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A Note on the Wick of the Casson Handles.

V. POÉNARU (*) - C. TANASI (**)

ABSTRACT - A smooth non compact 4-manifold V^4 such that $\mathrm{int}\, V^4 = R^4_{\text{standard}}$ and $\partial V^4 = S^1 \times \mathrm{int}\, D^2$ can be indentified by its «wick», which is a PROPER embedded $\Sigma = S^1 \times [0, \infty) \subset R^4_{\text{standard}}$. This paper provides an explicit description for the wick of the simplest Casson handles, in terms of a recursive non periodic infinite sequence of cobordisms. We also discuss how this could be plugged into an appropriate topological quantum field theory which should detect differentiably wild ends and, in particular, prove that the simplest Casson handle is smoothly wild. The conjectural T.Q.F.T. invariant, takes the following form for tame ends $h(R^4, \Sigma) = \{H_0 \xrightarrow{\text{id}} H_0 \xrightarrow{\text{id}} H_0 \xrightarrow{\text{id}} \ldots\}$ for a certain Hilbert space H_0.

1. Introduction.

Casson handles are a very special class of smooth non compact 4-manifolds V^4 with nonempty boundary, such that $\overset{\circ}{V}{}^4 = R^4_{\text{standard}}$ and $\partial V^4 = S^1 \times \overset{\circ}{D}{}^2$; these have been introduced by A. Casson about 20 years ago, as a substitute for the Whitney trick in dimension four. These manifolds play actually a key role in 4-dimensional topology. M. Freedman's main step in his proof of the 4-dimensional Poincaré Conjecture [5] was to show that any Casson handle is, *topologically speaking*, a handle of index two (without lateral surface), i.e. $V^4 \underset{\text{TOP}}{=} D^2 \times \overset{\circ}{D}{}^2$. But once the work

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of Donaldson (and others, see [4]) is also taken into account it becomes clear that among the Casson handles there have to be some which are smoothly wild, in the sense that they are not *smoothly* compactifiable to a copy of B^4_{standard}. This is a purely existential result, and for some time there was not a single Casson handle for which the issue of smooth wildness versus smooth tameness could be effectively settled. This situation has changed since [3]. But it is still not yet known, to the best of our knowledge, whether all Casson handles are smoothly wild and/or how many Casson handles there are, up to diffeomorphism. The present paper is a first step of a program for giving effective criteria for smooth wildness of Casson handles (and of other related objects). We start by showing that to each Casson handle one can associate canonically a «wick» which is a pair $(R^4_{\text{standard}}, \Sigma)$ where Σ is the image of a smooth proper embedding $S^1 \times [0, \infty) \hookrightarrow R^4$; the Casson handle can be completely reconstructed from its wick.

Most of this paper is devoted to the *explicit description* of the wick of the simplest Casson handle; this is the content of our theorem 1 from section 3 below. Casson handles, or rather the standard constructions leading to them, have a moduli space which is a Cantor set. The word «simplest» is well defined in terms of this Cantor set (see section 1 below). We have restricted ourselves for the time being, to this simplest case but, with some more work, the process can be applied to any other Casson handle; this will be object of a future paper. The point is that the wick (R^4, Σ) is a less esoteric object than V^4.The conjectural idea is that appropriate topological quantum field theories which would give criteria for the smooth tameness of (R^4, Σ) could be devised. The theories in question would have to verify self-consistence requirements which go one step beyond the ones described by M. Atiyah in [1].

We wish to thank Ron Stern for his encouragements and Louis Funar for helpful conversations.

2. An alternative view of Casson handles.

We will start with a general definition. We will call a *sort of link* a 4-dimensional smooth non compact 4-manifold V^4 with non-empty boundary, which is such that

$$(2.1) \qquad \overset{\circ}{V}{}^4 = R^4_{\text{standard}}, \qquad \partial V^4 = \bigcup_1^{\alpha} S^1_i \times \overset{\circ}{D}{}^2_i .$$

It is understood that the parameterization of the boundary correspond to the null-framing and, a priori at least, $1 \leqslant \alpha \leqslant \infty$.

Here is the simplest example.

Consider, to begin with, any usual link $\sum_1^\alpha S_i^1 \subset S^3 = \partial B^4_{\text{standard}}$, and the tubular neighbourhood $N \subset S^3$ of $\sum_1^\alpha S_i^1$ with $\alpha < \infty$. Clearly $B^4 - \left(\partial B^4 - \overset{\circ}{N}\right)$ is a sort of link, and a sort of link of this type will be called DIFFERENTIABLY TAME. In other terms a sort of link (with finitely many boundary components) is differentiably tame if it can be smoothly compactified to a copy of the standard 4-ball.There is also a weaker notion of topologically tame sort of links, which is obvious.

Here is a more exotic example.

Any Casson handle C.H. is a sort of link with $\alpha = 1$ (i.e. it is a SORT OF KNOT). Via Freedman's work we know that any C.H. is topologically tame (actually topologically standard) and via Donalson and Freedman we know that there must exist C.H.'s which are not differentially standard, since otherwise 4-dimensional differential topology would not present exotic features like the existence of non standard R^4's a.s.o. But if a C.H. is not DIFF standard then it is also DIFF wild, by which we mean that it cannot be smoothly compactified into a copy of B^4. The argument is very easy: if the C.H. comes from a usual DIFF link, then the link in question has to be trivial and hence the C.H. has to be DIFF standard.

Not only do sort of links appears in the context of the 4-dimensional Poincaré conjecture, but sort of links with ∞^{tely} many boundary components appear in the context of the first author's work on the 3-dimensional Poincaré conjecture [13]. The issue of DIFF tameness versus DIFF wildness is essential in this last context.

Here is a very general construction attached to sort of links. For simplicity's sake we will restrict ourself to sort of knots ($\alpha = 1$), from now on.

So, given V^4 let us consider some tubular neighbourhood of ∂V^4

$$\partial V^4 \times [0, 1] \overset{t}{\subset} V^4, \qquad \text{with} \qquad \partial V^4 \times 1 = \partial V^4.$$

If $S^1 = S^1 \times 1 \subset S^1 \times \overset{\circ}{D}{}^2 = \partial V^4$ is the central curve of ∂V^4, we will consider the pair

(2.2) $$(\operatorname{int} V^4, \, t(S^1 \times [0, 1))$$

which, after reparameterization, is made out of R^4_{standard} together with the image of a PROPER smooth embedding $S^1 \times [0, 1) \hookrightarrow R^4$. Our convention is that «PROPER» means inverse image of compact is compact and «proper» means $(X, \partial X) \subset (Y, \partial Y)$. Given such a pair, let us denote it by

$$(2.3) \qquad (R^4, \Sigma).$$

We will call $w(V^4) \underset{\text{def}}{=} (R^4, \Sigma)$, or just Σ, the wick of V^4.

