RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

LEONARDO BILIOTTI

On the automorphism group of a second order structure

Rendiconti del Seminario Matematico della Università di Padova, tome 104 (2000), p. 63-70

<http://www.numdam.org/item?id=RSMUP_2000__104__63_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 104 (2000)

On the Automorphism Group of a Second Order Structure.

LEONARDO BILIOTTI (*)

Introduction.

This paper will deal with some property of the automorphism group of special second order structures which are associated to semisimple flat homogeneous spaces. We will consider a second order structure Q, i.e. a subbundle of the 2-frame bundle $P^2(M)$ over a smooth manifold M, which is associated to a semisimple flat homogeneous space L/L_0 with dim M=dim L/L_0. Such structures have been intensitevely studied by Ochiai ([3]) and generalize some well-known structures, such as projective and conformal ones. We will confine ourself to the case when M is a reductive homogeneous space and Q is a second order structure, that it has a Cartan connection which is preserved by any automorphism; such class of semisimple flat homogeneous space L/L_0, has been classificated by Ochiai and the classification will be given in Table 1.

In section 1 we will briefly recall basic facts about semisimple flat homogeneous space according to the paper by Ochiai [3] to which we shall refer throughout the following.

In section 2 we will prove our main theorem, which can be formulated as follows:

THEOREM 1. *Let $M = G/K$ be a reductive homogeneous space and let Q be a second order structure over M associated to a semisimple flat homogeneous space L/L_0 where (L, L_0) belongs to Table 1. If G acts as an automorphism group of Q, then there exists a G-invariant torsion-free affine connection Γ belonging to Q.*

(*) Indirizzo dell'A.: IMECC-UNICAMP, 13083-970, Campinas (SP), Brasil, e-mail: biliotti@ime.unicamp.bz

TABLE 1.

	$K=R$	$K=C$	$K=Q$
l	$\mathfrak{sl}(p+q;R)$	$\mathfrak{sl}(p+q;C)$	$\mathfrak{su}^*(2p+2q)$
g_0	$\mathfrak{sl}(p;R)+\mathfrak{sl}(q;R)+R$	$\mathfrak{sl}(p;C)+\mathfrak{sl}(q;C)+C$	$\mathfrak{su}^*(2p)+\mathfrak{su}^*(2q)+R$
l		$\mathfrak{su}(n,n;C)$	
g_0		$\mathfrak{sl}(p,C)+R$	
l	$\mathfrak{so}(n,n;R)$	$\mathfrak{so}(2n;C)$	$\mathfrak{so}^*(4n)$
g_0	$\mathfrak{gl}(n;R)$	$\mathfrak{gl}(n;C)$	$\mathfrak{u}^*(2n)$
l	$\mathfrak{so}(p+1,q+1;R)$	$\mathfrak{so}(p+q+2,C)$	
g_0	$\mathfrak{so}(p+q;R)+R$	$\mathfrak{so}(p+q;C)+C$	
l	$\mathfrak{sp}(n;R)$	$\mathfrak{sp}(n;C)$	$\mathfrak{sp}(n,n)$
g_0	$\mathfrak{gl}(n;R)$	$\mathfrak{gl}(n;C)$	$\mathfrak{su}(2n)+R$

Our result generalizes the one in [4], where only the projective structures are taken into consideration. As far as we know, it is still unknown, which of the special semisimple flat homogeneous space considered by Ochiai have some meaningful geometric interpretation, besides the projective and conformal ones. In any case, our result greatly simplifies the study of transitive group G of automorphism, since it realize G as a group of affine transformation of some torsionfree connection.

1. Preliminaries.

Let L/L_0 be a connected homogeneous space on which a semisimple Lie group L acts effectively and transitively. The homogeneous space L/L_0 is called *semisimple flat homogeneous space* if the Lie algebra l of L has a graded structure $l=g_{-1}\oplus g_0\oplus g_1$, $[g_i, g_j]\subset g_{i+j}$, such that $g_0\oplus\oplus g_1$ is the Lie algebra of L_0; we may therefore write $L_0=G_0\cdot G_1$, where G_0 is a closed subgroup with Lie algebra g_0 and G_1 is a connected Lie group corresponding to g_1; more precisely G_0 is a normalizer of g_i, i.e. $G_0=\{a\in\in L_0\colon Ad(a)(g_i)=g_i\}$, $i=-1, 0, 1$ (see [3]).

