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Tensor fields of type (0,2) on linear frame bundles and cotangent bundles

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Tensor fields of type (0,2) on linear frame bundles and cotangent bundles (*).

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ABSTRACT - To any (0,2)-tensor field on the linear frame bundle, respectively on the cotangent bundle, we associate a global matrix function when a linear connection or a Riemannian metric on the base manifold is given. Based on this fact, natural (0,2)-tensor fields on frame and cotangent bundles are defined and characterized by means of well known algebraic results. In the symmetric case, our classification agrees with the one given by Sekizawa and Kowalski-Sekizawa. However, we do not make use of the theory of differential invariants.

1. Introduction.

In [4], [6] and [7] the authors defined natural symmetric tensor fields of type (0,2) on the linear frame bundle, respectively on the cotangent bundle, when the base manifold is endowed with a linear connection or a Riemannian metric. They also gave a complete classification of them, by means of the theory of differential invariants ([3], [5]). Using the well known fact that the frame bundle is naturally parallelizable, when the base manifold is endowed with a linear connection, and following the methods described in [1], we associate to any (0,2)-tensor field on the frame bundle (section 2), respectively on the cotangent bundle (section 4) a global matrix function. This matrix representation allows us to define and classify (sections 3 and 5) —from a simple point of view— what

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we also call *natural* (0,2)-tensor fields with respect to linear connections and Riemannian metrics. Actually, one of the advantages of this approach is that it lets us to obtain elementary proofs of the classification problems (Proposition 3.2 and Theorem 5.1).
Even though our way of defining naturality is quite different from that in [4], [6] and [7], we arrive at the same results as the one obtained by Kowalski-Sekizawa and Sekizawa (Remarks 3.1, 3.2, 5.1).

Throughout, all geometric objects are assumed to be differentiable, i.e. C^∞.

2. (0,2)-tensor fields on frame bundles.

Let M be a manifold of dimension $n \geqslant 2$ and for any point $p \in M$, let M_p be the tangent space of M at p. Let $\pi : TM \to M$ and $\boldsymbol{P} : LM \to M$ be respectively the tangent bundle and the linear frame bundle over M.

For a fixed linear connection ∇ on M, let $\boldsymbol{K} : T(TM) \to TM$ be the connection map induced by ∇.

In order to explicit our global matrix point of view and compare the main results in [4] and [6] with the one obtained by us (remarks 3.1 and 3.2), we shall describe in this section well known objects defined on LM in terms of $\boldsymbol{K}$.

Let us recall that for any $p \in M$ and any vector $v \in M_p$, the restriction $\boldsymbol{K}_v = \boldsymbol{K}|_{(TM)_v} : (TM)_v \to M_p$ is a surjective linear map characterized by the fact that for any vector field Y on M such that $Y(p) = v$, it satisfies $\boldsymbol{K}_v(Y_{*p}(w)) = \nabla_w Y$, where $Y_{*p} : M_p \to (TM)_v$ denotes the differential map of Y at p. For details, see [2].

The linear map $\pi_{*v} \times \boldsymbol{K}_v : (TM)_v \to M_p \times M_p$ defined by $\pi_{*v} \times \boldsymbol{K}_v(b) = (\pi_{*v}(b), \boldsymbol{K}(b))$ is an isomorphism that maps isomorphically the horizontal subspace H_v (= kernel of $\boldsymbol{K}_v$) onto $M_p \times \{0_p\}$ and the vertical subspace V_v (= kernel of π_{*v}) onto $\{0_p\} \times M_p$, where 0_p stands for the zero vector.

For $j = 1, \ldots, n$, let $\pi_j : LM \to TM$ be the projection map $\pi_j(p, \boldsymbol{e}) = \boldsymbol{e}_j$, where $\boldsymbol{e} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$. Let $\theta = (\theta^i)$ be the canonical form on LM and $\omega = (\omega^i_j)$ be the connection form of ∇. Hence, the 1-forms $\theta^1, \ldots, \theta^n$ on LM are defined by

$$\text{(2.1)} \qquad \boldsymbol{P}_{*(p,\boldsymbol{e})}(b) = \sum_{i=1}^{n} \theta^i(p, \boldsymbol{e})(b).\, \boldsymbol{e}_i$$

and the 1-forms ω^i_j on LM are defined —for $i, j = 1, \ldots, n$— in terms of

$\boldsymbol{K}$ by

$$(2.2) \qquad \boldsymbol{K}((\pi_j)_{*(p,\boldsymbol{e})}(b)) = \sum_{i=1}^{n} \omega_j^i(p,\boldsymbol{e})(b).\,\boldsymbol{e}_i$$

