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Blow-up of Oriented Boundaries.

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ABSTRACT - Blow-up techniques are used in Analysis and Geometry to investigate local properties of various mathematical objects, by means of their observation at smaller and smaller scales. In this paper we deal with the blow-up of sets of finite perimeter and, in particular, subsets of $\boldsymbol{R}^n$ $(n \geq 2)$ with prescribed mean curvature in L^p. We prove some general properties of the blow-up and show the existence of a «universal generator», that is a set of finite perimeter that generates, by blow-up, any other set of finite perimeter in $\boldsymbol{R}^n$. Then, *minimizing* sets are considered and for them we derive some results: more precisely, Theorem 3.5 implies that the blow-up of a set with prescribed mean curvature in L^n either gives *only* area-minimizing cones with the same *surface measure* or produces a family of area-minimizing cones whose surface measures fill a continuum of the real line, while Theorem 3.9 states a sufficient condition for the uniqueness of the tangent cone to a set with prescribed mean curvature in L^p, $p > n$.

Introduction.

Blow-up techniques are widely used in Analysis and Geometry to investigate, for example, the local properties of various mathematical objects: in order to obtain information about the behavior of a surface near a point, it may be useful to observe the surface at smaller and smaller scales; in many interesting cases, this «magnification process» can help to discover some remarkable asymptotic properties.

In the theory of sets of locally finite perimeter (*Caccioppoli sets*) this technique consists in «enlarging» a given set E of finite perimeter in $\boldsymbol{R}^n$ with respect to a point $x_0 \in \partial E$, thus constructing a sequence of dilations

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with «magnification factors» increasing towards ∞. Usually, thanks to compactness results, one can obtain from this sequence a limit set F, which will be called a *blow-up* of E (with respect to x_0: written $F \in \mathcal{BU}(E, x_0)$).

It is well known that, starting from a point of the reduced boundary of E (i.e. if $x_0 \in \partial^* E$) one gets in the limit the so called *tangent half-space to E at x_0* (see e.g. Theorem 3.7 in [7] or Section2.3 in [13]).

Also, if E is a set of least perimeter (with respect to compact variations) then one gets an asymptotic *area-minimizing cone* F, which is a half-space if and only if x_0 is a regular point of ∂E (see [13], Section2.6.1).

Various and deep results, obtained by several authors from 1960 to 1970, have yielded the proof of a fundamental regularity theorem, which says that an area-minimizing boundary in $\boldsymbol{R}^n$ can be decomposed in a real, analytic submanifold of codimension 1 and a closed, singular set of Hausdorff dimension at most $n-8$ (De Giorgi - Federer's Theorem, cfr. [7] and [13]). Nevertheless, the problem of the uniqueness of the tangent cone to a least-area boundary (at a singular point) is still open (see, for instance, [1]).

Indeed, regularity results of the previous type remain true in more general situations: for example, in the case of oriented boundaries with prescribed mean curvature in L^p, $p > n$ ([11], [12]), while in the case $p < n$ they are clearly false (actually, it is proved in [4] that *every* set of finite perimeter has curvature in L^1). The «borderline» case $p = n$ is more elusive and only recently (see [9]) an example of a set $E \subset \boldsymbol{R}^2$ with prescribed mean curvature in L^2, having a singular point on ∂E, has been found; however, the study of the case $p = n$ is far from its conclusion and offers many interesting cues (see however the recent work of L.Ambrosio and E.Paolini [3], where a new regularity-type result is proved): for instance, is every blow-up of a subset of $\boldsymbol{R}^n$, with boundary of prescribed mean curvature in L^n, still a minimizing cone?

In the present work, which is part of our Ph.D. thesis ([10]) to which we refer for a more detailed discussion, we will be especially concerned with this kind of problem, and more generally we will investigate the set $\mathcal{BU}(E, x_0)$ of all blow-ups of a given set E with respect to the point x_0.

We now briefly describe the contents of the following sections.

In Section 1 we introduce the main definitions and recall some well-known properties which will be useful in the following.