PROPOSITION 1. 1) *Up to diffeomorphism the pair* (R^4, Σ) *is independent of the choice of* t.

2) *Taking the wick establishes a bijection between* $\{$*the set of sort of knots, up to diffeomorphism*$\} \underset{\approx}{\overset{w}{\to}} \{$*the set of pairs* (R^4, Σ) *up to diffeomorphism*$\}$.

PROOF. It is well known that two tubular neighbourhoods

$$\partial V^4 \times [0, 1] \xrightarrow{t_1, t_2} V^4$$

(with $t_\varepsilon | V^4 \times 1$ the identity map of ∂V^4 to itself) are related by a global isotopy which leaves ∂V^4 fixed. This establishes point 1). In order to prove 2) if suffices to shows that given a pair (R^4, Σ) there is a well-defined sort of knot V^4, of which (R^4, Σ) is the wick. Now, Σ is the image of a smooth PROPER embedding

$$\Sigma = S^1 \times [0, 1) \overset{\varphi}{\hookrightarrow} R^4$$

to which we can again apply the tubular neighbourhood theorem: this tells us that there is another smooth PROPER embedding (which is unique up to isotopy)

$$\Sigma \times D^2 \overset{\Phi}{\hookrightarrow} R^4$$

such that $\Phi | (\Sigma \times \text{center}\, D^2) = \varphi$. Let us consider the disjointed union

$$X^4 = R^4 + (S^1 \times [0, 1] \times \overset{\circ}{D}{}^2)$$

and the quotient space of X^4 which is characterized by identifying each

$(x, y, \varrho) \in S^1 \times [0, 1] \times \overset{\circ}{D}{}^2$ (i.e. $x \in S^1$, $t \in [0, 1]$, $\varrho \in \overset{\circ}{D}{}^2$) with $\Phi(x, y, \varrho)$. This *is* a sort of link V^4 having our original (R^4, Σ) as its wick. Notice also that our description

$$X^4 = R^4 \underbrace{\cup}_{S^1 \times [0, 1) \times \overset{\circ}{D}{}^2} \left(S^1 \times [0, 1] \times \overset{\circ}{D}{}^2\right)$$

gives as an effective smooth atlas for X^4.

We recall now briefly some fact about Casson handles, just so that our terminology can be fixed; for a much more detailed description, the reader can consult [8], [5], [14].

We start by considering a generally immersed 2-disk $D_1^2 \xrightarrow{j} R^4$, where R^4 is oriented. Our immersion has, let us say, p_1 self intersections with intersection number $+1$ and n_1 self intersection points with intersection number -1. Let N_1 be the tubular neighbourhood $N_1 = Nbd(jD_1^2)$ in R^4. The obvious pair, which is called a one floor tower

$$(N_1, \underbrace{j(\partial D_1^2) \times D^2}_{\text{we call this } \partial_- N_1} \subset \partial N_1),$$

is completely characterized by (p_1, n_1). On $\partial N_1 - \partial_- N_1$ there are $p_1 + n_1$ canonically framed circles $C_j (j = 1, \ldots, p_1 + n_1)$ corresponding to the self intersection (see the fig. 2.1).

One can get a tower with two floors by considering $p_1 + n_1$ one-floor

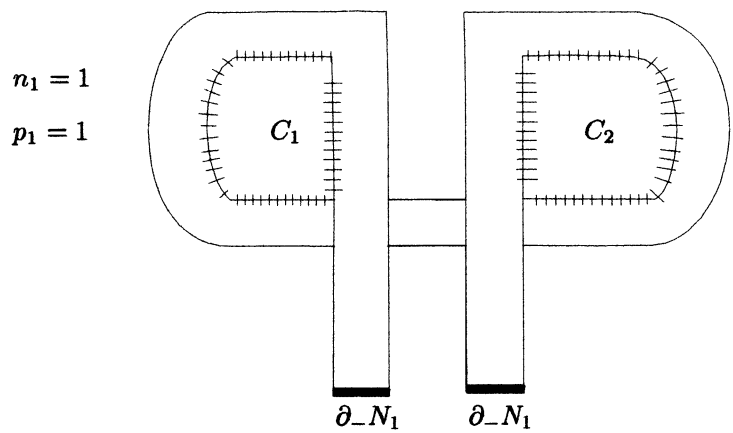


Fig. 2.1.

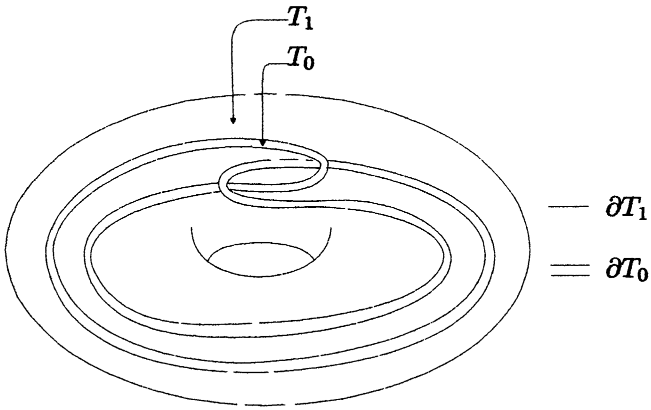


Fig. 2.2. $T_i \subset \operatorname{int} T_{i+1}$.

towers $N_{1,1}, \ldots, N_{1,p_1+n_1}$ each with their own (p_{1i}, n_{1i}), $(i = 1, \ldots, p_1 + + n_1)$ and glueing each N_{1i} to N_1 by identifying $\partial_- N_{1i}$ to the tubular neighbourhood of C_i, endowed with the canonical framing.