for any $(p, \boldsymbol{e}) \in LM$ and $b \in (LM)_{(p,\boldsymbol{e})}$. Since the $n+n^2$ 1-forms are linearly independent everywhere, we denote with $H_1, \dots, H_n$, $V_1^1, \dots, V_n^1, \dots, V_1^n, \dots, V_n^n$ the vector fields on LM, dual to $\theta^1, \dots, \theta^n$, $\omega_1^1, \dots, \omega_1^n, \dots, \omega_n^1, \dots, \omega_n^n$.

Hence, any vector field X on LM may be written as

$$X = \sum_{i=1}^{n} x^i.\,H_i + \sum_{i,j=1}^{n} x_i^j.\,V_j^i$$

where $x^i, x_i^j \colon LM \to \mathbb{R}$ are the differentiable mappings defined by $x^i = \omega^i(X)$ and $x_i^j = \omega_i^j(X)$.

Let us denote with ${}^{\nabla}X \colon LM \to \mathbb{R}^{n+n^2}$ the map

$${}^{\nabla}X = (x^1, \dots, x^n, x_1^1, \dots, x_1^n, \dots, x_n^1, \dots, x_n^n).$$

Now, let G be a (0,2)-tensor field on LM. We construct $(n+1)^2$ matrix functions

$$(G_{ij}), \quad (G_{ij}^l), \quad ({}^lG_{ij}), \quad (G_{ij}^{lm}) \colon LM \to \mathbb{R}^{n \times n}$$

by setting, for $l, m = 1, \dots, n$,

$$(2.6) \qquad \begin{cases} G_{ij} = G(H_i, H_j), & G_{ij}^l = G(H_i, V_j^l) \\ {}^lG_{ij} = G(V_i^l, H_j), & G_{ij}^{lm} = G(V_i^l, V_j^m). \end{cases}$$

The differentiable function ${}^{\nabla}G \colon LM \to \mathbb{R}^{(n+n^2)\times(n+n^2)}$ defined in the block form as

$${}^{\nabla}G = \begin{bmatrix} (G_{ij}) & (G_{ij}^1) & \cdots & (G_{ij}^n) \\ ({}^1G_{ij}) & (G_{ij}^{11}) & \cdots & (G_{ij}^{1n}) \\ \vdots & \vdots & \ddots & \vdots \\ ({}^nG_{ij}) & (G_{ij}^{n1}) & \cdots & (G_{ij}^{nn}) \end{bmatrix}$$

will be called the *matrix of G with respect to ∇*.

Clearly, for any pair of vector fields X, Y on LM, one gets the global matrix representation

$$G(X, Y) = {}^{\nabla}X \, . {}^{\nabla}G \, . ({}^{\nabla}Y)^t \tag{2.7}$$

where $({}^{\nabla}Y)^t$ denotes the transpose of ${}^{\nabla}Y$.

In fact, if $G_{(p,e)}: (LM)_{(p,e)} \times (LM)_{(p,e)} \to \mathbb{R}$ denotes the bilinear form induced by G on $(LM)_{(p,e)}$, the matrix ${}^{\nabla}G(p, e)$ is just the matrix of $G_{(p,e)}$ with respect to the basis

$$\{H_1(p,e),\dots,H_n(p,e), V_1^1(p,e),\dots, V_n^1(p,e),\dots, V_1^n(p,e),\dots, V_n^n(p,e)\}$$

Since the main results to compare are expressed by using horizontal and vertical lifts of vectors, we conclude this section showing how the vector fields H_i, V_j^i (using our terminology) are described by means of these two liftings.

For $1 \leqslant j \leqslant n$, let $V^j_{(p,e)}$ be the subspace spanned by $\{V_1^j(p, e), \dots, V_n^j(p, e)\}$ and $H_{(p,e)}$ the horizontal subspace spanned by $\{H_1(p, e), \dots, H_n(p, e)\}$. Then it follows that $(\pi_j)_{*(p,e)}: H_{(p,e)} \oplus \oplus V^j_{(p,e)} \to H_{e_j} \oplus V_{e_j} = (TM)_{e_j}$ is an isomorphism which maps isomorphically $H_{(p,e)}$ onto H_{e_j} and $V^j_{(p,e)}$ onto V_{e_j}.