Let M be a manifold of dimension $n=\dim L/L_0$. We denote by $P^r(M)$ the r-frame bundle over M with structure group $G^r(n)$; $P^1(M)$ is usually called the frame bundle and $G^1(n)$ is isomorpic to $GL(n, R)$. It turns out that L_0 is realizable as a subgroup of $G^2(n)$ and $G_0=L_0\cap G^1(n)$.

Now, let P be a G_0-structure over M, i.e a G_0-reduction of $P^1(M)$, and let Γ be a G_0-connection without torsion, i.e. a torsionfree connection Γ

on $P^1(M)$ such that Γ has values into the Lie algebra of G_0, when restricted to P. By a result due to Kobayashi (see [2]), there exists a one-to-one correspondence between affine connections without torsion and admissible cross-section, i.e. an application $s: P^1(M) \to P^2(M)$, satisfying $s(ua) = s(u)\, a$, $\forall a \in G^1(n)$. Hence each G_0-connection without torsion on P gives a G_0-subbundle $s(P)$ of $P^2(M)$. Since $G_0 \subset L_0 \subset G^2(n)$, we have a L_0-subbundle $Q(\Gamma)$ of $P^2(M)$ obtained by extending the structure group of $s(P)$ from G_0 to L_0.

The above consideration allows to define an equivalence relation, called L_0-equivalence, in the set $A(P)$, the set of G_0-connection without torsion of P, as follows: if Γ, Θ are element of $A(P)$, then

$$\Gamma \text{ is equivalent to } \Theta \text{ if and only if } Q(\Gamma) = Q(\Theta).$$

The L_0-subbundle $Q(\alpha)$ of $P^2(M)$, where α is any class of $A(P)$, will be called a L_0-structure of second order. The L_0-equivalence is a generalization of some well-know structures such as projective and conformal; we will briefly recall some facts about such structures:

Projective Geometry.

In this case L/L_0 is a projective space, G_0 is $GL(n, \boldsymbol{R})$ and the L_0-equivalence is the projective equivalence: two torsionfree connection ω, $\omega' \in A^1(P)$ are equivalent if and only if there exists $p: P^1(M) \to \boldsymbol{R}^n$ such that:

$$\omega - \omega' = \theta p + (p\theta)\, id\,,$$

where θ is a canonical form of $P^1(M)$. Geometricaly, the condition above, means that ω and ω' have the same geodesic up to reparametrization. The projective geometry studies the invariant properties of a projective equivalence class (see [2] for detailed exposition).

Conformal Geometry.

In this case L/L_0 is the n-dimensional Möbius space, G_0 is $CO(n)$ and the L_0-equivalence relation is the conformal equivalence: ω is equivalent to ω' if and only if there exists $p: Q \to \boldsymbol{R}^n$, where Q is a $CO(n)$-structure, such that

$$\omega - \omega' = p\theta - p^t \theta^t - (\theta p)\, id\,.$$

It can be proved that, given a $CO(n)$-structure on M, then its first prolongation is a conformal structure as a subbundle of $P^2(M)$ and viceversa. On the other hand, $CO(n)$-structure on M are in a natural one to one correspondence with conformal equivalence class of Riemannian metrics on M. We refer to [2] for a detailed exposition.

We now fix a L_0-structure of second order Q, relative to a G_0-structure P and a G_0 torsionfree connection Γ and we consider the group $\mathrm{Aut}(M, Q)$ of the automorphism of such structure; a diffemorphism ϕ belongs to $\mathrm{Aut}(M, Q)$ if and only if its natural lift $\phi^{(2)}$ to $P^2(M)$ preserves Q. It is clear that any ϕ in $\mathrm{Aut}(M, Q)$ acts as a permutation on $A(P)$. For some special class of semisimple flat homogeneous space, Ochiai proved the existence and uniqueness of a special *normal* Cartan connection ω_Q on Q, which is automatically preserved by any automorphism in $\mathrm{Aut}(M, Q)$. This theorem is a generalization of existence and uniqueness of *normal* Cartan connection of projective and conformal structures; he showed that the existence of a *normal* Cartan connection is related to the vanishing of certain Spencer cohomology group; we will give all such Lie algebras l, in Table 1.