Towers with any finite number of floors are defined, inductively, in an obvious way. When the construction is iterated indefinitely and when any boundary piece except for $\operatorname{int} \partial_- N_1$ itself is deleted we get, by definition, a Casson handle. The whole construction is completely characterized by the sequence (p_1, n_1), (p_{1i}, n_{1i}) with $i = 1, 2, \ldots, p_1 + n_1$, $(p_{\alpha\beta\gamma}, n_{\alpha\beta\gamma})$, ... We will restrict ourselves from now to the «simplest Casson handle» V^4 $(p_1 = 1, n_1 = 0)$, $(p_{11} = 1, n_{11} = 0)$, $(p_{111} = 1, n_{111} = = 0)$, ... i.e. to the case where at each new floor we have exactly one self intersection point with intersection number $+1$. We concentrate, from now, on this particular V^4, but it is not hard to extend our methods to the other Casson handles. We will give now a description of V^4 which is slightly different from the one we just sketched. Start with the simplest Whitehead manifold Wh^3 defined by $Wh^3 = \bigcup_{n=1}^{\infty} T_i$ where $T_1 \subset \ldots \subset T_i \subset \subset T_{i+1} \subset \ldots$ is the sequence of nested tori from fig. 2.2. It is well known that $Wh^3 \times (0, 1)$ is the standard R^4 and one way to define our simplest Casson Handles is to start with the following description of $Wh^3 \times \times (0, 1)$

$$Wh^3 \times (0, 1) = \left(\mathring{T}_1 \times [-1, 1)\right) \cup \left(\bigcup_{n=2}^{\infty} \mathring{T}_n \times (-n, 1)\right),$$

and then change $\mathring{T}_1 \times [-1, 1)$ into $\mathring{T}_1 \times [-1, 1]$, thereby getting

$$(2.4) \quad V^4 = \left(\mathring{T}_1 \times [-1,1]\right) \cup \left(\bigcup_{n=2}^{\infty} \mathring{T}_n \times (-n,1)\right) \supset$$

$$\supset \bigcup_{n=1}^{\infty} \mathring{T}_n \times (-n, 1) = \mathring{V}^4 = R^4$$

where, obviously, $\partial V^4 = \mathring{T}_1 \times 1$. More precisely we have

PROPOSITION 2. 1) *There is a diffeomorphism*

$$\text{C.H.} = V^4 = \left(\mathring{T}_1 \times [-1, 1]\right) \cup \left(\bigcup_2^{\infty} \mathring{T}_n \times (-n, 1)\right).$$

2) *As a consequence of* 1), *the wick of the Casson handle* V^4 *is*

$$w(V^4) = \Big(\underbrace{\bigcup_2^{\infty} \mathring{T}_n \times [-n, 1)}_{R^4},\ S^1 \times [-1, 1)\Big);$$

here $S^1 \times [-1, 1)$ *is the image of* PROPER *embedding* $S^1 \times [0, \infty) \overset{\varphi}{\hookrightarrow} R^4$, *and* S^1 *is the middle curve of* T_1.

PROOF. It is convenient to give a slightly different view of Wh^3. For each $i = 1, 2, \dots$ we will consider an immersed disk $D_i \overset{g_i}{\to} T_{i+1}$ such that (with g_i denoted generically by g) $g(\partial D_i)$ is the middle curve $S^1 \times pt \subset \subset S^1 \times D^2 = T_i$ of the $i\,'th$ solid torus T_i of Wh^3, $g(D_i) \subset \operatorname{int} T_{i+1}$ and $g(D_i)$ is a spine of T_{i+1} (see fig. 2.3).

Figure (2.4) represents D_n at the source, the double points being dotted. On the figure in question we can see a canonical simple closed curve $C_n \subset g_n D_n$. It is understood that the set gD_n is very thin with respect to gD_{n+1} for each n; in more precise terms gD_n is concentrated inside a thin tubular neighbourhood of $g\partial D_{n+1}$ and the double points of $g \,|\, (D_n \cup \cup D_{n+1})$ are like in fig. 2.5 (see also fig. 2.3). So $D_1 \cup \dots \cup D_n \subset \mathring{T}_{n+1}$ is itself a spine and Wh^3 is the open regular neighbourhood of the rather wild space $D_1 \cup \dots \cup D_n \cup \dots$ (see also [16]).

All this having been said, let us return to our simplest Casson handle V^4 and consider its successive towers

$$(\overline{\operatorname{int} N_1}, \operatorname{int} \partial_- N_1) \subset (\overline{\operatorname{int} N_2}, \operatorname{int} \partial_- N_2) \subset \dots$$

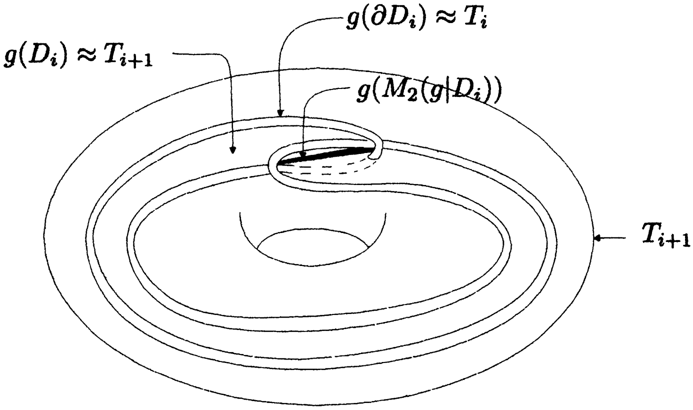


Fig. 2.3. $g(D_i) \swarrow T_{i+1}$, i.e. $g(D_i)$ is a spine of T_{i+1}. By $M_2(g)$ we denote the double points of g.

where $\overline{\operatorname{int} N_1} = \operatorname{int} N_1 \cup \operatorname{int} \partial_- N_1$ is the closure of $\operatorname{int} N_1$ in V^4, and where

$$(V^4, \partial V^4) = \bigcup_n (\overline{\operatorname{int} N_n}, \operatorname{int} \partial_- N_1).$$

Using our way of viewing Wh^3, via D_1, $D_2 \ldots$, we can easily check the following facts

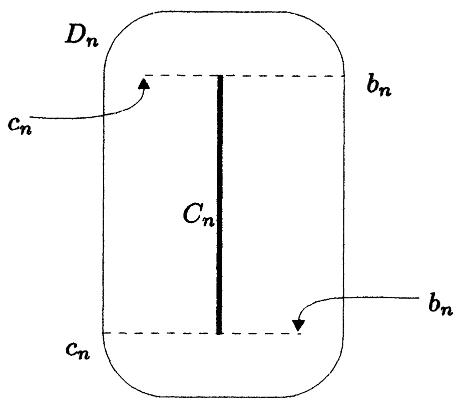


Fig. 2.4.