If $z \in M_p$ the j-th vertical lift of z to LM at (p, e) is the unique $z^{v,j}(p, e) \in V^j_{(p,e)}$ which satisfies

$$(\pi_j)_{*(p,e)}(z^{v,j}(p, e)) = (\pi_{*e_j} \times \boldsymbol{K}_{e_j})^{-1}(0_p, z)$$

and the horizontal lift of z to LM at (p, e) is the unique $z^h(p, e) \in H_{(p,e)}$ —which for any $1 \leqslant j \leqslant n$— satisfies

$$(\pi_j)_{*(p,e)}(z^h(p, e)) = (\pi_{*e_j} \times \boldsymbol{K}_{e_j})^{-1}(z, 0_p).$$

From (2.1) and (2.2) it follows

$$H_i(p, e) = (e_i)^h(p, e) \qquad V_i^j(p, e) = (e_i)^{v,j}(p, e). \tag{2.5}$$

EXAMPLES. For a given ∇ on M, let G^d and G^h be the diagonal and horizontal lifts of ∇. Using the way in which these tensor fields are described in [6], from (2.3) and (2.5) one gets:

$$
{}^{\nabla}G^{d} = \begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix} \qquad {}^{\nabla}G^{h} = \begin{bmatrix} 0 & I & \cdots & I \\ I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ I & 0 & \cdots & 0 \end{bmatrix}
$$

3. Natural (0,2)-tensor fields on LM.

If $GL(n, \mathbb{R})$ denotes the general linear group, for any $(p, \boldsymbol{e}) \in LM$ with $\boldsymbol{e} = \{e_1, \ldots, e_n\}$, let $\boldsymbol{R}_{(p,\boldsymbol{e})}\colon GL(n, \mathbb{R}) \to LM$ be the map defined by $\boldsymbol{R}_{(p,\boldsymbol{e})}(a) = (p, \boldsymbol{e}\,.\,a)$, where $\boldsymbol{e}\,.\,a = \{u_1, \ldots, u_n\}$ with $u_j = \sum_{i=1}^{n} a_j^i\,.\,\boldsymbol{e}_i$ if

$$
a = \begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{bmatrix}.
$$

Let $N = LM \times GL(n, \boldsymbol{R})$ and $\psi : N \to LM$ be the projection map defined by $\psi(p, u, \xi) = \boldsymbol{R}_{(p,u)}(\xi)$.

For any $a \in GL(n, \boldsymbol{R})$, let $\boldsymbol{R}_a\colon N \to N$ be the map defined by $\boldsymbol{R}_a(p, u, \xi) = (p, u\,.\,a, a^{-1}\,.\,\xi)$; then $\psi \circ \boldsymbol{R}_a = \psi$.

Now, let ∇ be a linear connection on M and let G be a (0,2)-tensor field on LM. If $T = {}^{\nabla}G \circ \psi$, then T is a differentiable map, satisfying for any $a \in GL(n, \boldsymbol{R})$

$$
(3.1) \qquad T \circ \boldsymbol{R}_a = T.
$$

On the other hand, if $T : N \to \boldsymbol{R}^{(n+n^2)\times(n+n^2)}$ is a differentiable map that satisfies (3.1), we define G via (2.7) by setting ${}^{\nabla}G(p, \boldsymbol{e}) = T(p, \boldsymbol{e}, I)$ for any $(p, \boldsymbol{e}) \in LM$.

Hence, ${}^{\nabla}G \circ \psi = T$ and we get a one to one correspondence «${}^{\nabla}G \leftrightarrow T$» between (0,2)-tensor fields on LM and differentiable mappings $T : N \to \boldsymbol{R}^{(n+n^2)\times(n+n^2)}$ satisfying (3.1).

We say that T is the *associated matrix* to ${}^{\nabla}G$.

DEFINITION 3.1. A (0,2)-tensor field G on LM will be called *natural with respect to* ∇ if T depends only on the variable «ξ».

For example, the tensor fields G^d and G^h mentioned in section 2, are natural.

REMARK 3.1. Clearly, G is natural with respect to ∇ if and only if ${}^{\nabla}G$ is constant. Looking at the main result in [6], one sees —via (2.3) and (2.5)— that the concept of natural symmetric tensor field introduced by Sekizawa agrees with the one given by us.