In Section 2, some general properties of the set $\mathcal{BU}(E, x_0)$ are firstly established; then we show the existence of a *universal generator* M, i.e. a set of finite perimeter such that $\mathcal{BU}(M, 0)$ contains all sets of finite perimeter in $\boldsymbol{R}^n$. In Section3 we firstly deal with the blow-up of subsets of $\boldsymbol{R}^n$ with prescribed mean curvature in L^n (indeed, satisfying the weaker minimality condition (3.1)). In particular, we prove that, if E is a subset of this kind, then the following alternative holds (Theorem 3.5): either $\mathcal{BU}(E)$ is exclusively made of area-minimizing cones with a common *surface measure* (i.e. the perimeter of any such cone in the unit ball of $\boldsymbol{R}^n$ is constant), or there exists a family $\{C_\lambda, \lambda \in [l, L], l < L\}$ of area-minimizing cones in $\boldsymbol{R}^n$, such that for all $\lambda \in [l, L]$, C_λ has surface measure $= \lambda$. Then we state a sufficient condition (Theorem 3.9) for the uniqueness of the tangent cone to a set with prescribed mean curvature in L^p, $p > n$. We remark that the existence of a family of area-minimizing cones whose surface measures fill a continuum, as well as the uniqueness of the tangent cone to an area-minimizing boundary, are, in their full generality, still open problems.

1. Preliminaries.

For $E, A \subset \boldsymbol{R}^n$, with $n \geq 2$, A open and E measurable, the *perimeter* of E in A is defined as follows:

$$P(E, A) = \sup\left\{ \int_E div\, g(x)\, dx : g \in C_0^1(A; \boldsymbol{R}^n),\ \|g\|_\infty \leq 1 \right\}.$$

This definition can be extended to any Borel set $B \subset \boldsymbol{R}^n$ by setting

$$P(E, B) = \inf\{P(E, A) : B \subset A, A \text{ open}\}.$$

For further properties of the perimeter we refer to [2], [7], [13].

We say that E is a set of locally finite perimeter (or a Caccioppoli set) if $P(E, A) < \infty$ for every bounded open set $A \subset \boldsymbol{R}^n$. We denote by $B_r(x)$ the open Euclidean n-ball centered in $x \in \boldsymbol{R}^n$ with radius $r > 0$, whose Lebesgue measure is $\omega_n r^n$, and by $\alpha_E(x, r)$ the *perimeter of E normalized in $B_r(x)$*, i.e.

$$\alpha_E(x, r) = r^{1-n} P(E, B_r(x)).$$

Given $F, V, A \subset \boldsymbol{R}^n$, with A open and bounded, we define the following distance between F and V in A:

$$dist(F, V; A) = |(F \triangle V) \cap A| ,$$

where $F \triangle V = (F \cup V) \setminus (F \cap V)$ and $|\cdot|$ is the Lebesgue measure in $\boldsymbol{R}^n$. Obviously we intend that F and V coincide when they differ for a set with zero Lebesgue measure. In the following we will say that a sequence $(E_h)_h$ of measurable sets *converges in* $L^1_{loc}(\boldsymbol{R}^n)$ *(or simply converges)* to a measurable set E (in symbols, $E_h \to E$) if and only if

$$dist(E_h, E; A) \xrightarrow[h \to \infty]{} 0$$

for all open bounded sets $A \subset \boldsymbol{R}^n$.

We define *deviation from minimality* of a Caccioppoli set E in an open and bounded set A the function

$$\psi(E, A) = P(E, A) - \inf \{P(F, A) : F \triangle E \subset\subset A\},$$

where $F \triangle E \subset\subset A$ means that $F \triangle E$ is relatively compact in A. In particular, the condition $\psi(E, A) = 0$ says that E has least perimeter (with respect to compact variations) in A.

DEFINITION 1.1. *Let $E, F \subset \boldsymbol{R}^n$ be Caccioppoli sets and $x_0 \in \boldsymbol{R}^n$ arbitrarily chosen. We say that F is a blow-up of E with respect to x_0 (or $F \in \mathcal{BU}(E, x_0)$) if and only if there exists a monotone increasing sequence $(\lambda_h)_h$ of positive real numbers tending to infinity and such that, if we put $E_h = x_0 + \lambda_h(E - x_0)$, the sequence E_h converges to F.*

From now on, without loss of generality, we will only consider the case $x_0 = 0 \in \partial E$ (where ∂E is the so called *measure-theoretical boundary* of E, i.e. the closed subset of the topological boundary whose elements x satisfy $0 < |E \cap B_r(x)| < |B_r(x)| = \omega_n r^n$ for all $r > 0$), because the blow-up with respect to internal and external points is, respectively, $\boldsymbol{R}^n$ and the empty set, thus not really interesting.