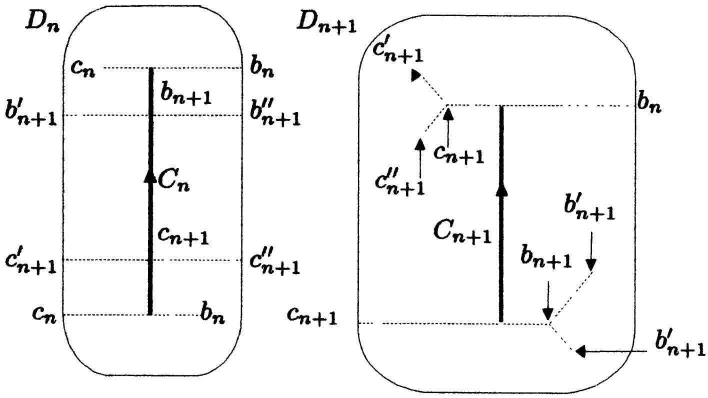


Fig. 2.5. $M_2(g|(D_n \cup D_{n+1})$.

i) we have

$$A_m \overset{\text{def}}{=} (\overline{\operatorname{int} N_m}, \operatorname{int} \partial_- N_1) =$$

$$= \left(\mathring{T}_1 \times [-1, 1] \cup \bigcup_{n=2}^{m} \mathring{T}_n \times (-n, 1), \mathring{T}_1 \times (+1) \right) \overset{\text{def}}{=} B_m .$$

ii) The following diagram involving the natural inclusions $A_m \subset \subset A_{m+1}$, $B_m \subset B_{m+1}$ commutes

$$\begin{array}{ccc} A_m & = & B_m \\ \downarrow & & \downarrow \\ A_{m+1} & = & B_{m+1} . \end{array}$$

3. An explicit description of the wick.

We will think of R^4 as being the infinite union

$$R^4 = B^4 \cup (S^3 \times [1, 2]) \cup (S^3 \times [2, 3]) \cup \ldots \cup (S^3 \times [n, n+1]) \cup \ldots$$

with $\partial B^4 = S^3 \times 1$, and we will consider the Hopf link $C + C'$ in S^3 i.e.

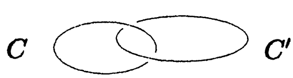


Fig. 3.1.

When we go from S^3 to $S^3 \times n$ this will be denoted by $C_n + C'_n$. If we replace $C + C'$ by a tubular neighbourhood $NC_n + NC'_n$ we can consider a *first satellite* of C_n denoted by $SC_n \subset NC_n$ and which is the one given by the Whitehead link from figure 3.2.

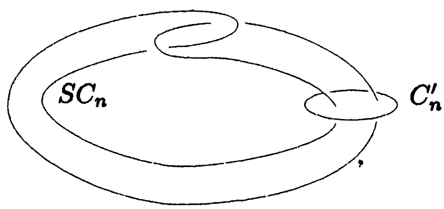


Fig. 3.2. $SC_n + C'_n$.

Similarly we define higher satellites like the link from figure 3.3

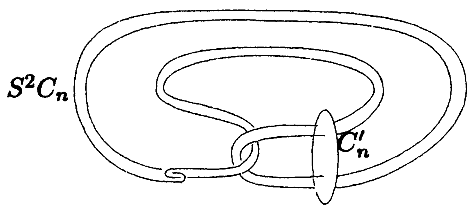


Fig. 3.3. $S(SC_n) + C'_n = S^2 + C'_n$.

and, more generally $S^p C_n + C'_n$ for any $p \in Z^+$.

We can also take several parallel copies of C'_n and introduce links like the one from figure 3.4 i.e. pass to things like $S^p C_n + qC'_n$ for arbitrary $p, q \in Z^+$; no other links will be considered for the time being. It should be stressed that the subscripts n do not refer to different knots but just to the sphere $S^3 \times n$ where the corresponding knot is being considered.

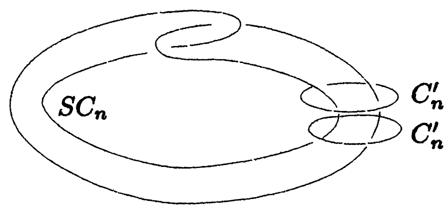


Fig. 3.4. $SC_n + 2C'_n$.

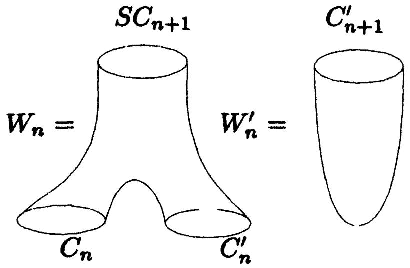


Fig. 3.5. $\sigma_n = W_n + W'_n$.

Later in this section we will describe explicitely a smooth pair of manifolds, namely the «basic cobordism»

(3.2) $$(S^3 \times [n, n+1], \sigma_n)$$

which, up to diffeomorphism, is supposed to be independent of n.

Here σ_n is a 2-dimensional proper submanifold of $S^3 \times [n, n+1]$, which consists of two disjointed pieces, $\sigma_n = W_n + W'_n$, with W_n a pair of pants and W'_n an unknotted disk, such that $\partial W_n = C_n + C'_n + SC_{n+1}$ and $\partial W'_n = C'_{n+1}$. In this last formula, $C_n + C'_n \subset S^3 \times n$ is the Hopf link (i.e. fig. 3.1, if we forget the subscript) while, $SC_{n+1} + C'_{n+1} \subset S^3 \times (n+1)$ is the link from figure 3.2, again modulo the subscript. At the source, σ_n is like in figure 3. 5. The situation at the target is represented very schematically in figure 3. 6. Here $N(n+1) \subset S^3 \times (n+1)$ is a thin tubular neighbourhood of C_{n+1}, containing the satellite SC_{n+1} and taking the form $N(n+1) = C_{n+1} \times \beta_{n+1}$ where

(3.3) $$\beta_1 \subset \beta_2 \subset \ldots \subset \beta_n \subset \beta_{n+1} \subset \ldots \subset R^2$$

is a sequence of concentric disks of small bounded radius, which is given once and for all.