Let $\langle\,,\rangle$ be a Riemannian metric on M, ∇ the Levi-Civita connection of $\langle\,,\rangle$ and $\mathcal{O}(M)$ the bundle of orthonormal frames over $(M, \langle\,,\rangle)$. Let $N = \mathcal{O}(M) \times GL(n, \boldsymbol{R})$ and $\psi : N \to LM$ be the projection map defined by

$$\psi(p,\, u,\, \xi) = \boldsymbol{R}_{(p,\, u)}(\xi)\,.$$

For any $a \in \mathcal{O}(n)$ (orthogonal group), let $\boldsymbol{R}_a \colon N \to N$ be the map defined by $\boldsymbol{R}_a(p,\, u,\, \xi) = (p,\, u.\, a,\, a^{-1}.\, \xi)$; then, $\psi \circ \boldsymbol{R}_a = \psi$.

Finally, let us consider the family of mappings $T : N \to \boldsymbol{R}^{(n+n^2)\times(n+n^2)}$ satisfying, for any $a \in \mathcal{O}(n)$

$$T \circ \boldsymbol{R}_a = T\,. \tag{3.2}$$

Any (0,2)-tensor field G on LM defines a differentiable map T satisfying (3.2). Reciprocally, if T is a differentiable map that satisfies (3.2), we define G —via (2.7)— as follows: applying the Gram-Schmidt process to any $(p, \boldsymbol{e}) \in LM$, one gets an orthonormal basis $(p, \bar{\boldsymbol{e}}) \in \mathcal{O}(M)$ and a differentiable map $\alpha : LM \to GL(n, \boldsymbol{R})$ that associates each $(p, \boldsymbol{e}) \in LM$ with the only matrix verifying $\boldsymbol{e} = \bar{\boldsymbol{e}}\,.\,\alpha(p, \boldsymbol{e})$. Set ${}^{\nabla}G(p, \boldsymbol{e}) = T(p, \bar{\boldsymbol{e}},\, \alpha(p,\, \boldsymbol{e}))$, for any $(p,\, \boldsymbol{e}) \in LM$.

Hence, ${}^{\nabla}G \circ \psi = T$ and we get a one to one correspondence «${}^{\nabla}G \leftrightarrow T$» between (0,2)-tensor fields on LM and differentiable mappings satisfying (3.2).

We say that T is the *associated matrix* to ${}^{\nabla}G$ with respect to $\langle\,,\rangle$.

DEFINITION 3.2. We say that G is *natural with respect to* $\langle\,,\rangle$ if the associated matrix T depends only on the variable «ξ».

The following result is well known in the literature and will be needed.

LEMMA 3.1. *Let $S \subset \boldsymbol{R}^{n\times n}$ be the submanifold consisting of all positive definite symmetric matrices and $\varphi : S \to S$ the map defined by $\varphi(x) = x^{1/2}$ (the square root). Then φ is differentiable.*

We shall now characterize all natural (0,2)-tensor fields on LM with respect to Riemannian metrics on M.

PROPOSITION 3.2. *Let $\langle,\rangle$ be a Riemannian metric on M, ∇ the Levi-Civita connection of $\langle,\rangle$ and G a (0,2)-tensor field on LM. The following are equivalent:*

(1) *G is natural with respect to $\langle,\rangle$.*

(2) *Setting ${}^{\nabla}G = (G_{\alpha\beta}): LM \to \boldsymbol{R}^{(n+n^2)\times(n+n^2)}$ where $\alpha, \beta = 1, \dots, n+n^2$, there exist differentiable functions $f_{\alpha,\beta}: S \to \boldsymbol{R}$ such that $G_{\alpha,\beta}(p, \boldsymbol{e}) = f_{\alpha,\beta}((\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle))$, where $\boldsymbol{e} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$.*

PROOF. Let T be the associated matrix to ${}^{\nabla}G$ and $(p, u, \xi) \in \boldsymbol{N}$. Since T satisfies (3.2), we have for any $a \in \mathcal{O}(n)$

$$ {}^{\nabla}G \circ \psi(p, u, \xi) = T(p, u, \xi) = T(p, u.\, a, a^{-1}.\, \xi). \tag{3.3}$$

Hence, if $a \in \mathcal{O}(n)$ and $b \in S$ are the unique matrices that satisfy $\xi = a.\, b$, we obtain

$$ {}^{\nabla}G \circ \psi(p, u, \xi) = T(p, u.\, a, b) \tag{3.4}$$

(1) $\Rightarrow$ (2). Let $(p, \boldsymbol{e}) \in LM$ and $(p, u, \xi) \in \boldsymbol{N}$ such that $\boldsymbol{e} = u.\, \xi = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$, then $\xi^t.\, \xi = (\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle)$. Now, (2) follows inmediately from (3.4) since $b = \varphi(\xi^t.\, \xi) = \varphi((\langle \boldsymbol{e}_i, \boldsymbol{e}_j\rangle))$, φ is differentiable and $T(p, u.\, a, b) = T(b)$.

