t,\, x,\, \varrho,\, \theta)|/\varrho \leq Cr^{-\varepsilon}/a_\mu(x,\, \theta) \leq C' r^{-\varepsilon}$ for a suitable constant $C' > > 0$. So, for a fixed $\varepsilon_0 \in (0,\, 1)$, we can find r so large to have $|g(t,\, x,\, \varrho,\, \theta)| \leq \varepsilon_0 \varrho$. Taking this fact into account, an application of the Mac Laurin expansion for the exponential function yields

$$|e^{g(t,\, x,\, \varrho,\, \theta)} - 1| \leq \frac{e^{\varepsilon_0 \varrho} - 1}{\varepsilon_0 \varrho} |g(t,\, x,\, \varrho,\, \theta)|\,.$$

Hence, in view of (4.9), we get

$$|e^{g(t,x,\varrho,\theta)}-1| \leqslant Ct^{\frac{\varepsilon}{\mu}}\,\frac{e^{\varepsilon_0\varrho}-1}{\varepsilon_0\varrho}\left(\frac{\varrho}{a_\mu(x,\theta)}\right)^{\frac{\mu-\varepsilon}{\mu}}. \tag{4.10}$$

As it concerns the function $\overline{\phi}(t, x, \varrho, \theta)$, again on the integration domain, by Remark 4.2, we have

$$t^{\frac{k}{\mu}}\,|\overline{\phi}(t,x,\varrho,\theta)| \leqslant C''\left(\frac{\varrho}{a_\mu(x,\theta)}\right)^{\frac{k}{\mu}} \tag{4.11}$$

for a suitable constant $C''>0$. We return now to the integrals which appear in the right-hand side of (4.8). The most internal integral can be replaced by an integral $\int_0^{+\infty}$ if one multiplies the function under the integral sign by the characteristic function of $[tr^\mu a_\mu(x,\theta), +\infty)$. So, granted (4.10) and (4.11), the new function to integrate is dominated by

$$\frac{CC''}{\varepsilon_0\mu}a_\mu(x,\theta)^{\frac{-|q|-k+\varepsilon}{\mu}-1}\varrho^{\frac{|q|+k-\varepsilon}{\mu}-1}e^{-\varrho}(e^{\varepsilon_0\varrho}-1),$$

which is clearly integrable with respect to ϱ, θ. This concludes the proof in the case $\varepsilon \leqslant |q|+k$.

When $\varepsilon > |q|+k$, we must prove that

$$\int_{\sigma(\xi)\geqslant r}(e^{-ta(x,\xi)}-e^{-ta_\mu(x,\xi)})\,\phi(x,\xi)\,d\xi = O(1) \qquad \text{as } t\to 0^+.$$

In fact, in this case we have $a(x,\xi)-a_\mu(x,\xi) = O(|\xi|_q^{\mu-\varepsilon'})$ as $|\xi|_q\to+\infty$, with $\varepsilon' = |q|+k$. Therefore it follows from the first part of the proof that (4.8) holds with ε replaced by ε', which is our thesis. The lemma is proved. ■

Proof of Theorem 4.1. With the notations of the proof of Theorem 3.1, by (3.7) we have $\mathrm{Tr}\, e^{-tA}-\mathrm{Tr}\,U(t) = O(t)$ as $t\to 0^+$. Hence, it suffices to prove that (4.1) holds with $\mathrm{Tr}\; e^{-tA}$ replaced by $\mathrm{Tr}\; U(t)$. On the other hand, by the well-known formula for the kernel of a pseudo-differential operator, for $t>0$ we can write $\mathrm{Tr}\; U(t) = \int_M a(t,x)\,dx$ where, for any fixed integer $N\geqslant 1$, $a(t,x)$ can be expressed in a local chart $\mathcal{O}$ as sum of the terms

$$I_1(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-ta(x, \xi)} d\xi ,$$

$$I_2(t, x) = (2\pi)^{-n} \sum_{j=1}^{N-1} \int_{\mathbb{R}^n} u_j(t, x, \xi)\, d\xi ,$$

$$I_3(t, x) = (2\pi)^{-n} \int_{\mathbb{R}^n} p_N(t, x, \xi)\, d\xi ,$$

with $p_N(t, x, \xi) = u(t, x, \xi) - \sum_{j=0}^{N-1} u_j(t, x, \xi)$.

Now we establish asymptotic formulas as $t \to 0^+$ for I_1, I_2, I_3, which hold uniformly for x in an arbitrarily fixed compact subset K of $\mathcal{O}$.

If we take $N = \frac{|q|}{\theta} + 1$, by (3.6) we see at once that $I_3(t, x)$ is $O(1)$ as $t \to 0^+$ (we have used the fact that the function $\sigma(\xi)^k$ is integrable in $|\xi| > R > 0$ if $k < -|q|$).

