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Stability Estimates for a Linearized Muskat Problem.

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ABSTRACT - In this paper we study the problem obtained by the linearization (in a particular geometry) of the classic Muskat problem, a free boundary problem which models the piston-like displacement of oil by water in a porous medium. We show that the stability of the linear problem depends on a parameter which is the ratio of the viscosities of the two fluids (this fact agrees with the experimental behavior of the displacement front). We also prove that the ill-posed case can be stabilized (according to Tikhonov) prescribing a priori bounds on the solutions.

1. Introduction.

One of the commonly employed devices in oil recovery consists in the forced injection of water into the oil reservoir. The resulting underground motion, called piston-like displacement, consists in the displacement of one fluid (the oil) by another (water). It was modelled in 1934 by M. Muskat [11, 12] by means of a free boundary problem where the normal velocity of the interface is proportional to the normal derivative of the solution of a Dirichlet problem for an elliptic operator.

Let $\Omega = \Omega_1(t) \cup \Omega_2(t) \cup \Sigma(t)$ with $\partial\Omega = \Sigma_1 \cup \Sigma_2$ where $\Omega_1(t)$, $\Omega_2(t)$ are regions occupied by oil and water respectively and $\Sigma(t)$ is the interface.

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Let also

$$a_1 = \frac{k}{\phi\mu_w} > 0\,, \qquad a_2 = \frac{k}{\phi\mu_o} > 0\,, \qquad \mathrm{e}\gamma = \frac{a_1}{a_2}$$

where

k = permeability of the porous medium to the fluid,
ϕ = porosity of the medium,
μ_w = viscosity of water,
μ_o = viscosity of oil.

Then we define

$$a(t) = \begin{cases} a_1 & \text{in } \Omega_1(t) \\ a_2 & \text{in } \Omega_2(t) \end{cases}$$

and let u be the fluid pressure; it solves, at any fixed time, the following transmission problem

$$\text{(1)} \qquad \begin{cases} \operatorname{div}(a(t)\,\nabla u) = 0 & \text{in } \Omega \\ u = 1 & \text{on } \Sigma_1 \\ u = 0 & \text{on } \Sigma_2 \end{cases}$$

whose surface of discontinuity $\Sigma(t)$ moves in time following the evolution equation

$$\text{(2)} \qquad \begin{cases} v_{\vec{n}}(t) = -a_1 \dfrac{\partial u_1}{\partial \vec{n}}_{|\Sigma(t)} = -a_2 \dfrac{\partial u_2}{\partial \vec{n}}_{|\Sigma(t)}\,, \\ \Sigma(0) = \Sigma_0 \end{cases}$$

where $v_{\vec{n}}$ is the normal velocity of $\Sigma(t)$.

The Muskat problem consists in equations (1) and (2).

The aspect of this problem we are most interested in is the analysis of the free boundary motion. This point of view can be justified by the experimental fact that, during the displacement of a fluid contained in a porous medium by another less viscous one, the displacement front may become unstable. In the present case, when $\mu_w < \mu_0$, protuberances occur that may advance through the average front (this effect is commonly referred to as *fingering*). An understanding of the fingering phenomenon is crucial as its occurrence poses severe limitations to the effective oil recovery from the reservoir. The process commonly used to avoid this phenomenon consists in increasing the viscosity of the water injected in the reservoir by means of some additives.

This problem was previously studied in the sixties by N. Verigin [15], J. S. Aronofsky [3] and A. E. Scheidegger [13, 14]. Then, around 1990, L. Jiang, Z. Chien and J. Liang [8, 9] published some significant papers on a weak formulation of the multidimensional case, and on an approximating Muskat model. Recently J. Mossino and F. Abergel have published some important papers on the argument. Some extension of the Muskat problem to the parabolic case has been studied by W. Fulk and R. B. Guenther [7], J. Cannon and A. Fasano [5] and L. C. Evans [6].

In this paper we study the relationship between the ratio of the viscosities and the stability of the linear problem derived from the Muskat problem.

When the geometry of the system is simple, the Muskat problem has particular solutions (more or less explicit).