(2) $\Rightarrow$ (1). Clearly, (2) implies that ${}^{\nabla}G \circ \psi(p, u, \xi) = (f_{\alpha,\beta}(\xi^t.\, \xi))$. Consequently T depends only on «ξ».

EXAMPLES. Let G^d and G^h be the diagonal and horizontal lifts of $\langle,\rangle$. Using the way in which these tensor fields are described in [4], from (2.3) and (2.5) one gets:

$$ {}^{\nabla}G^d = \begin{bmatrix} A & 0 & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A \end{bmatrix} \qquad {}^{\nabla}G^h = \begin{bmatrix} 0 & A & \cdots & A \\ A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A & 0 & \cdots & 0 \end{bmatrix} $$

where $A : LM \to \boldsymbol{R}^{n\times n}$ is the positive definite symmetric matrix function defined by $A(p, \boldsymbol{e}) = (\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle)$ if $\boldsymbol{e} = \{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$.

REMARK 3.2. Comparing the main result of [4] with proposition above —having in mind (2.3) and (2.5)— it follows that the concept of *natural symmetric tensor field* introduced by Kowalski-Sekizawa agrees (in the symmetric case) with the one given by us.

REMARK 3.3. Proposition shows how to construct (0,2)-tensor fields on LM when M is endowed with a Riemannian metric $\langle\, ,\rangle$. In fact, if $f : S \to \mathbb{R}^{(n+n^2)\times(n+n^2)}$ is any differentiable function, define G via (2.4) by setting

$$^{\nabla}G(p, \boldsymbol{e}) = f((\langle \boldsymbol{e}_i, \boldsymbol{e}_j \rangle))$$

for any $(p, \boldsymbol{e}) \in LM$. If, in addition —for any $x \in S$— $f(x)$ is a positive definite symmetric matrix, then G is a Riemannian metric on LM and the curvatures of (LM, G) may be computed in terms of f.

REMARK 3.4. Proposition above is also true replacing the Levi-Civita connection by any Riemannian connection since no torsion-free assumption is needed.

4. (0,2)-tensor fields on cotangent bundles.

For any $p \in M$, let M_p^* be the dual space of M_p. Let $\pi : T^*M \to M$ be the cotangent bundle of M and ∇ a linear connection on M. The connection ∇ defines a differentiable map $\boldsymbol{K}^* : T(T^*M) \to T^*M$ called the dual connection map. For any $p \in M$ and any co-vector $w \in M_p^*$ the restriction $K_w^* = K^* |_{(T^*M)_w} : (T^*M)_w \to M_p^*$ is a surjective linear map, characterized by the fact that for any 1-form ω on M such that $\omega(p) = w$ and any vector $v \in M_p$, it satisfies $K_w^*(\omega_{*p}(v)) = \nabla_v \omega$, where $\omega_{*p} : M_p \to (T^*M)_w$ denotes the differential map of ω at p.

The linear map $\pi_{*w} \times \boldsymbol{K}_w^* : (T^*M)_w \to M_p \times M_p^*$ defined by $\pi_{*w} \times \boldsymbol{K}_w^*(b) = (\pi_{*w}(b), \boldsymbol{K}^*(b))$ is an isomorphism that maps the horizontal subspace H_w (= kernel of $\boldsymbol{K}_w^*$) onto $M_p \times \{0_p\}$ and the vertical subspace V_w (= kernel of π_{*w}) onto $\{0_p\} \times M_p^*$, where 0_p denotes indistinctly the zero vector and the zero co-vector.

Since $(T^*M)_w = H_w \oplus V_w$, any vector field X on T^*M may be written in the form $X = X^h + X^v$, where

(4.1) $$X^h(w) = (\pi_{*w} \times K_w^*)^{-1}(\pi_{*w}(X(w)), 0_p)$$

and

(4.2) $$X^v(w) = (\pi_{*w} \times K_w^*)^{-1}(0_p, K^*(X(w)))\,.$$

Let L^*M be the co-frame bundle over M and $\psi : N = L^*M \times R^n \to \to T^*M$ the map defined by

(4.3) $$\psi(p, u^*, \xi) = \sum_{i=1}^{n} \xi_i . u^i$$

where $u^* = \{u^1, \ldots, u^n\}$ is a basis for M_p^* and $\xi = (\xi_1, \ldots, \xi_n)$.