Moreover, we will write B_r, $P(E, r)$, $\alpha_E(r)$, $dist(F, V; r)$, $\psi(E, r)$, $P(E)$ and $\mathcal{BU}(E)$ instead of, respectively, $B_r(0)$, $P(E, B_r)$, $\alpha_E(0, r)$, $dist(F, V; B_r)$, $\psi(E, B_r)$, $P(E, \boldsymbol{R}^n)$ and $\mathcal{BU}(E, 0)$.

Properties of ψ.

(P.1) If A and B are open sets and $A \subset B$, then $\psi(E, A) \leq \psi(E, B)$.

(P.2) $\psi(\cdot, A)$ is lower semicontinuous with respect to convergence in $L^1_{loc}(A)$.

(P.3) Given an open and bounded subset A of $\boldsymbol{R}^n$, suppose that E_h and $\psi(E_h, A)$ converge, respectively, to E and $\psi(E, A)$, and moreover that $P(E, \partial B) = 0$ for some open set B relatively compact in A. Then

$$P(E_h, B) \to P(E, B).$$

(P.4) $\psi(tE, r) = t^{n-1}\psi(E, r/t)$, for every $t, r > 0$.

For the proof see, e.g., [16] and [17].

Properties of α_E

(P.5) $\alpha_E(r)$ is lower semicontinuous on $(0, \infty)$ and has bounded variation on every compact interval $[a, b] \subset (0, \infty)$.

(P.6) $\alpha_E(r)$ is left continuous on $(0, \infty)$ and for all $r > 0$

$$\alpha_E(r^+) - \alpha_E(r) = r^{1-n} P(E, \partial B_r).$$

(P.7) $\alpha_{tE}(r) = \alpha_E(r/t)$ for every $r, t > 0$. In particular, if E is a cone of vertex 0 (that is, $tx \in E$ for all $x \in E$ and $t > 0$) then α_E is a constant function coinciding with the perimeter of E in the unit ball: in the following we will refer to this quantity as the *surface measure* of the cone E.

To prove (P.5), (P.6) and (P.7) simply observe that $P(E, r)$ is non-decreasing in $(0, \infty)$, that $\lim_{s \to r^-} P(E, s) = P(E, r)$ and finally that $P(E, \overline{B_r}) = P(E, r) + P(E, \partial B_r)$.

«Mixed» properties.

(P.8) For every $s > r > 0$ we have

$$\alpha_E(s) - \alpha_E(r) + (n-1)\int_r^s t^{-n}\psi(E, t)\, dt \geqslant 0\,;$$

a straightforward consequence of this inequality and of property (P.1) is that, if $\psi(E, R) = 0$ for some $R > 0$, then α_E is non-decreasing on $(0, R)$.

(P.9) For almost every $s > r > 0$ it holds that

$$\left[\int_{\partial B_1} |\chi_E(sx) - \chi_E(rx)|\, d\mathcal{H}^{n-1}(x) \right]^2 \leq$$

$$\leq 2 \left[\alpha_E(s) - \alpha_E(r) + (n-1) \int_r^s t^{-n} P(E, t)\, dt \right] \cdot$$

$$\cdot \left[\alpha_E(s) - \alpha_E(r) + (n-1) \int_r^s t^{-n} \psi(E, t)\, dt \right];$$

here χ_E denotes the characteristic function of E (more precisely, its trace in the sense of BV-function theory, see e.g. [7], chapter 2) and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. One can observe that the left-hand side of this inequality vanishes for almost every $s > r > 0$ if and only if E coincides, up to a negligible set, with a cone with vertex 0. It follows that, if $\psi(E, R) = 0$ and α_E is constant on $(0, R)$, then E is equivalent to a cone inside B_R. On the other hand, one checks immediately that the set $S \subset \mathbf{R}^2$ defined by

$$S = \{ t(\cos(\log t + \alpha), \sin(\log t + \alpha)) : t > 0,\ 0 < \alpha < \pi \}, \tag{1.1}$$

which is «trapped» between two antipodal logarithmic spirals, verifies $\alpha_S(r) = 2\sqrt{2}$ for all $r > 0$. This shows the importance of the minimality condition $\psi(E, R) = 0$ to ensure the conicity of E inside B_R.