Next we introduce the *connected* surface

(3.4) $$\Sigma_n = W_n \bigcup_{C'_n} W'_{n-1} \subset S^3 \times [n-1, n+1],$$

which is a cylinder with $\partial \Sigma_n = C_n + SC_{n+1}$. For each Σ_n we consider the tubular neighbourhood

(3.5) $$\Sigma_n \times \beta_n \subset S^3 \times [n-1, n+1],$$

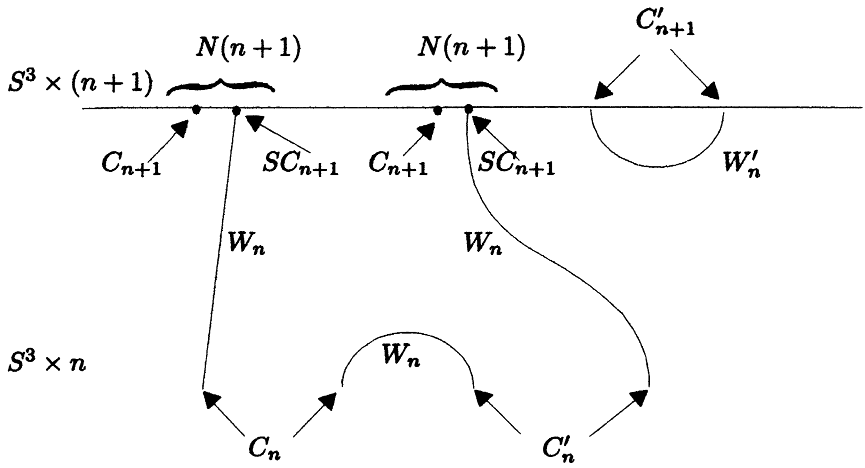


Fig. 3.6. $(S^3 \times [n, n+1], \sigma_n = W_n + W'_n)$.

At the target the top part is $(SC_{n+1} \times \beta_n, SC_{n+1}) \subset S^3 \times (n+1)$, while the botton part is $(C_n \times \beta_n, S^{n-1} C_n) \subset S^3 \times n$, where it is understood that the β_n's are such that

$$\text{(3.5bis)} \qquad SC_{n+1} \subset \Sigma_n \times \beta_n | S^{n+1} \subset \text{int} \underbrace{(\Sigma_{n+1} \times \beta_{n+1}) | S^{n+1}}_{\text{this is } N(n+1)}.$$

Figure 3.7, which is to be compared with figure 3.6, represents very schematically the *«periodic» infinite sequence of fat cobordisms*

$$\text{(3.6)} \qquad (\Sigma_1 \times \beta_1) \cup (\Sigma_2 \times \beta_2) \cup \ldots \cup (\Sigma_n \times \beta_n) \cup \ldots$$

with the glueing pattern being dictated by (3.5bis). It will be convenient now, notationally speaking, to think of $\Sigma_n \times \beta_n$ (see (3.5)) as being the image of an embedding

$$\text{(3.7)} \qquad (C_n \times [0,1] \times \beta_n; C_n \times 0 \times \beta_n, C_n \times 1 \times \beta_n) \xrightarrow{f_n}$$

$$\xrightarrow{f_n} (S^3 \times [n-1, n+1]; S^3 \times (n-1), S^3 \times (n+1)),$$

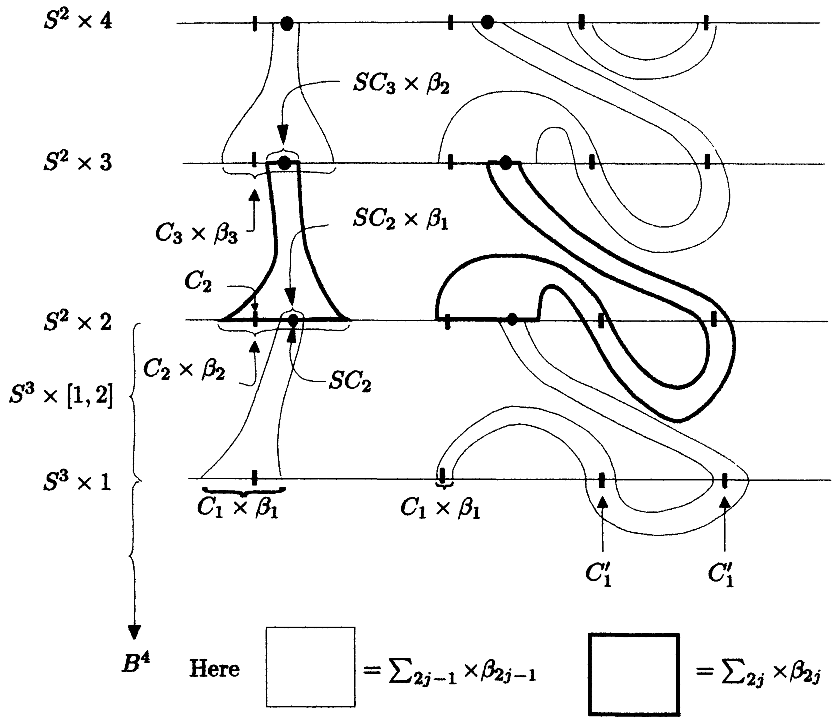


Fig. 3.7. $\bigcup_1^\infty \Sigma_n \times \beta_n \subset B^4 \cup \bigcup_1^\infty S^3 \times [i, i+1] = R^4$.

where $f_n(C_n \times [0, 1] \times \star_n) = \Sigma_n$, $f_n(C_n \times 0 \times \beta_n) = C_n \times \beta_n$ and hence also $f_n(C_n \times 0 \times \star_n) = C_n$ while $f_n(C_n \times 1 \times \star_n) = SC_{n+1}$. If we start with the canonical embedding $S^{n-1} C_n \subset C_n \times \beta_n$ and go from there to

$$S^{n-1} C_n \times [0, 1] \subset \underbrace{C_n \times [0, 1] \times \beta_n}_{\text{source of } f_n},$$

then, since $f_n(C_n \times 1) = SC_{n+1}$, we can isotope the fs, without contradicting any of the things already said, so that

$$f_n(S^{n-1} C_n \times 1) = S^n C_{n+1} = f_{n+1}(S^n C_{n+1} \times 0). \tag{3.8}$$

This is suggested by figure 3.8. So, in the spirit of figure 3.7 we can glue $f_n(S^{n-1} C_n \times [0, 1))$ to $f_{n+1}((S^n C_{n+1} \times [0, 1))$ and define the *ape-*

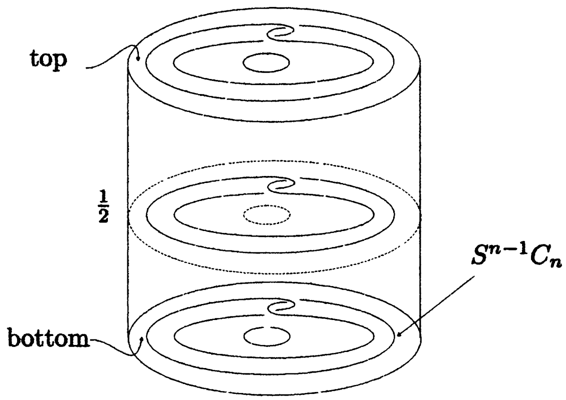


Fig. 3.8. $C_n \times [0, 1] \times \beta_n$, source of map f.

riodic surface

(3.9) $\Sigma = f_1(C_1 \times [0, 1]) \cup f_2(SC_2 \times [0, 1]) \cup \ldots \cup f_n(S^{n-1}C_n \times [0, 1]) \ldots \subset$

$$\subset (\Sigma_1 \times \beta_1) \cup (\Sigma_2 \times \beta_2) \cup (\Sigma_3 \times \beta_3) \cup \ldots \subset B^4 \bigcup_i^{\infty} S^3 \times [n, n+1] = R^4$$

which is the image of a PROPER embedding $S^1 \times [0, 1) \subset R^4$, with $\partial \Sigma = C_1$. Unlike (3.6) which is essentially periodic, (3.9) is not, but its geometry is dictated by the very precise algorithm which one has to iterate in going from n to $n+1$, and which we will explain below.