Concerning $I_2(t, x)$, it can be written, by the expression for u_j given in (3.5), as sum over $j = 1, \ldots, \frac{|q|}{\theta}$ and $h = 2, \ldots, 2j$ of the terms

$$R_{j,h}(t, x) = t^h \int_{\mathbb{R}^n} e^{-ta(x, \xi)} S^{h\mu - \theta j} d\xi .$$

In view of Lemma 4.5 with $k = h\mu - \theta j$, we have

$$R_{j,h}(t, x) = I_{j,h}(t, x) + O(t^{\min\left\{h, \frac{-|q| + \theta j + \varepsilon}{\mu}\right\}})$$

with

$$I_{j,h}(t, x) = t^h \int_{\sigma(\xi) \geq 1} e^{-ta_\mu(x, \xi)} S^{h\mu - \theta j} d\xi .$$

On the other hand, there exists a constant $\overline{C} > 0$ such that for $x \in K$

$$|I_{j,h}(t, x)| \leq \overline{C} t^h \int_{\sigma(\xi) \geq 1} e^{-ta_\mu(x, \xi)} \sigma(\xi)^{h\mu - \theta j} d\xi \leq \overline{C} t^h \int_{\mathbb{R}^n} e^{-ta_\mu(x,\xi)} \sigma(\xi)^{h\mu - \theta j} d\xi .$$

We stress that $h\mu - \theta j > -|q|$, and therefore Lemma 4.5 ensures the estimates $I_{j,h}(t) = O(t^{\frac{-|q| + \theta j}{\mu}})$. Hence, for every $j = 1, \ldots, \frac{|q|}{\theta}$ and $h = 2, \ldots, 2j$ we get $R_{j,h}(t) = O(t^{-\frac{|q|}{\mu} + \frac{\theta}{\mu}})$.

As far as I_1 is concerned, again by Lemma 4.5 with $\phi \equiv 1$ we have

$$(2\pi)^{-n} \int_{\mathbf{R}^n} e^{-ta(x,\xi)} d\xi = (2\pi)^{-n} \int_{\sigma(\xi) \geq 1} e^{-ta_\mu(x,\xi)} d\xi + O(t^{\min\left\{0, \frac{-|q|+\varepsilon}{\mu}\right\}})$$

$$= (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-ta_\mu(x,\xi)} d\xi + O(t^{\min\left\{0, \frac{-|q|+\varepsilon}{\mu}\right\}}).$$

Taking Lemma 4.3 and (4.5) into account and adding together the contributions given by every chart by a partition of unity, we obtain (4.1). ■

THEOREM 4.6. (*Weyl's Formula*) *Let M be a compact Riemannian q-manifold and let A be an operator satisfying* (3.1). *Then the corresponding counting function $N(\lambda)$ can be estimate as follows:*

$$N(\lambda) = C\lambda^{\frac{|q|}{\mu}} + o(\lambda^{\frac{|q|}{\mu}}) \quad \text{as } \lambda \to +\infty, \tag{4.12}$$

where C is defined in (4.2).

PROOF. Taking Theorem 4.1 into account, apply the Karamata's Theorem ([18], pg. 89). ■

EXAMPLE 4.7. Consider the differential operator $P = \partial_{\theta_1}^4 - \partial_{\theta_2}^2$ on the flat foliated torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z} \oplus 2\pi\mathbb{Z}$, with leaves given by the circumferences of equation $\theta_2 = \text{const.}$; we have $P \in L^{4,(1,2)}(\mathbb{T}^2)$. Its quasi-homogeneous principal symbol is given by $p(\xi_1, \xi_2) = \xi_1^4 + \xi_2^2$, and therefore P is quasi-elliptic. Moreover P is self-adjoint; in fact it can be written as $P = Q^* Q = QQ^*$ where $Q = \partial_{\theta_2} - \partial_{\theta_1}^2 \in L^{2,(1,2)}(\mathbb{T}^2)$ is the heat operator in two variables. Now, one sees at once that $Pe^{i(n_1\theta_1 + n_2\theta_2)} = p(n_1, n_2)e^{i(n_1\theta_1 + n_2\theta_2)}$ for all $(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z}$, giving all the eigenvalues of P, hence a direct calculus of the integral in the Weyl's formula shows that

$$\#\{(n_1, n_2) \in \mathbb{Z} \oplus \mathbb{Z} \mid n_1^4 + n_2^2 \leq \lambda\} =$$

$$= N(\lambda) \sim \frac{1}{3}\sqrt{\frac{2}{\pi}}\,\Gamma\left(\frac{1}{4}\right)^2 \lambda^{\frac{3}{4}} \quad \text{as } \lambda \to +\infty.$$

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