(P.10) If $\psi(E, R) = 0$ for some $R > 0$, then $P(E, \partial B_r) = 0$ for every $r \in (0, R)$, hence $P(E, r)$ and $\alpha_E(r)$ are continuous on $(0, R)$.

For the proof of (P.8) and (P.9) see, e.g., [13], chapter 5.

Property (P.10) is a consequence, for example, of formula (17.5) of [15]; here we give a very simple and direct proof (suggested to us by M. Miranda) which combines De Giorgi's Regularity Theorem with the fact that, if S and M are two hypersurfaces of class C^2, with mean curvature $H_S \neq H_M$ at all points of $S \cap M$, then necessarily one has

$$\mathcal{H}^{n-1}(S \cap M) = 0. \tag{1.2}$$

To show this, we proceed in the following way. Fix $x_0 \in S \cap M$, then, up to a translation and a rotation of the coordinate system, we can assume that $x_0 = 0$ and that S and M are, near 0, the graphs of two functions ϕ_S

and ϕ_M, of class C^2 over an open set A of $\boldsymbol{R}^{n-1}$, and with $\phi_S(0) = = \phi_M(0) = 0$. Now, set $\psi(y) = \phi_S(y) - \phi_M(y)$ for all $y \in A$ and consider its gradient $\nabla\psi(0) = \nabla\phi_S(0) - \nabla\phi_M(0)$. If $\nabla\psi(0) \neq 0$ then, by the implicit function theorem, the intersection $S \cap M$ is locally a submanifold of codimension 2, thus $\mathcal{H}^{n-1}$-negligible. If $\nabla\psi(0) = 0$, then we can suppose, without loss of generality, that $\nabla\phi_S(0) = \nabla\phi_M(0) = 0$; in this case, we have that $\Delta\phi_S(0) = (n-1)\,H_S(0)$ and $\Delta\phi_M(0) = (n-1)\,H_M(0)$ (where $\Delta\phi$ is the laplacian of ϕ), hence $\Delta\psi(0) \neq 0$ (for example, > 0). Therefore, $(\partial^2\psi/\partial y_i^2) > 0$ near $y = 0$ for some $i \in \{1, \ldots, n-1\}$. This forces ψ to be (locally) strictly convex in the direction of the i-th axis, hence every line parallel to the i-th axis intersects the zero set of ψ in at most 2 points, that is, this set is $\mathcal{H}^{n-1}$-negligible. This proves (1.2). The conclusion of (P.10) follows from (1.2) with $S = \partial B_r$ and $M = \partial^* E$ ($H_S = r^{-1}$, $H_M = 0$) and from the identity $P(E, \partial B_r) = \mathcal{H}^{n-1}(\partial^* E \cap \partial B_r)$ (see e.g. [7], chapter 4). ■

2. General properties of the blow-up.

In this Section we first prove some general properties of the family $\mathcal{BU}(E)$, also giving some examples, and finally we construct a «universal generator», that is a set $M \subset \boldsymbol{R}^n$ of finite perimeter for which $\mathcal{BU}(M)$ contains any other set of finite perimeter in $\boldsymbol{R}^n$.

PROPOSITION 2.1. *Choose* $F \in \mathcal{BU}(E)$, *then* $\varrho F \in \mathcal{BU}(E)$ *for all* $\varrho > 0$.