We consider the case when the domain is a strip in $\mathbb{R}^n$

$$\Omega = \{(\underline{x}, y) \in \mathbb{R}^{n+1} : \underline{x} \in \mathbb{R}^n \text{ e } 0 < y < L\}$$

Here the surface $\Sigma(t)$ can be described by the function

$$y = \sigma(\underline{x}, t)$$

and, if initially it is a plane orthogonal to the y-axis, $\sigma_0(\underline{x}) = c_0$, the problem has the particular solution, independent on $\underline{x}$,

$$\sigma(\underline{x}, t) = \frac{-\gamma L + \sqrt{[\gamma L + (1-\gamma)\, c_0]^2 + 2a_1(1-\gamma)\, t}}{(1-\gamma)} =: c(t),$$

that is to say the free boundary remains always an orthogonal y-axis plane; this can be shown by routine calculations.

The Muskat problem can be linearized near this solution: taking

$$\sigma(\underline{x}, t) = c(t) + \varepsilon \varrho(\underline{x}, t)$$

and taking the first order approximation to all the functions in the equations (1) and (2) we can find the problem satisfied by the perturbation $\varrho(\underline{x}, t)$.

In this paper we study the properties of this problem: we will find that it turns out to be well-posed or ill-posed (just like the backward heat diffusion problem) depending on wheter the parameter γ is smaller or greater than one. When $\gamma > 1$ we can stabilize the problem according to

Tikhonov's definition: in fact we will prove the stability of a class of solutions with a priori bound on the gradient.

2. The linearized problem.

We notice that in problem (1) the time appears only as a parameter. For every fixed $t > 0$, we introduce the auxiliary unknown function

$$\psi(t) := \frac{\partial u_1}{\partial \vec{n}}_{|\sigma(\underline{x},\, t)}$$

so that problem (1) can be reformulated as follows:

$$(3) \qquad \begin{cases} \Delta u_1 = 0 & \text{in } \Omega_1 \\ u_1 = 1 & \text{on } y = 0 \\ \dfrac{\partial u_1}{\partial \vec{n}} = \psi & \text{on } \Sigma \end{cases} \qquad \begin{cases} \Delta u_2 = 0 & \text{in } \Omega_2 \\ u_2 = 1 & \text{on } y = L \\ \dfrac{\partial u_2}{\partial \vec{n}} = \gamma\psi & \text{on } \Sigma \end{cases}$$

with the condition

$$(4) \qquad u_1 = u_2 \ \text{ on } \Sigma .$$

When $\sigma(\underline{x},\, t) = c(t)$ problem (3), (4) takes the solution:

$$\begin{cases} u_{1,\,0}(\underline{x},\, y,\, t) = 1 + \psi_0(t)\ y \\ u_{2,\,0}(\underline{x},\, y,\, t) = \gamma\psi_0(t)(y - L), \end{cases}$$

with

$$\psi_0(t) = -\frac{1}{c(t) + \gamma(L - c(t))} .$$

Then we suppose that the relevant quantities of the problem have an expansion of this type

$$\sigma(\underline{x},\, t) = c(t) + \varepsilon\varrho(\underline{x},\, t) + o(\varepsilon)$$

$$u_1(\underline{x},\, y,\, t) = u_{1,\,0}(y,\, t) + \varepsilon v_1(\underline{x},\, y,\, t) + o(\varepsilon)$$

$$u_2(\underline{x},\, y,\, t) = u_{2,\,0}(y,\, t) + \varepsilon v_2(\underline{x},\, y,\, t) + o(\varepsilon)$$

$$\psi(\underline{x},\, t) = \psi_0 + \varepsilon\chi(\underline{x},\, t) + o(\varepsilon) .$$

Put such expressions in problem (3), (4) and keep the first order approximation; then taking the Fourier transform of (3) and (4), by routine calculation [see 10] we get

$$\hat{\chi}(\underline{\xi}, t) = a(\underline{\xi}, t)\, \hat{\varrho}(\underline{\xi}, t)$$

where

$$a(\underline{\xi},t) = (1-\gamma)\frac{1}{c(t)+\gamma(L-c(t))}\cdot\frac{|\underline{\xi}|}{\mathrm{Th}\,(c(t)|\underline{\xi}|)+\gamma\,\mathrm{Th}\,[(L-c(t))|\underline{\xi}|]}. \tag{5}$$