The family of maps R_a: $N \to N$, $a \in GL(n, R)$, given by

(4.4) $$R_a(p, u^*, \xi) = (p, u^*a, \xi . a^{-1})$$

where $u^*a = \left\{\sum_{i=1}^{n} a_i^1 . u^i, \ldots, \sum_{i=1}^{n} a_i^n . u^i\right\}$ if $a = \begin{bmatrix} a_1^1 & \cdots & a_n^1 \\ \vdots & & \vdots \\ a_1^n & \cdots & a_n^n \end{bmatrix}$ defines the action of $GL(n, R)$ on N. Clearly, $\psi \circ R_a = \psi$.

If $u = \{u_1, \ldots, u_n\}$ is a basis for M_p, we denote with $u^* = \{u^1, \ldots, u^n\}$ the dual basis. For $1 \leq i \leq n$, let e_i, e_{n+i}: $N \to T(T^*M)$ be the mappings defined by

(4.5) $$e_i(p, u^*, \xi) = (\pi_{*w} \times K_w^*)^{-1}(u_i, 0_p)$$

and

(4.6) $$e_{n+i}(p, u^*, \xi) = (\pi_{*w} \times K_w^*)^{-1}(0_p, u^i)$$

where $w = \psi(p, u^*, \xi)$.

Making use of local coordinates, it is easy to check that these functions are differentiable. By construction, $\{e_1(p, u^*, \xi), \ldots, e_n(p, u^*, \xi)\}$ and $\{e_{n+1}(p, u^*, \xi), \ldots, e_{2n}(p, u^*, \xi)\}$ are, respectively, basis for H_w and V_w.

Therefore, if X is a vector field on T^*M, one gets, from (4.1) and (4.2), that

(4.7) $$X \circ \psi(p, u^*, \xi) = \sum_{l=1}^{2n} x^l(p, u^*, \xi) . e_l(p, u^*, \xi)$$

where $x^l\colon N \to \mathbf{R}$ are defined, for $w = \psi(p, u^*, \xi)$, by

$$x^i(p, u^*, \xi) = u^i(\pi_{*w}(X(w))) \tag{4.8}$$

and

$$x^{n+i}(p, u^*, \xi) = \boldsymbol{K}^*_w(X(w))(u_i). \tag{4.9}$$

Let ${}^\nabla X = (x^1, \dots, x^{2n})\colon N \to \mathbf{R}^{2n}$ be the map induced by X and $\{e_1, \dots, e_{2n}\}$. Now, let G be a (0,2)-tensor field on T^*M. If X, Y are vector fields on T^*M and $w \in T^*M$, then $G(X, Y)(w) = G_w(X(w), Y(w))$, where $G_w\colon (T^*M)_w \times (T^*M)_w \to \mathbf{R}$ is the bilinear form induced by G on $(T^*M)_w$. Hence, we can define a differentiable matrix function ${}^\nabla G\colon N \to \mathbf{R}^{2n \times 2n}$ as follows: if $(p, u^*, \xi) \in N$ and $w = \psi(p, u^*, \xi)$, let ${}^\nabla G(p, u^*, \xi)$ be the matrix of G_w with respect to the basis $\{e_1(p, u^*, \xi), \dots, e_{2n}(p, u^*, \xi)\}$.

We shall call ${}^\nabla G$ the matrix of G with respect to ∇. From (4.7) one gets the global matrix representation

$$G(X, Y) \circ \psi = {}^\nabla X . {}^\nabla G . ({}^\nabla Y)^t \tag{4.10}$$

where $({}^\nabla Y)^t$ denotes the transpose of ${}^\nabla Y$. Writing ${}^\nabla G$ in the block form ${}^\nabla G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$ where $A_i\colon N \to \mathbf{R}^{n \times n}$, one sees from (4.5) and (4.6) that they satisfy the following $GL(n, \mathbf{R})$-invariance properties

$$A_1 \circ \mathbf{R}_a = (a^{-1})^t . A_1 . a^{-1} \tag{4.11}$$

$$A_2 \circ \mathbf{R}_a = (a^{-1})^t . A_2 . a^t \tag{4.12}$$

$$A_3 \circ \mathbf{R}_a = a . A_3 . a^t \tag{4.13}$$

$$A_4 \circ \mathbf{R}_a = a . A_4 . a^{-1} \tag{4.14}$$

Hence, for a fixed linear connection on M, we get a one to one correspondence ${}^\nabla G \leftrightarrow T$ between (0,2)-tensor fields on T^*M and differentiable matrix functions $T = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}\colon N \to \mathbf{R}^{2n \times 2n}$ where each A_i $(1 \leq i \leq 4)$ satisfies respectively (4.11), (4.12), (4.13) and (4.14).