PROOF. Following the definition of $\mathcal{BU}(E)$, let us consider a sequence $E_h = \lambda_h E$ converging to F and fix $\varrho > 0$ and a bounded open set A. The new sequence $\widehat{E}_h = \varrho\lambda_h E$ is such that

$$|(\widehat{E}_h \triangle \varrho F) \cap A| = \varrho^n \left|(E_h \triangle F) \cap \varrho^{-1} A\right| \underset{h}{\rightarrow} 0\,,$$

that is $\widehat{E}_h$ tends to ϱF. ■

It follows immediately from Proposition 2.1 that, if E admits a unique blow-up (that is $\mathcal{BU}(E) = \{F\}$) then F is a cone with vertex 0. Of course, if C is a cone with vertex 0 then $\mathcal{BU}(C) = \{C\}$.

PROPOSITION 2.2. *$\mathcal{BU}(E)$ is closed with respect to convergence in* $L^1_{loc}(\boldsymbol{R}^n)$.

PROOF. Consider a sequence F_h contained in $\mathcal{BU}(E)$ and convergent to a set $W \subset \boldsymbol{R}^n$. For every $h \in \boldsymbol{N}$ there exists a sequence $E_i^h = \lambda_i^h E$ convergent to F_h and an index i_h with the property

$$dist(E_{i_h}^h, F_h; h) < h^{-1}.$$

Hence, the new sequence $E_h = \lambda_{i_h}^h E$ must converge to W, that means $W \in \mathcal{BU}(E)$. ■

PROPOSITION 2.3. $\mathcal{BU}(F) \subset \mathcal{BU}(E)$ *for every* $F \in \mathcal{BU}(E)$.

PROOF. It is an immediate consequence of Proposition 2.1 and 2.2. ■

PROPOSITION 2.4. *If* $E = \varrho E$ *for some* $\varrho > 0$, $\varrho \neq 1$, *that is,* E *is* self-homothetic, *then*

$$\mathcal{BU}(E) = \{\sigma E, \sigma > 0\}.$$

PROOF. Since $E = \varrho E$ implies $E = \varrho^{-1} E$, we can restrict ourselves to the case $\varrho > 1$. It is easy to see that $E_h := \varrho^h E = E$ for every $h \in \boldsymbol{N}$, thus necessarily $E \in \mathcal{BU}(E)$ and, from Proposition 2.1, the inclusion $\supseteq$ follows.

On the other hand, for every $F \in \mathcal{BU}(E)$ there exists a sequence $\lambda_h \uparrow \infty$, such that $E_h := \lambda_h E$ converges to F. Now, fix h and consider $i_h \in \boldsymbol{Z}$ satisfying $\varrho^{i_h} \leqslant \lambda_h < \varrho^{i_h+1}$, then set $\sigma_h = (\lambda_h / \varrho^{i_h}) \in [1, \varrho)$; recalling that $E = \varrho^j E \forall j \in \boldsymbol{Z}$, one obtains

$$\sigma_h E = \sigma_h \varrho^{i_h} E = \lambda_h E,$$

hence $\sigma_h E$ converges to F. Up to subsequences, σ_h tends to $\sigma \in [1, \varrho]$, therefore $\sigma_h E$ converges to σE and, since the limit is unique, $F = \sigma E$. ■

When a set E satisfies the condition $E \in \mathcal{BU}(E)$, we will say that E is *asymptotically self-homothetic*. It follows from Proposition 2.4 that self-homothetic sets are asymptotically self-homothetic, the converse being false in general, as will be clear later on.

REMARK 2.5. Let us consider now a Caccioppoli set E, with $0 \in \partial E$, and suppose there exist c and $R > 0$ such that

$$\alpha_E(r) \leqslant c \tag{2.1}$$

for all $0 < r < R$. Then, taken a sequence $(\lambda_h)_h$ as above, there exists a subsequence, again denoted $(\lambda_h)_h$, whose corresponding «magnified sets» E_h converge to some limit set F. In fact, for each $s > 0$, we have

$$P(E_h, s) = \lambda_h^{n-1} P(E, s\lambda_h^{-1}) = s^{n-1} \alpha_E(s\lambda_h^{-1}) \leq cs^{n-1}$$

for every h such that $\lambda_h > s/R$. We can then apply a classical compactness theorem and the conclusion follows. In particular, (2.1) guarantees that $\mathcal{BU}(E)$ is not empty.

We have seen (Proposition 2.1) that the presence of a non-conical blow-up $F \in \mathcal{BU}(E)$ implies the presence of all enlarged sets ϱF. On the other hand, we may ask what happens when $\mathcal{BU}(E)$ contains two or more elements.