The linearization of the Cauchy problem (2) leads to the following problem

$$\begin{cases} \dfrac{\partial\varrho}{\partial t} = -T_{a(t)}\varrho \\ \varrho(\underline{x}, 0) = \varrho_0(\underline{x}) \end{cases} \tag{6}$$

where

$$T_{a(t)}\varrho = \int_{\mathbf{R}^n} e^{i\underline{x}\cdot\underline{\xi}} a(\underline{\xi}, t)\, \hat{\varrho}(\underline{\xi}, t)\, d\underline{\xi}. \tag{7}$$

Denoting by $\hat{\varrho}(\underline{\xi}, t)$ the partial Fourier transform with respect to $\underline{x}$ of $\varrho(\underline{x}, t)$ we get

$$\begin{cases} \dfrac{\partial}{\partial t}\hat{\varrho}(\underline{\xi}, t) = -a(\underline{\xi}, t)\, \hat{\varrho}(\underline{\xi}, t) \\ \hat{\varrho}(\underline{\xi}, t) = \hat{\varrho}_0(\underline{\xi}) \end{cases}$$

whose solution is

$$\hat{\varrho}(\underline{\xi}, t) = \hat{\varrho}_0(\underline{\xi}) \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau\right). \tag{8}$$

Taking the inverse Fourier transform in (8) we obtain the formal solution to problem (6).

Let us now formulate problem (6) in the appropriate functional spaces.

DEFINITION. *We say that* $u \in H^s(\mathbb{R}^n)$, *with* $s \in \mathbb{R}^+$, *if*

$$\int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^s \, |\hat{u}|^2 d\underline{\xi} < \infty$$

where $\hat{u}$ *is the Fourier transform of* u.

The linearized Muskat problem:

we are given $\varrho_0 \in H^s(\mathbb{R}^n)$; find $\varrho \in C([0, T], H^s(\mathbb{R}^n))$ such that (6) is satisfied.

Now, saying that the problem is well posed means that the operator

$$\begin{aligned} A : C([0, T], H^s(\mathbb{R}^n)) &\to H^s(\mathbb{R}^n) \\ \varrho(\underline{x}, t) &\mapsto \varrho_0(\underline{x}) \end{aligned},$$

which maps the solution to the date, is bijective and that the inverse operator

$$\begin{aligned} A^{-1}: \quad H^s(\mathbb{R}^n) &\to C([0, T], H^s(\mathbb{R}^n)) \\ \varrho_0(\underline{x}) &\mapsto \varrho(\underline{x}, t) \end{aligned}$$

is continuous.

To study the linearized operator it is advisable to investigate first the properties of the symbol $a(\underline{\xi}, t)$, given by (5).

Let us define for convenience

$$\alpha(t) = c(t) + \gamma(L - c(t)).$$

– If $\gamma = 1$ then $a(\underline{\xi}, t) = 0$.

We note that in this case the solution is

$$\varrho(\underline{x}, t) = \varrho_0(\underline{x}) \qquad \text{for every } t \in [0, T],$$

that is, when the fluids have the same viscosity, for the linearized problem the interface moves at a constant velocity conserving the same shape.

– If $\gamma < 1$ then $a(\underline{\xi}, t) > 0$ for every $(\underline{\xi}, t)$.

It is an increasing function of $|\underline{\xi}|$ with positive minimum $b(t) = (1 - \gamma)\dfrac{a_1}{\alpha(t)^2}$ at $|\underline{\xi}| = 0$ decreasing in time and the oblique asymptote $g(t) = \dfrac{(1-\gamma)}{(1+\gamma)} \cdot \dfrac{a_1}{\alpha(t)} |\underline{\xi}|$ with an inclination which decreases in time.

– If $\gamma > 1$ $a(\underline{\xi}, t) < 0$ per ogni $(\underline{\xi}, t)$.

It is a decreasing function of $|\underline{\xi}|$ with negative maximum $b(t) = (1-\gamma)\frac{a_1}{\alpha(t)^2}$ at $|\underline{\xi}| = 0$ decreasing in time and the oblique asymptote with an inclination which decreases in time.

Then we obtain some estimates of $|a(\underline{\xi}, t)|$ independent on time.

LEMMA 1. *There exist positive constants m and M, independent on time, such that the inequalities*

$$m(1+|\underline{\xi}|^2)^{1/2} \leqslant |a(\underline{\xi}, t)| \leqslant M(1+|\underline{\xi}|^2)^{1/2}$$

hold for every $t \in [0, T]$.