Notice that

(3.10) $$\Sigma \cap S_n^3 = S^{n-1}C_n + 2^{n-1}C_n'.$$

For $n = 2, 3, 4$ this collection of links is displayed in figure 3.9. It is worthwhile mentioning that although these links look superficially very simple, their Jones polynomials are incredibly complicated. This stems from the fact that the operation of taking satellites is highly non trivial, from the standpoint of the polynomials in question. The formulae involving the polynomials for satellites involve, a priori, an infinite sequence of quantum group invariants (see also [11]). H. Morton and S. Strickland have used their machinery [12], in order to calculate for us the polynomials for the first three of our links. They have shown, that for $SC + 2C'$

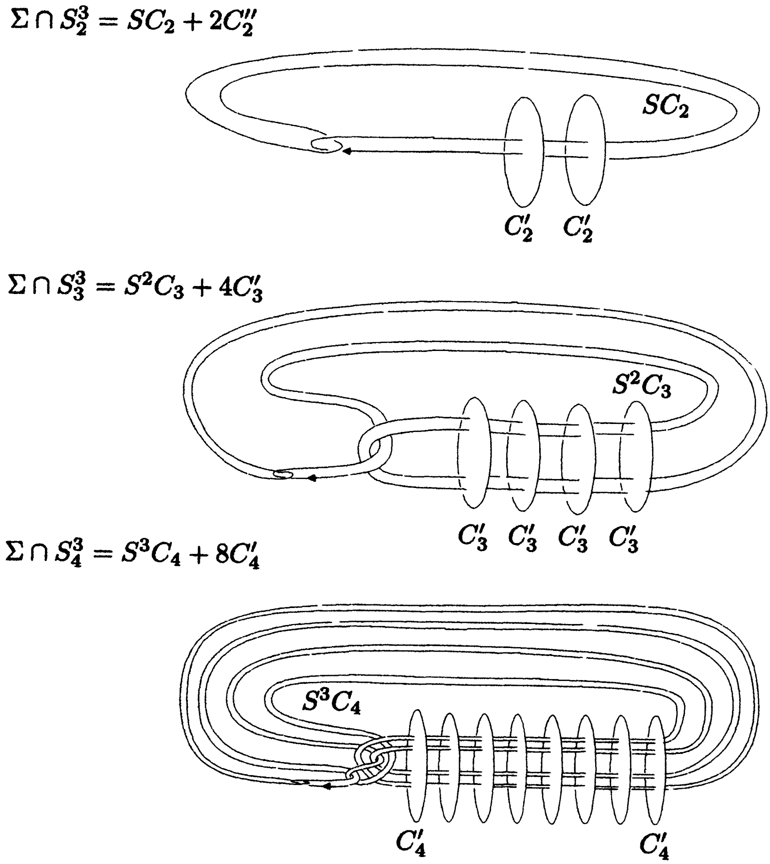


Fig. 3.9.

the polynomial is

$$-t^5 + t^4 + t^2 + 1 + t^{-1} - t^{-2} + t^{-3}$$

while for $S^2C + 4C'$ it is

$$-t^{14} + t^{13} + 2t^{11} - t^{10} - 3t^9 + 6t^8 - t^7 - 5t^5 - 2t^4$$
$$+5t^3 - t^2 + 6t + 1 - 2t^{-1} - t^{-2} + 3t^{-3} + 2t^{-4} + 3t^{-5}$$
$$+3t^{-6} - 3t^{-9} + 4t^{-10} + t^{-11} - t^{-12} - t^{-14} + t^{-15}.$$

For $S^3 C + 8C'$ the polynomial is

$$-t^{44}+t^{43}+t^{42}+t^{41}-t^{40}-t^{39}-t^{38}-2t^{37}-6t^{35}+11t^{34}$$
$$+7t^{33}+15t^{32}-10t^{31}-22t^{30}-21t^{29}-20t^{28}+30t^{27}+8t^{26}$$
$$+83t^{25}+12t^{24}-36t^{23}-98t^{22}-82t^{21}+95t^{20}+62t^{19}+101t^{18}$$
$$-48t^{17}-108t^{16}-88t^{15}+7t^{14}+70t^{13}+58t^{12}-4t^{11}-69t^{10}$$
$$+14t^{9}+20t^{8}+184t^{7}+45t^{6}+15t^{5}+30t^{4}-55t^{3}+60t^{2}-10t$$
$$-68-85t^{-1}-8t^{-2}+7t^{-3}+55t^{-4}-33t^{-5}+58t^{-6}+26t^{-7}$$
$$+29t^{-8}+110t^{-9}-40t^{-10}+t^{-11}-20t^{-12}-11t^{-13}+59t^{-14}$$
$$+19t^{-15}-15t^{-16}-73t^{-17}-53t^{-16}+57t^{-19}+81t^{-20}+20t^{-21}$$
$$-75t^{-22}-40t^{-23}-6t^{-24}+88t^{-25}+61t^{-26}+6t^{-27}-32t^{-28}$$
$$-117t^{-29}-21t^{-30}+6t^{-31}+89t^{-32}+49t^{-33}+t^{-34}-20t^{-35}$$
$$-50t^{-36}-t^{-37}-16t^{-38}+15t^{-39}+8t^{-40}+17t^{-41}+8t^{-42}$$
$$-8t^{-43}-7t^{-44}-9t^{-45}+6t^{-46}-2t^{-47}+t^{-48}+t^{-49}+2t^{-50}+t^{-51}$$
$$-t^{-52}-t^{-53}-t^{-54}+t^{55}.$$

Here is type main technical result of the present note.

THEOREM 1. *The pair (R^4, Σ) is the wick of the Whitehead manifold V^4.*

PROOF. We will give the description of the «basic cobordism» $(S^3 \times \times [n, n+1], \sigma_n)$; once this has been understood the rest of the proof is a process of direct geometrical checking, which we leave to the reader.