Let E be a Caccioppoli set verifying (2.1) and let $F, V \in \mathcal{BU}(E)$. Suppose that for some $r > 0$ we have $dist(F, V; r) = d > 0$, then consider two sequences λ_h and μ_h such that $\lambda_h \leq \mu_h < \lambda_{h+1}$ and, moreover,

$$\lim_{h \to \infty} dist(\lambda_h E, F; r) = \lim_{h \to \infty} dist(\mu_h E, V; r) = 0 .$$

For h sufficiently large we have $dist(\lambda_h E, F; r) < d/4$ and $dist(\mu_h E, V; r) < d/4$; this last relation, together with the triangle inequality, implies that $dist(\mu_h E, F; r) > 3d/4$, so there must be $t_h \in (\lambda_h, \mu_h)$ with the property $dist(t_h E, F; r) = d/2$, because the function $dist(tE, F; r)$ is continuous with respect to the t variable. As a further consequence of the triangle inequality, we get $dist(t_h E, V; r) \geq d/2$, hence the sequence $E_h = t_h E$ converges, up to subsequences, to a blow-up W satisfying

$$dist(W, F; r) = \frac{d}{2}$$

and, consequently,

$$dist(W, V; r) \geq \frac{d}{2},$$

thanks to the continuity of $dist$ with respect to convergence in $L^1_{loc}(\boldsymbol{R}^n)$.

More generally, for every $\sigma \in (0, 1)$ it is possible to find $W_\sigma \in \mathcal{BU}(E)$ verifying

$$dist(W_\sigma, F; r) = \sigma d \quad \blacksquare$$

The following proposition is a straightforward generalization of the previous facts.

PROPOSITION 2.6. *If E verifies* (2.1), *then $\mathcal{B}\mathcal{U}(E)$ contains either a single cone or infinitely many elements.*

For further information on this and related topics we refer to [10].

It seems useful, at this point, to recall some meaningful examples:

(i) If E is an open set of class C^1, that is, E is locally the subgraph of a class C^1 function, then the blow-up of E at any point $x \in \partial E$ is the tangent half-space to E at x. More generally, this is true for the blow-up of any Caccioppoli set at every point of its *reduced boundary* (see [7], Theorem 3.7).

(ii) On the other hand, it is easy to find examples of self-homothetic sets with Lipschitz boundary that are not cones, for which, according to Proposition 2.4, $\mathcal{B}\mathcal{U}$ contains infinitely many elements; for instance, as in [14], one can take the subgraph of a sawtooth function with homothetically increasing teeth: just set

$$E = \{(x, y) \in \boldsymbol{R}^2 \colon y < f(x)\},$$

where

$$f(x) = \sum_{i=-\infty}^{+\infty} 2^i f_0(2^{-i} x),$$

$$f_0(x) = \begin{cases} x - 1 & \text{if } x \in [1, 3/2], \\ 2 - x & \text{if } x \in [3/2, 2], \\ 0 & \text{otherwise}; \end{cases}$$

it is now immediate to see that $E = 2E$.

Another (non-Lipschitz) example of this kind is the «logarithmic spiral» S defined in (1.1): actually, it is quite easy to check that $S = e^{2\pi} S$.

(iii) «Bilogarithmic spiral» (see [9] or [2], chapter 1): it is a 2-dimensional example of a set E for which $\mathcal{BU}(E)$ is made of all half-planes passing through 0. We will recall it in Section 3.

We can now ask the following question: given a set F of finite perimeter in $\boldsymbol{R}^n$, is it possible to find E such that $F \in \mathcal{BU}(E)$? The next construction leads to an affirmative answer.