The differentiability of G —for T given— follows from (4.10) and the fact that ψ is a submersion.

EXAMPLES. Let θ be the canonical 1-form on T^*M and $d\theta$ the exterior derivative of θ, also called the 2-form on T^*M. They are defined by

$$\theta(X)(w) = w(\pi_{*w}(X(w))) \tag{4.15}$$

and

$$d\theta(X, Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X, Y]) \tag{4.16}$$

for any vector fields X, Y on T^*M and any co-vector $w \in T^*M$.

If $\otimes$ denotes the tensor product, set $G_1 = \theta \otimes \theta = \theta^2$. Let G_2 and G_3 be the (0,2)-tensor fields on T^*M defined by

$$G_2(X, Y) = d\theta(Y^v, X^h), \qquad G_3(X, Y) = G_2(Y, X) \tag{4.17}$$

for any vector fields X, Y on T^*M.

The corresponding ${}^\nabla G$ matrices are given by

$${}^\nabla G_1 = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \quad {}^\nabla d\theta = \begin{pmatrix} B_1 & -I \\ I & 0 \end{pmatrix}$$

where $A_1(p, u^*, \xi) = (\xi^t) \cdot \xi$, $B_1(p, u^*, \xi) = (\psi(p, u^*, \xi)(\boldsymbol{T}(u_i, u_j)))$ and $\boldsymbol{T}$ denotes the torsion tensor of ∇. The first matrix follows immediately from the definition of G_1, whereas the value of ${}^\nabla d\theta$ can be checked, for example, by using local coordinates. As one observes, the matrices ${}^\nabla G_i$ do not depend on $(p, u^*) \in L^*M$.

5. Natural (0,2)-tensor fields on T^*M.

Definition 5.1. A (0,2)-tensor field G on T^*M will be called *natural with respect to* ∇ if its matrix ${}^\nabla G$ depends only on ξ.

As we pointed out in [1], the only natural (0,2)-tensor field on the tangent bundle with respect to a linear connection is the null tensor. In contrast, on the cotangent bundle the set of natural (0,2)-tensor fields defines a three dimensional real vector space.
The following result characterizes all natural (0,2)-tensor fields with respect to linear connections.

THEOREM 5.1. *Let ∇ be a linear connection on M and G a (0,2)-tensor field on T^*M. The following are equivalent*

(1) *G is natural with respect to ∇*

(2) *There exist constants* λ_1, λ_2, λ_3 *such that*

$$G = \lambda_1 \, . \, G_1 + \lambda_2 \, . \, G_2 + \lambda_3 \, . \, G_3 .$$

PROOF. (2) $\Rightarrow$ (1). This is clear, since G_1, G_2 and G_3 are natural with respect to ∇.

(1) $\Rightarrow$ (2). Since ${}^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$ depends only on ξ, each matrix function A_i can be viewed as a function $A_i\colon \boldsymbol{R}^n \to \boldsymbol{R}^{n \times n}$. From (4.4) and (4.11)-(4.14) it follows that

$$A_1(\xi \, . \, a) = a^t . \, A_1(\xi) \, . \, a \tag{5.1}$$

$$A_2(\xi \, . \, a) = a^t . \, A_2(\xi) \, . (a^{-1})^t \tag{5.2}$$

$$A_3(\xi \, . \, a) = a^{-1} . \, A_3(\xi) \, . (a^{-1})^t \tag{5.3}$$

$$A_4(\xi \, . \, a) = a^{-1} . \, A_4(\xi) \, . \, a \tag{5.4}$$

for any $a \in GL(n, \boldsymbol{R})$ and $\xi \in \boldsymbol{R}^n$. In particular, for any $a \in \mathcal{O}(n)$ and $\xi \in \boldsymbol{R}^n$ these functions A_i $(i = 1, \ldots, 4)$ satisfy

$$A_i(\xi \, . \, a) = a^t . \, A_i(\xi) \, . \, a \tag{5.5}$$

Lemma 3.1 of [1], implies that there exist differentiable functions α_i, $\beta_i\colon [0, +\infty) \to \boldsymbol{R}$ $(1 \leq i \leq 4)$ such that