PROOF. Using the expression of $g(t)$ and $b(t)$ we easily derive the following estimate

$$\frac{1}{2}(|b(t)| + |g(t)|) \leqslant |a(\underline{\xi}, t)| \leqslant (|b(t)| + |g(t)|)$$

that is

$$|1-\gamma|\frac{a_1}{2\alpha(t)}\left(\frac{1}{\alpha(t)} + \frac{|\underline{\xi}|}{1+\gamma}\right) \leqslant |a(\underline{\xi}, t)| \leqslant |1-\gamma|\frac{a_1}{\alpha(t)}\left(\frac{1}{\alpha(t)} + |\underline{\xi}|\right).$$

In the case $\gamma < 1$ since $\frac{1}{L} \leqslant \frac{1}{\alpha(t)} \leqslant \frac{1}{\alpha(0)}$ e $\frac{1}{2} < \frac{1}{1+\gamma}$ we have

$$|1-\gamma|\frac{a_1}{4L}\left(\frac{2}{L} + |\underline{\xi}|\right) \leqslant |a(\underline{\xi}, t)| \leqslant |1-\gamma|\frac{a_1}{\alpha(0)}\left(\frac{1}{\alpha(0)} + |\underline{\xi}|\right),$$

which easily implies that: there exist positive constants $c_1, c_2 > 0$ such that

$$c_1(1+|\underline{\xi}|) \leqslant |a(\underline{\xi}, t)| \leqslant c_2(1+|\underline{\xi}|).$$

So by using

$$(1+|\underline{\xi}|^2)^{1/2} \leqslant (1+|\underline{\xi}|) \leqslant 2(1+|\underline{\xi}|^2)^{1/2}$$

we arrive at the desired result.

In the case $\gamma > 1$ since $\frac{1}{\alpha(0)} \leqslant \frac{1}{\alpha(t)} \leqslant \frac{1}{L}$ we have

$$|1-\gamma|\frac{a_1}{2(1+\gamma)\alpha(0)}\left(\frac{1+\gamma}{\alpha(0)} + |\underline{\xi}|\right) \leqslant |a(\underline{\xi}, t)| \leqslant |1-\gamma|\frac{a_1}{L}\left(\frac{1}{L} + |\underline{\xi}|\right)$$

and proceeding as in the previous case we complete the proof. ■

COROLLARY. *$T_{a(t)}$ is an elliptic pseudodifferential operator of the first order, for every $t \in [0, T]$.*

3. Stability of the linearized problem.

By lemma 1 we easily obtain the following result.

THEOREM 1. *For every $\gamma \neq 1$ and for every $s \in \mathbb{R}^n$ the operator*

$$T_{a(t)}: H^s(\mathbb{R}^n) \to H^{s-1}(\mathbb{R}^n)$$

defined by

$$T_{a(t)}\varrho = \int e^{i\underline{x}\underline{\xi}} a(\underline{\xi}, t)\, \hat{\varrho}(\underline{\xi}, t)\, d\underline{\xi}$$

is bounded with bounded inverse.

Moreover there exist two positive constants $k_1, k_2 > 0$ independent on time, such that

$$k_1 \|T_{a(t)}\varrho\|_{H^{s-1}} \leq \|\varrho\|_{H^s} \leq k_2 \|T_{a(t)}\varrho\|_{H^{s-1}}.$$

PROOF. From the definition of the operator $T_{a(t)}$ it follows that

$$\widehat{T_{a(t)}\varrho} = a\hat{\varrho}$$

and recalling that when $\gamma \neq 1$ is $a(\underline{\xi}, t) \neq 0$, we have

$$\hat{\varrho} = \frac{\widehat{T_{a(t)}\varrho}}{a}.$$

$$\|T_{a(t)}\varrho\|^2_{H^{s-1}} = \int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^{s-1} |\widehat{T_{a(t)}\varrho}|^2 d\underline{\xi} = \int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^{s-1} |a(\underline{\xi}, t)|^2 |\hat{\varrho}|^2 d\underline{\xi} \leq$$

$$\leq M^2 \int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^{s-1} (1 + |\underline{\xi}|)^2 |\hat{\varrho}|^2 d\underline{\xi} = M^2 \|\varrho\|^2_{H^s}.$$