Consider first the immersion $D^2 \xrightarrow{j} R^3$ given by figure 3.10. The arc $[a, b]$ from the figure is the unique double line of j (at the target). The clasp $[a, b]$ is supposed to be the common image of two disjointed arcs

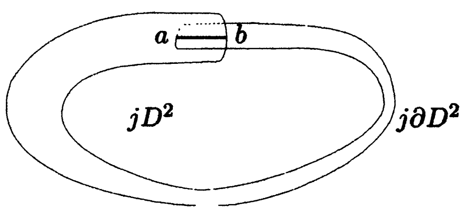


Fig. 3.10.

$[\alpha', \beta''] + [\alpha'', \beta'] \subset D^2$ where $\alpha', \beta' \in \partial D^2$. We can change j into an embedding $D^2 \xrightarrow{I} R^3 \times R = R^4$, without budging $j | \partial D^2$, by lifting $(\alpha', \beta'']$ in the positive t-direction and at the same time lowering $(\alpha'', \beta']$. More precisely we can get a commutative diagram

$$\begin{array}{ccc} & & R^3 \times R \\ & \overset{I}{\nearrow} & \downarrow \\ D^2 & \xrightarrow[j]{} & R^3 \end{array} \tag{3.11}$$

where I is an embedding, and which is such that $I(p) = (jp, 0)$ for every $p \in \partial D^2$. In the plane which is spanned by the $[a, b]$ direction and the t-direction, we have the situation from figure 3.11.

We consider now a very thin tubular neighbourhood N of jD^2 and the «vertical» cylinder

$$Y = \{(jp, t), \text{ with } p \in \partial D^2, t \in R_+\} \subset R^4.$$

These objects are suggested in figure 3.12.

It is not hard to see that $\partial N \cap Y$ is the Hopf link $C_n + C_n'$, with C_n coming essentially from ∂D^2 and with $C_n' = \{$*the boundary of the fiber of* $N \to ID^2$ at $I\beta''\}$; this is also suggest in figure 3.12.

In figure 3.13 we consider two nested embeddings

$$jD^2 \subset T_{n+1} \subset T_{n+2}.$$

We can also assimilate T_{n+1} with an embedded curve and extend it to an immersion $D^2 \xrightarrow{j_1} T_{n+2}$ analogous to the extension $D^2 \xrightarrow{j} T_{n+1}$ of $\partial D^2 \xrightarrow{j|\partial D^2} T_{n+1}$. This can be lifted to an embedding $D^2 \xrightarrow{I_1} T_{n+2} \times R$ on the same lines as (3.11) and we will consider a thin tubular neighbourhood

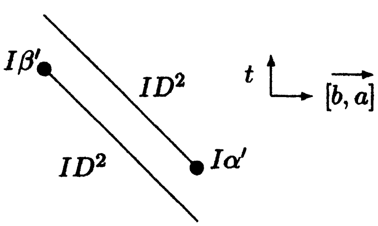


Fig. 3.11.

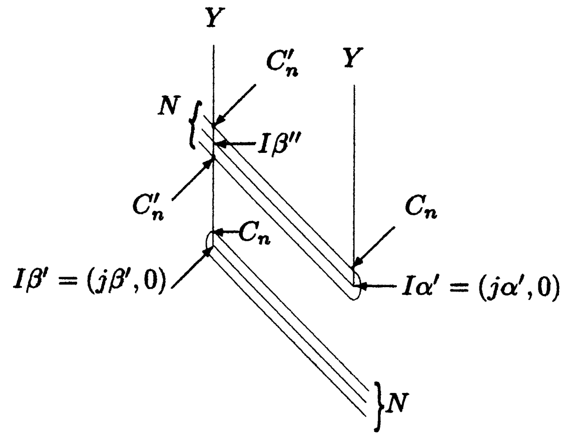


Fig. 3.12.

N_1 of $I_1 D^2$, containing N in its interior. With this we have

$$Y \cap \partial N_1 = SC_{n+1} + 2C'_{n+1}$$

and $\Big(Y \cap \underbrace{\overline{N_1 - N}}_{\text{this is our } S^3 \times [n,\, n+1]}\Big)$ consists of a pair of pants W_n like in figure 3.5 and of two unknotted disks bounding the two copies of C'_{n+1}. By keeping only one of them and denoting it by W'_n we have obtained our desired «basic cobordism»

$$(S^3 \times [n,\, n+1],\, \sigma_n = W_n + W'_n)\,.$$

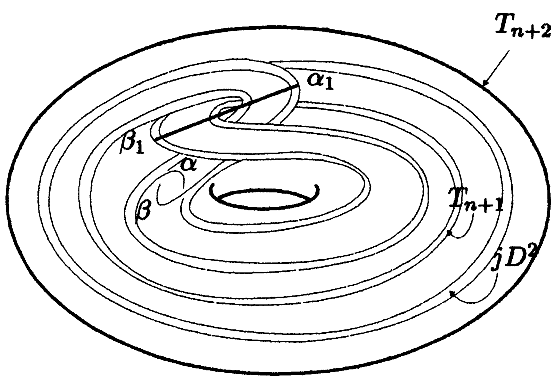


Fig. 3.13.

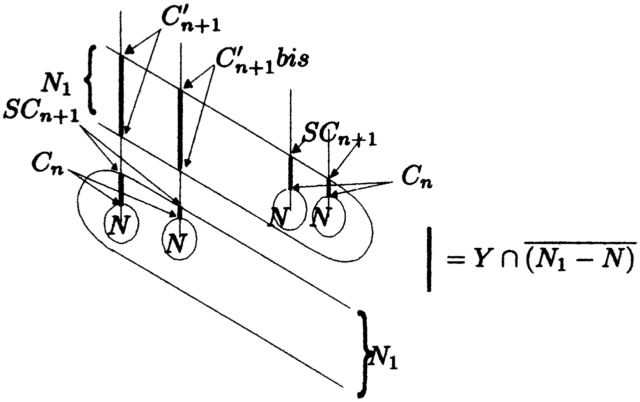


Fig. 3.14.