$$A_i(\xi) = \alpha_i(|\xi|^2) \, . \, I + \beta_i(|\xi|^2) \, . (\xi^t) \, . \, \xi \tag{5.6}$$

for any $\xi \in \boldsymbol{R}^n$. Here, $|\;|$ denotes the norm induced by the usual scalar product $\langle , \rangle$ on $\boldsymbol{R}^n$. Equality (5.6) applied to any $a \in GL(n, \boldsymbol{R})$ implies, from (5.1) to (5.4), that $\alpha_1 = \beta_2 = \alpha_3 = \beta_3 = \beta_4 = 0$ and β_1, α_2, α_4 are constants λ_1, λ_2, λ_3.

REMARK 5.1. The Riemannian extension of ∇ to T^*M is defined in the literature as the (0,2)-tensor field $\tilde{G}$ on T^*M given by $\tilde{G} = G_2 + G_3$. From the theorem one sees that a symmetric (0,2)-tensor field is natural with respect to ∇ if and only if there exist constants a, b such that $G = a \, . \, \tilde{G} + b \, . \, \theta^2$. Hence, from the main result in [7], it follows that the concept of natural symmetric (0,2)-tensor field with respect to linear connections introduced by Sekizawa , agrees with the one given by us.

Let $\langle , \rangle$ be a Riemannian metric on M, ∇ the Levi-Civita connection of $\langle , \rangle$ and $\mathcal{O}^*(M)$ the bundle of orthonormal co-frames over $(M, \langle , \rangle)$. Let now $N = \mathcal{O}^*(M) \times \boldsymbol{R}^n$ and $\psi : N \to T^*M$ be the projection map $\psi(p, u^*, \xi) = \sum_{i=1}^{n} \xi_i u^i$.

The family of maps $\boldsymbol{R}_a: N \to N$, $a \in \mathcal{O}(n)$, given by $\boldsymbol{R}_a(p, u^*, \xi) = (p, u^*. a, \xi. a^t)$ defines the action of $\mathcal{O}(n)$ on N and verifies $\psi \circ \boldsymbol{R}_a = \psi$. Just as in section 4 —with $\mathcal{O}(n)$ replacing $GL(n, \boldsymbol{R})$ and considering each of the basis $u = \{u_1, \dots, u_n\}$ to be orthonormal— we get, for any (0,2)-tensor field G on T^*M, the global matrix representation

$$G(X, Y) \circ \psi = {}^{\nabla}X \,.\, {}^{\nabla}G \,.\, ({}^{\nabla}Y)^t$$

where ${}^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$ and the functions $A_i: N \to \boldsymbol{R}^{n \times n}$ satisfy, for any $a \in \mathcal{O}(n)$ and $1 \leq i \leq 4$,

$$(5.7) \qquad A_i \circ \boldsymbol{R}_a = a \,.\, A_i \,.\, a^t.$$

Hence, for a fixed Riemannian metric on M, we get a one to one correspondence ${}^{\nabla}G \leftrightarrow T$ between (0,2)-tensor fields on T^*M and the differentiable matrix functions $T = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}: N \to \boldsymbol{R}^{2n \times 2n}$, where each A_i satisfies (5.7).

DEFINITION 5.2. A (0,2)-tensor field G on T^*M will be called *natural with respect to* $\langle , \rangle$ if its matrix ${}^{\nabla}G$ depends only on ξ.

According to (5.7), Lemma 3.1 in [1] implies

THEOREM 5.2. *Let $\langle , \rangle$ be a Riemannian metric on M and G a (0,2)-tensor field on T^*M. The following are equivalent*

(1) *G is natural with respect to $\langle , \rangle$*

(2) *If ${}^{\nabla}G = \begin{pmatrix} A_1 & A_2 \\ A_4 & A_3 \end{pmatrix}$, with $A_i: \boldsymbol{R}^n \to \boldsymbol{R}^{n \times n}$, there exist differentiable functions $\alpha_i, \beta_i: [0, +\infty) \to \boldsymbol{R}$ such that $A_i(\xi) = \alpha_i(|\xi|^2) \,.\, I + \beta_i(|\xi|^2) \,.\, \xi^t \,.\, \xi$.*

REMARK 5.2. From remark 3.1 in [1], it follows that natural (0,2)-tensor fields on tangent and cotangent bundles with rspect to Riemannian metrics have the same matrices ${}^{\nabla}G$.

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