Moreover

$$\|\varrho\|^2_{H^s} = \int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^s |\hat{\varrho}|^2 d\underline{\xi} = \int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^s \frac{|\widehat{T_{a(t)}\varrho}|^2}{|a(\underline{\xi}, t)|} d\underline{\xi} \leq$$

$$\leqslant \int_{\mathbb{R}^n} \frac{(1+|\underline{\xi}|^2)^s}{m^2(1+|\underline{\xi}|^2)} |\widehat{T_{a(t)}\varrho}|^2 d\underline{\xi} =$$

$$= \frac{1}{m^2} \int_{\mathbb{R}^n} (1+|\underline{\xi}|^2)^{s-1} |\widehat{T_{a(t)}\varrho}|^2 d\underline{\xi} = \frac{1}{m^2} \|T_{a(t)}\varrho\|_{H^{s-1}}.$$

Then, by taking $k_1 = \frac{1}{M}$ and $k_2 = \frac{1}{m}$, we complete the proof. ■

To go on the following lemma is useful.

LEMMA 2. *If $\gamma < 1$ then the estimate*

$$\exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau\right) \leqslant \exp(-\lambda t)$$

holds, where $\lambda = \frac{a_1(1-\gamma)}{L^2} > 0$.

PROOF. We know that if $\gamma < 1$

$$\inf_{\mathbb{R}} a(\underline{\xi}, \tau) = (1-\gamma)\frac{a_1}{\alpha(\tau)^2} \qquad \text{for every } \tau \in [0, T].$$

Hence

$$\int_0^t a(\underline{\xi}, \tau)\, d\tau \geqslant \int_0^t (1-\gamma)\frac{a_1}{\alpha(\tau)^2} d\tau \geqslant \int_0^t (1-\gamma)\frac{a_1}{L^2} = (1-\gamma)\frac{a_1}{L^2} t. \qquad ■$$

From the expression

$$\widehat{\varrho}(\underline{\xi}, t) = \widehat{\varrho}_0(\underline{\xi}) \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau\right)$$

we obtain the following result:

THEOREM 2. *If $\gamma < 1$ then*

$$\|\varrho(t)\|_{H^s} \leqslant e^{-\lambda t} \|\varrho_0\|_{H^s} \quad \textit{for every } s \in \mathbb{R}^n.$$

If $\gamma > 1$ then

$$\|\varrho(t)\|_{H^s} > \|\varrho_0\|_{H^s} \quad \textit{for every } s \in \mathbb{R}^n.$$

PROOF. If $\gamma < 1$ then

$$|\widehat{\varrho}(\underline{\xi}, t)| = \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau\right) |\widehat{\varrho}_0(\underline{\xi})| \leq e^{-\lambda t} |\widehat{\varrho}_0(\underline{\xi})| .$$

Hence

$$\|\varrho(t)\|^2_{H^s} = \int (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}(\underline{\xi}, t)|^2 d\underline{\xi} \leq e^{-2\lambda t} \int (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}_0(\underline{\xi})|^2 d\underline{\xi}.$$

On the other hand, if $\gamma > 1$, recalling that $a(\underline{\xi}, t) < 0$, we have

$$\exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau\right) > 1$$

and then

$$|\widehat{\varrho}(\underline{\xi}, t)| = \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau\right) |\widehat{\varrho}_0(\underline{\xi})| > |\widehat{\varrho}_0(\underline{\xi})| .$$

Hence

$$\|\varrho(t)\|^2_{H^s} = \int (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}(\underline{\xi}, t)|^2 d\underline{\xi} >$$

$$> \int (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}_0(\underline{\xi})|^2 d\underline{\xi} = \|\varrho_0\|^2_{H^s}. \qquad \blacksquare$$

THEOREM 3. *If* $\gamma < 1$ *then problem* (6) *is well-posed.*
If $\gamma > 1$ *then problem* (6) *is ill-posed.*

PROOF. By theorem 2 it follows that, if $\gamma < 1$, for every $\varrho_0(\underline{x}) \in H^s(\mathbb{R}^n)$ there exists a unique corresponding inverse image $\varrho(\underline{x}, t)$ such that

$$\|\varrho(t)\|_{H^s} \leq e^{-\lambda t} \|\varrho_0\|_{H^s} \text{ for every } t \in [0, T].$$