We recall that a Cartan connection in a bundle Q is a 1-form ω on Q with values in the Lie algebra l of L such that:

1. $\omega(A^*) = A$, for all $A \in g_0 \oplus g_1$;
2. $R_a^* \omega = Ad(a^{-1})\, \omega$, for all $a \in L_0$;
3. $\omega(X) = 0$ if and only if $X = 0$.

Here we denote by A^* the fundamental vector field corresponding on element A of the Lie algebra $g_0 \oplus g_1$. By definition of Cartan connection it follows in particular that ω gives an absolute parallelism on Q; by a classical Theorem of Kobayashi the mapping

$$\Psi_u \colon G \to Q$$

$$g \to g^{(2)}(u),$$

where u is any fixed element of Q, gives a closed embedding of G into Q.

The aim of this note is to give a simple criterium on the Lie group structure of $G = \mathrm{Aut}(M, Q)$ in order that there exists an affine connection Γ in $A(P)$ left fixed by G or, is equivalent, a fixed point by the action of G on $A(P)$. The existence of such fixed connection greatly simplifies the study of the G-action on M.

We consider the case when $M = G/K$, where G is a Lie group and K is a Lie closed subgroup of G. We will assume that $G \subset \mathrm{Aut}(M, Q)$, where Q

is a second order structure on $M = G/K$ and $M = G/K$ is a reductive, that is, if $\boldsymbol{g}$ and $\boldsymbol{k}$ are the Lie algebras of G and K respectively, there is $Ad(K)$-invariant subspace $\boldsymbol{m}$ of $\boldsymbol{g}$ so that

$$\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{m} .$$

In the next section we will give a proof of the main theorem.

2. Proof of the Theorem 1.

Let ξ be a projection of G over $G/K = M$, π the projection of Q over $G/K = M$, ω a *normal* Cartan connection on Q and, since ω is l-valued form, we put $\omega = \omega_{-1} \oplus \omega_0 \oplus \omega_1$. For every $u \in Q$, we get an embedding of G into Q given by

$$\Psi_u : G \to Q ,$$

$$g \to g^{(2)}(u) ,$$

where $g^{(2)}$ is a lift of the trasformation g of $G/K = M$ to an automorphism of the bundle Q. We use that embedding to define, for every $u \in Q$, a subspace H_u of $T_u Q$ as follows:

$$H_u = (\Psi_u)_* [e](Ad(g)(\boldsymbol{m})) ,$$

where g is any element of G with $\xi(g) = \pi(u)$; the definition is well-posed because the subspace $Ad(g)(\boldsymbol{m})$ does not depend on the choice of element g thanks to the reductivity. We can easily prove, see [4], that H_u is isomorphic, by the differential of π, to $T_{\pi(u)} M$ and the distribution H is G-invariant and L_0-invariant. Our aim is to show that

$$\Upsilon_0 = \{u \in Q : \mathrm{Tr}(\omega_0 |_u(X)) = 0, \ \forall X \in H_u\} ,$$

is a principal G_0-subbundle isomorphic to P; if we prove this, then the restriction of Θ_0 to Υ_0, where Θ_0 is the $\boldsymbol{gl}(n; \boldsymbol{R})$-component of the canonical form Θ of $P^2(M)$, gives a torsionfree G_0-connection, which is G-invariant.

We fix a point $u \in Q$ then there exist unique vector $X_1, \dots, X_n$ of H_u that is $\omega_{-1}(X_i) = e_i$, where $e_1, \dots, e_n$ is a base of g_{-1}; that is because $\pi_* |H_u$ is an isomorphism onto $T_\pi(u) M$. We can identify the Lie algebra g_0 as a Lie subalgebra of $\boldsymbol{gl}(g_{-1})$, see [3], as follows:

$$X \to \boldsymbol{ad}(X) |_{g_{-1}} .$$

Now we consider the bilinear application $B:(X, Y) \to \mathrm{Tr}([X, Y])$; this is a duality between g_{-1} and g_1 because $-2B(X, Y) = \boldsymbol{B}(X, Y)$, where $\boldsymbol{B}$ is the Killing-Cartan form of l (see [3]). We can costruct, thanks to the duality, an element $y \in g_1$ in the following way:

$$\phi(a^i e_i) = a^i \mathrm{Tr}(\omega_0 |_u(X_i)) = B(y, a^i e_i),$$

and we claim that if we put $w = \exp(y) \in L_0$, then

$$\mathrm{Tr}(\omega_0 |_{uw}(X)) = 0, \quad \forall X \in H_{uw}.$$

Indeed we have $H_{uw} = (R_w)_* H_u$ and

$$\omega_0 |_{uw}((R_w)_* X)) = \omega_0 |_u(X) - [y, \omega_{-1} |_u(X)],$$

because ω is a Cartan connection and l is a graded Lie algebra. On the other hand $\mathrm{Tr}([y, \omega_{-1} |_u(X)])) = B(y, \omega_{-1} |_u(X))$ from the definition of y.

Now let w_1 and w_2 in L_0 such that $\mathrm{Tr}(\omega_0 |_{uw_i} = 0)$, $\forall X \in H_{uw_i}$, i=1,2, and note that $uw_2 = uw_1 w_1^{-1} w_2$; we can write $w_1^{-1} w_2 = j_0 \exp(x)$, for some $x \in g_1$ and $j_0 \in G_0$ so for every $X \in H_{uw_1}$ we have

$$\omega_0 |_{uw_2}((R_{w_1^{-1} w_2})_* X) = (Ad(\exp(-x) j_0^{-1})(\omega(X)))_{g_0}, \quad \forall X \in H_{uw_1},$$

where $(\ )_{g_0}$ means the g_0-component. Hence

$$0 = \mathrm{Tr}(g_0^{-1} \omega_0 |_{uw_1}(X)\, g_0) - \mathrm{Tr}([x, Ad(j_0^{-1})(\omega_{-1} |_{uw_1}(X))],$$

by the action of Ad into the Lie algebra g_0. Now we recall that the g_1 values 1-form $\omega_1 |_{uw_1}: H_{uw_1} \to g_{-1}$ is surjective, G_0 normalizes g_i i=-1, 0, 1 in L_0 and $\mathrm{Tr}(\omega_0 |_{uw_1}(X) = 0)$. Hence $x = e$ and $w_1^{-1} w_2 \in G_0$; more precisely we have shown that $\forall u \in Q$ there exist a unique element $\eta \in G_1$ such that $\mathrm{Tr}(\omega_{u\eta}(X)) = 0$, $\forall X \in H_{u\eta}$. So we can define a differential map

$$\lambda : Q \to L_0/G_0,$$

where for every $u \in Q$ we define $\lambda(u)$ to be the class $[h]$ in L_0/G_0 of any element $h \in L_0$ with

$$\mathrm{Tr}(\omega_0 |_{uh}(X)) = 0, \quad \forall X \in H_{uh}.$$

It is easy to check that λ is a L_0-equivariant map, i.e. $\lambda(ug) = g^{-1} \lambda(u)$, $\forall u \in Q$ and $\forall g \in L_0$. We note that $\Upsilon_0 = \lambda^{-1}([e])$; we obtain that Υ_0 is a G_0-subbundle by the following general result.

LEMMA 2.1. *Let $(Q(M, L_0), M, \pi)$ be a principal fibre bundle over M with structure group L_0. Suppose now G_0 to be a closed subgroup of L_0 and suppose there is a differential map λ*

$$\lambda : Q(M, L_0) \to L_0/G_0$$

that is L_0-equivariant, i.e. $\lambda(ug) = g^{-1}\lambda(u)$, $\forall u \in Q(M, L_0)$ and $\forall g \in L_0$. Then $\Upsilon_0 = \lambda^{-1}(id)$ with projection $\pi|_{\Upsilon_0}$ is a G_0-principal bundle.

The proof can be visited in [4].

