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Finitely Generated Soluble Groups with an Engel Condition on Infinite Subsets.

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ABSTRACT - In this note, we prove that, in every finitely generated soluble group G, $G/Z_2(G)$ is finite if and only if in every infinite subset X of G there exist different x, y such that $[x, y, y] = 1$.

B. H. Neumann proved in [9] that a group G is centre-by-finite if and only if every infinite subset X of G contains two different commuting elements. This answered a question posed by Paul Erdös. Extensions of problems of this type are studied in [1], [4], [5], [8] and [11].

We denote by $E(\infty)$ (respectively, $N(\infty)$) the class of groups G such that, every infinite subset X of G, contains different elements x and $y \in X$ such that $[x,_k y] = 1$ (respectively, $\langle x, y \rangle$ is nilpotent of class at most k) for some $k = k(x, y) \geq 1$. If the integer k is the same for all infinite subsets of G, we say that G is in the class $E_k(\infty)$ (respectively, $N_k(\infty)$).

It is easy to see that the above classes are closed with respect to forming subgroups and homomorphic images.

In [6] J. C. Lennox and J. Wiegold studied the class $N(\infty)$ and proved that a finitely generated soluble group is in $N(\infty)$ if and only if it is finite-by-nilpotent.

Also, in [7] P. Longobardi and M. Maj studied the class $E(\infty)$ and proved that a finitely generated soluble group is in $E(\infty)$ if and only if it is finite-by-nilpotent. Moreover, they proved that a finitely generated soluble group G is in $E_2(\infty)$ if and only if $G/R(G)$ is finite, where $R(G)$ is

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the characteristic subgroup of G consisting of all right 2-Engel elements of G.

In [2] and [3] C. Delizia proved that, a finitely generated soluble (or residually finite) group G is in $N_2(\infty)$ if and only if $G/Z_2(G)$ is finite.

Here we prove the following:

THEOREM. *Let G be a finitely generated soluble group. Then* $G \in E_2(\infty)$ *if and only if* $G/Z_2(G)$ *is finite.*

PROOF. Let G be a finitely generated soluble $E_2(\infty)$-group. By Theorem 1 of [7], G contains a finite normal subgroup N such that G/N is torsion-free nilpotent. Now by Theorem 2 of [7], $R(G)$ has finite index in G, where $R(G) = \{a \in G \mid [a, x, x] = 1 \text{ for all } x \in G\}$, thus $R(G)\,N/N$ has finite index in G/N. So $R(G)\,N/N$ is a torsion-free 2-Engel group, therefore by Theorem 7.14 in [10], $R(G)\,N/N$ is nilpotent group of class at most 2. Since G/N is torsion-free nilpotent and $R(G)\,N/N$ is of finite index in G/N, thus G/N is nilpotent group of class at most 2. We note that G is residually finite since it is a finitely generated nilpotent-by-finite group. Thus it contains a normal subgroup L of finite index such that $L \cap N = 1$. Now $[L, G, G] \leq N \cap L = 1$. Then $L \leq Z_2(G)$ as required to be shown.

Conversely, if $G/Z_2(G)$ is finite and $\{x_i \colon i \in I\}$ is an infinite set of elements of G, there exist $i, j \in I$ with $i \neq j$ such that $x_i \equiv x_j \bmod Z_2(G)$. Therefore $x_i x_j^{-1} = z \in Z_2(G)$, so $\langle x_i, x_j \rangle = \langle z, x_j \rangle$ is nilpotent of class at most 2. Hence $G \in N_2(\infty) \subset E_2(\infty)$.

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