Then

$$\|\varrho\|_{C([0, T], H^s)} =: \sup_{0 \leq t \leq T} \|\varrho(t)\|_{H^s} \leq \|\varrho_0\|_{H^s}.$$

If $\gamma > 1$ then the operator which maps $\varrho(\underline{x}, t)$ to $\varrho_0(\underline{x})$ is not surjective because a random choice of $\varrho_0 \in H^s$ does not guarantee that

$$\int (1+|\underline{\xi}|^2)^s \left| \hat{\varrho}_0(\underline{\xi}) \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau \right) \right|^2 d\xi < \infty \quad \text{for every } t \in [0, T].$$

To make it happen, for example, $\hat{\varrho}_0(\underline{\xi})$ has to go to zero faster than $\exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau \right)$ at infinity, and this is just a limitation on the choice of data.

But even if we were to limit the range of the operator in a suitable way, the problem would still not be stable.

In fact, since, for every $t \in [0, T]$, the function $\exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau \right)$ is not bounded on $\mathbb{R}$, it follows that no constant $c > 0$ such that

$$\|\varrho\|_{C([0,T], H^s)} \leq c \|\varrho_0\|_{H^s}$$

exists. ■

4. Stabilization.

In the case $\gamma > 1$ we will prove that the ill-posed problem can be stabilized using the Tikhonov stabilization technique: the continuity of the inverse linearized operator can be restored by prescribing some global bounds to solutions.

We will show that we can find a solution subspace $W \subset C([0,T], H^s(\mathbb{R}^n))$ with this property: if we call W_A the $W-$ image by A and A_W the restriction of A to the set W, the operator

$$\begin{aligned} B_W:\quad W_A \subset H^s(\mathbb{R}^n) &\to \quad W \\ \varrho_0(\underline{x}) &\mapsto \varrho(\underline{x}, t) \end{aligned}$$

which is the inverse of A_W, turns out to be bounded.

THEOREM 4. *Let*

$$\widetilde{W} = \{ f \in H^{s+1}(\mathbb{R}^n) : \|f'\|_{H^s} \leq E,$$

$$\textit{where } f' \textit{ is the distributional derivatives of } f \}.$$

Then, for every solution $\varrho(\underline{x}, t)$ *of the linearized Muskat problem such* *that* $\varrho(t) \in \widetilde{W}$, *and for every* $t \in [0, T]$, *we have*

$$\|\varrho(t)\|_{H^s}^2 \leqslant \|\varrho_0\|_{H^s}^2 e^{\eta + \beta\delta} + \frac{E^2}{\delta^2} \text{ for every } \delta > 0$$

where

$$\eta = \frac{2a_1(\gamma - 1)T}{L^2} \quad \text{and} \quad \beta = \frac{2a_1(\gamma - 1)T}{L}$$

PROOF. Saying that $\varrho(t) \in \widetilde{W}$ means that

$$\int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^s |\underline{\xi}|^2 |\widehat{\varrho}(\underline{\xi}, t)|^2 d\underline{\xi} \leqslant E^2 \text{ for every } t \in [0, T].$$

Furthermore, saying that ϱ is a solution means that $\varrho = B_W \varrho_0$, that is

$$\widehat{\varrho}(\underline{\xi}, t) = \widehat{\varrho}_0(\underline{\xi}) \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau \right).$$

Then

$$\|\varrho(t)\|_{H^s}^2 = \int_{\mathbb{R}^n} (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}(\underline{\xi}, t)|^2 d\underline{\xi} =$$

$$= \int_{|\underline{\xi}| < \delta} (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}(\underline{\xi}, t)|^2 d\underline{\xi} + \int_{|\underline{\xi}| > \delta} \frac{(1 + |\underline{\xi}|^2)^s |\underline{\xi}|^2 |\widehat{\varrho}(\underline{\xi}, t)|^2}{|\underline{\xi}|^2} d\underline{\xi} \leqslant$$

$$\leqslant \left| \exp\left(-\int_0^t a(\underline{\xi}, \tau)\, d\tau \right) \right|^2 \int_{|\underline{\xi}| < \delta} (1 + |\underline{\xi}|^2)^s |\widehat{\varrho}_0|^2 d\underline{\xi} +$$

$$+ \frac{1}{\delta^2} \int_{|\underline{\xi}| > \delta} (1 + |\underline{\xi}|^2)^s |\underline{\xi}|^2 |\widehat{\varrho}(\underline{\xi}, t)|^2 d\underline{\xi} \leqslant$$

$$\leqslant \|\varrho_0\|_{H^s}^2 \exp\left(-2 \int_0 t a(\underline{\xi}, \tau)\, d\tau \right) + \frac{E^2}{\delta^2}.$$