Now let Γ be a G_0 torsionfree connection on P belonging to an equivalence class α that generates Q, and let s be an admissible cross-section corresponding to Γ. We can define the map

$$F : P \to \Upsilon_0,$$

$$u \to s(u)\ \phi(s(u)),$$

where $\phi(s(u))$ is the unique element of G_1 for which

$$\mathrm{Tr}\,(\omega_0\,|_{s(u)\phi(s(u))}(X)) = 0\,, \quad \forall X \in H_{s(u)\phi(s(u))}\,.$$

Now, it is easy to check that F is a fibre bundle isomorphism; indeed we only need to prove that F is injective. If $u_1, u_2 \in P$ are such that $F(u_1) = F(u_2)$ then we can find $g_0 \in G_0$ with $u_2 = u_1 g_0$ and

$$s(u_1)\ \phi(s(u_1)) = s(u_2)\ \phi(s(u_2)) = s(u_1)\ g_0\, \phi(s(u_2))\,.$$

Hence $\phi(s(u_1)) = g_0\,\phi(s(u_2))$; since $\phi(s(u_i)) \in G_1$ for $i=1,2$ and $G_0 \cap G_1 = e$, then $g_0 = e$ and $u_1 = u_2$. Q.E.D.

We will briefly recall now the notations used in Table 1.

Let $\boldsymbol{K}$ denote the field of real number $\boldsymbol{R}$, or the complex field $\boldsymbol{C}$ or quaternions field $\boldsymbol{Q}$; in a natural way, $\boldsymbol{R} \subset \boldsymbol{C} \subset \boldsymbol{Q}$. For each element $x \in \boldsymbol{K}$ we define the element $\bar{x}$ and $\tilde{x}$ as follows:

If $x = x_0 + x_1 i + x_2 j + x_3 k$, with $x_i \in \boldsymbol{R}$ then

$$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k\,, \quad \tilde{x} = x_0 + x_1 i - x_2 j - x_3 k\,.$$

We use the following notation:

(1) $\boldsymbol{gl}(n; \boldsymbol{K}) = \{$all matrices of order n over the field $\boldsymbol{K}\}$;

(2) $\boldsymbol{sl}(n; \boldsymbol{K}) =$ the semisimple part of $\mathrm{gl}(n; \boldsymbol{K})$;

(3) $\boldsymbol{so}(p, q; \boldsymbol{K}) = \{A \in \boldsymbol{gl}(n; \boldsymbol{K}) : \tilde{A}^t I_{p,q} + I_{p,q} A = 0\}$, where the

matrix $I_{p,q}$ is:

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

$$\boldsymbol{so}(n;\boldsymbol{K}) = \boldsymbol{so}(n,0;\boldsymbol{K});$$

(4) $\boldsymbol{u}(p,q;\boldsymbol{K}) = \{A \in \boldsymbol{gl}(p+q;\boldsymbol{K}) : \overline{A}^t I_{p,q} + I_{p,q} A = 0\}$, $\boldsymbol{u}(n;\boldsymbol{K}) = \boldsymbol{u}(n,0,\boldsymbol{K})$;

(5) $\boldsymbol{su}(p,q;\boldsymbol{K}) = \boldsymbol{u}(p,q;\boldsymbol{K}) \bigcap \boldsymbol{sl}(p,q;\boldsymbol{K})$;

(6) $\boldsymbol{sp}(n;\boldsymbol{K}) = \{A \in \boldsymbol{gl}(n;\boldsymbol{K}) : \widetilde{A}^t J + \; + JA = 0\}$.

We remember also that the Lie algebra $l = g_{-1} \oplus g_0 \oplus g_1$.

REFERENCES

[1] S. Kobayashi - K. Nomizu, *Foundations of Differential Geometry*, Vol I, 1963, and Vol II, 1969, Wiley, New York.

[2] S. Kobayashi, *Trasformation groups in Differential geometry*, Springer, Berkley (1970).

[3] T. Ochiai, *Geometry Associated with Semisimple Flat Homogeneous Spaces*, American Mathematical Society, **152** (1970), pp. 159-193.

[4] F. Podestá, *Projective Structures on Reductive Homogeneous Spaces*, Proc. A.M.S., **109** (1990), pp. 1087-1096.

[5] E. Cartan, *Les Espaces a Connession conform*, Gauthier-Villar, Paris, (1955).

[6] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, Inc London (1978).

[7] F. Warner, *Foundations of Differential Manifold and Lie Groups*, Springer, Berlin (1972).

[8] S. Kobayashi - T. Nagano, *On Projective Connection*, J. Math. Mech., **13** (1964), pp. 215-236.

Manoscritto pervenuto in redazione il 15 gennaio 1999.