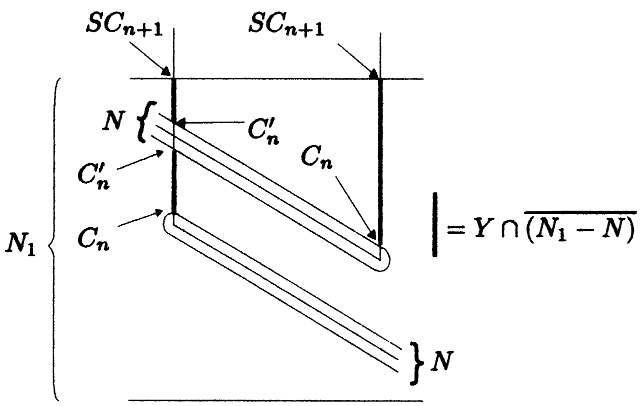


Fig. 3.15.

The relative situations of N, N_1, Y in a plane generated by $[\alpha_1, \beta_1]$ (see figure 3.13) and t, respectively in a plane generated by $[\alpha, \beta]$ and t are shown in the figures 3.14, 3.15.

4. Some conjectural ideas on topological quantum field theories (TQFT).

This highly speculative section owes a lot to the ideas of L. Funar [6], [7]. We have the hope that the work of P. Kronheimer and T. Mrowka [9],

[10], which extends the work of Donaldson and of Seiberg-Witten to the embedded surfaces in 4-manifolds and/or improved versions of Floer's theory (R. Stern) can be helpful in this context, after they have been appropriately extended. In [1] Atiyah gives axioms for topological quantum fields theories, examples of which are the Witten-V.Jones theory in dimension 3 and Donaldson's theory [4] in dimension 4. See also [15]. In these theories (TQFT) one starts with closed n-manifold Σ^n and with compact cobordisms W^{n+1}, these objects can have additional structures: orientations, submanifolds.... The theory associates with each Σ^n a «Hilbert space» $H(\Sigma^n)$ and to each cobordism $(W^{n+1}; \Sigma^n_{\text{in}}, \Sigma^n_{\text{out}})$ an operator $H(\Sigma^n_{\text{in}}) \xrightarrow{h(w)} H(\Sigma^n_{\text{out}})$. These objects have to fulfill a number of axioms. We simply recall here the condition that H of a disjointed union is the tensor product of the corresponding h's, which makes the theory really quantum, and also the fact that $h(\dots)$ satisfies an «obvious» associativity property. This is actually a deep condition which from the physical standpoint is related to relativistic invariance and/or to selfconsistency: by cutting arbitrarily a closed $(n+1)$-manifold M^{n+1} into codimension zero pieces the theory provides us with a well defined scalar $h(M^{n+1}) \in C$ the vacuum-vacuum expectation value of M^{n+1} (viewed as a quantum transition from one vacuum state to another one, by tunneling). Incidentally, achieving a meaningful physical Q.F.T. theory with such selfconsistency properties is a major project (for the bootstrap program this is the Graal...)

Here are some thoughts about a very conjectural TQFT which might possibly occur in our context. To begin with, this should be a TQFT on the lines of Atiyah's paper where the basic objects would be 4-dimensional cobordism W^4 endowed with properly embedded surfaces $S \subset W^4$. This means, in particular, that for every 3-dimensional link (M^3, L) we would have a Hilbert space $H(H^3, L)$ and for every (W^4, S) a linear map $H(\partial^0 W^4, S^0) \to H(\partial^1 W^4, S^1)$ where $S^\varepsilon = S \cap \partial^\varepsilon W^4$ with $\varepsilon = 0, 1$. So far this could very well be the Kronheimer-Mrowka theory and improved version of Floer's theory might come in handy too. But we would want some extra features for our conjectural theory, namely making sense of ends of open 4-manifolds endowed with a properly embedded surface. This would require some new axioms, in addition to the usual ones. Let us consider, to begin with an infinite sequence of cobordisms

$$(W^4(1), S(1)), \dots$$

with

$$\underbrace{(\partial^1 W^4(n),\, S^1(n))}_{\text{out}} = \underbrace{(\partial^0 W^4(n+1),\, S^0(n+1))}_{\text{in}}.$$

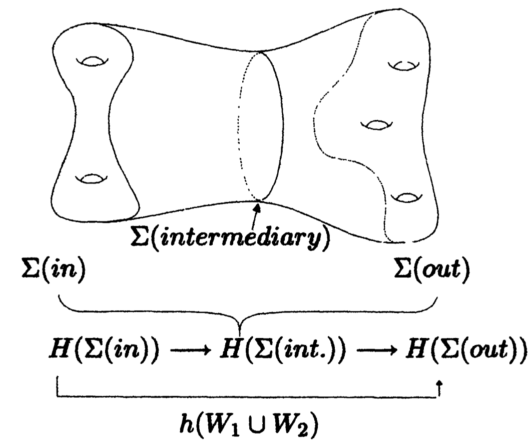
Our hypothetical TQFT would attach obviously to this an infinite sequence of linear maps

$$(A) \qquad H_0 \to H_1 \to H_2 \to \dots$$

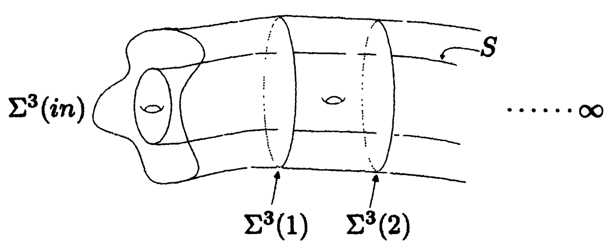
where H_n is the Hilbert space

$$H(\partial^0 W^4(n),\, S^0(n)) = H(\partial^1 W^4(n-1,\, S^1(n-1))).$$

So we shift now from the standard TQFT picture



To the more exotic picture.



where the possible physical meaning of ∞ is anybody's guess. If we want to make sense, in this context, of the end of $(W, S) = \left(\bigcup_n W^4(n),\, \bigcup_n S(n)\right)$

we need an axiom which with respect to the usual associativity property of TQFT would be like complete additivity versus additivity in measure theory. Here is a naive guess. Imagine we have some appropriate quotient space E of all the infinite sequences of linear map like (A), with the requirement that the class of our (A) in E, let us call it $H(W, S) \in E$, should depend only on the end of (W, S). Clearly in E it should not make any difference if we delete any number of H_i's from (A) provided, of course, that an infinite number is still left, and if we redefine afterwards the maps in the obvious way. But if $H(W, S)$ is supposed to be a nontrivial invariant, then our new requirement is clearly quite strong.

If we go back to our «sort of knots» and their wicks (R^4, Σ), then in the context of the naive guess, smooth tameness should be characterized by the property that

$$H(R^4, \Sigma) = \{H_0 \xrightarrow{\text{id}} H_0 \xrightarrow{\text{id}} H_0 \xrightarrow{\text{id}} \dots\}.$$

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