Moreover if $\gamma > 1$

$$\alpha(t) = c(t) + \gamma(L - c(t)) \geqslant L \text{ for every } t \text{ and}$$

$$-a(\delta, \tau) = \frac{a_1(\gamma - 1)}{\alpha(t)} \frac{\delta}{\operatorname{Th}(c(t)\,\delta) + \gamma \operatorname{Th}((L - c(t))\delta)} \leqslant$$

$$\leqslant \frac{a_1(\gamma - 1)}{\alpha(t)}\left(\frac{1}{\alpha(t)} + \frac{\delta}{1+\gamma}\right) \leqslant \frac{a_1(\gamma-1)}{L}\left(\frac{1}{L} + \delta\right).$$

Then

$$2\int_0^t -a(\underline{\delta}, \tau)\, d\tau \leqslant 2\int_0^T -a(\underline{\delta}, \tau)\, d\tau \leqslant 2\int_0^T \frac{a_1(\gamma-1)}{L}\left(\frac{1}{L} + \delta\right) =$$

$$= \frac{2a_1(\gamma-1)}{L}\left(\frac{1}{L} + \delta\right) T = \eta + \beta\delta\,.$$

Finally

$$\|\varrho_0\|^2_{H^s} \leqslant \|\varrho_0\|^2_{H^s} \exp\left(-2\int_0^t a(\underline{\xi}, \tau)\, d\tau\right) +$$

$$+ \frac{E^2}{\delta^2} \leqslant \|\varrho_0\|^2_{H^s} e^{\eta + \beta\delta} + \frac{E^2}{\delta^2} \quad \text{for every } \delta > 0\,. \qquad \blacksquare$$

THEOREM 5. *If*

$$W = \left\{ f \in C([0, T], H^s(\mathbb{R}^n)) \colon \int |\underline{\xi}|^2 |\hat{f}(\underline{\xi}, t)|^2 d\underline{\xi} \leqslant E^2 \text{ per ogni } t \in [0, T] \right\}$$

then the operator B_W: $W_A \subset H^s(\mathbb{R}^n) \to W$ *is bounded and the following estimate*:

$$\|B_W \varrho_0\| \leqslant (\|\varrho_0\| e^{\eta} + \frac{E^2\beta^2}{(\log \|\varrho_0\|)^2})^{1/2}$$

holds, for every $\varrho_0 \in W_A$ *such that* $0 < \|\varrho_0\| < 1$.

PROOF. Theorem 4 has shown that

$$\|\varrho(t)\|_{H^s} \leqslant \left(\|\varrho_0\|_{H^s}^2 e^{\eta + \beta\delta} + \frac{E^2}{\delta^2}\right)^{1/2} \quad \text{for every } \delta > 0$$

hence

$$\|B_W \varrho_0\|_{C([0,T], H^s)} = \|\varrho\|_{C([0,T], H^s)} =: \sup_{0 \leqslant t \leqslant T} \|\varrho(t)\|_{H^s} \leqslant$$

$$\leqslant \left(\|\varrho_0\|_{H^s}^2 e^{\eta + \beta\delta} + \frac{E^2}{\delta^2}\right)^{1/2} \quad \text{for every } \delta > 0 .$$

By choosing

$$\delta = -\frac{\log \|\varrho_0\|}{\beta} \quad \text{with } 0 < \|\varrho_0\| < 1$$

we obtain

$$\|B_W \varrho_0\|_{C([0,T], H^s)} \leqslant \left(\|\varrho_0\| e^{\eta} + \frac{E^2 \beta^2}{(\log \|\varrho_0\|)^2}\right)^{1/2} .$$

Because of

$$\lim_{\|\varrho_0\| \to 0} \left(\|\varrho_0\| e^{\eta} + \frac{E^2 \beta^2}{(\log \|\varrho_0\|)^2}\right)^{1/2} = 0 .$$

we have that B_W is bounded at the origin. ■

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