Let F be a set of finite perimeter in $\boldsymbol{R}^n$ and, for $r_h = (h!)^{-1}$ and $t_h = \sqrt{h} r_h^{-1} = h!\sqrt{h}$, define

$$E = \bigcup_{h \in \boldsymbol{N}} (t_h^{-1} F) \cap (B_{r_h} - \overline{B}_{r_{h+1}}).$$

Clearly E is contained in B_1 and has finite perimeter in $\boldsymbol{R}^n$:

$$\begin{aligned} P(E) &= \sum_{h=1}^{\infty} P(t_h^{-1} F, B_{r_h} - \overline{B}_{r_{h+1}}) + \sum_{h=1}^{\infty} P(E, \partial B_{r_h}) \\ &= \sum_{h=1}^{\infty} t_h^{1-n} P(F, B_{\sqrt{h}} - \overline{B}_{\sqrt{h}/(h+1)}) + \sum_{h=1}^{\infty} P(E, \partial B_{r_h}) \\ &\leq P(F) \cdot \sum_{h=1}^{\infty} t_h^{1-n} + n\omega_n \sum_{h=1}^{\infty} r_h^{n-1} < \infty . \end{aligned}$$

We have

$$\begin{aligned} &(t_h E) \cap (B_{\sqrt{h}} - \overline{B}_{\sqrt{h}/(h+1)}) = t_h (E \cap (B_{r_h} - \overline{B}_{r_{h+1}})) = \\ &t_h (t_h^{-1} F \cap (B_{r_h} - \overline{B}_{r_{h+1}})) = F \cap (B_{\sqrt{h}} - \overline{B}_{\sqrt{h}/(h+1)}), \end{aligned}$$

hence, for a fixed $R > 0$ and for all $h > R^2$

$$t_h E = F \text{ inside } B_R - \overline{B}_{\sqrt{h}/(h+1)}.$$

This shows that, as h tends to ∞, the sequence $E_h = t_h E$ converges to F (actually, in a stronger sense).

A much more general result can be obtained with a suitable extension of the preceding construction. Let $\{F_k\}_k$ be a countable family of sets of perimeter less than 2 and dense, with respect to the convergence in $L^1_{loc}(\boldsymbol{R}^n)$, into the class $\mathcal{P}$ of sets of perimeter less than 1. The existence of such a family can be proved starting from the density of polyhedral sets into the class of all *bounded* sets with finite perimeter, as originally shown by E. De Giorgi in [5]. Indeed, standard truncation arguments, coupled with the fact that (according to the isoperimetric inequality) any set of finite perimeter in $\boldsymbol{R}^n$ ($n \geq 2$) either has finite Lebesgue measure or is the complement of a set of finite measure, then give the sequence F_k (see [10] for more details). Now, define

$$M = \bigcup_{h \in N} (t_h^{-1} F_h) \cap (B_{r_h} - \overline{B}_{r_{h+1}}).$$

with r_h and t_h as above and note that

$$\begin{aligned} P(M) &= \sum_{h=1}^{\infty} P(t_h^{-1} F_h, B_{r_h} - \overline{B}_{r_{h+1}}) + \sum_{h=1}^{\infty} P(M, \partial B_{r_h}) \\ &= \sum_{h=1}^{\infty} t_h^{1-n} P(F_h, B_{\sqrt{h}} - \overline{B}_{\sqrt{h}/(h+1)}) + \sum_{h=1}^{\infty} P(M, \partial B_{r_h}) \\ &\leq 2 \cdot \sum_{h=1}^{\infty} t_h^{1-n} + n\omega_n \sum_{h=1}^{\infty} r_h^{n-1} < \infty . \end{aligned}$$

Taken E in the class $\mathcal{P}$, there exists a subsequence F_{h_m} converging to E, hence we can conclude that $M_m = t_{h_m} M$ converges to E, as $m \to \infty$. This means that $E \in \mathcal{B}\mathcal{U}(M)$ for every $E \in \mathcal{P}$. On the other hand, from Proposition 2.1, $\mathcal{B}\mathcal{U}(M)$ contains every set E of finite perimeter, because $\lambda E \in \mathcal{P}$ for some contraction factor $\lambda < 1$. In particular, it follows that M is *asymptotically self-homothetic*, that is

$$M \in \mathcal{B}\mathcal{U}(M),$$

but at the same time it cannot be *self-homothetic* (by virtue of Proposition 2.4).

We collect the preceding results in the following

PROPOSITION 2.7. (Universal generator) *There exists $M \subset \mathbf{R}^n$ of finite perimeter, such that*

$$\mathcal{B}\mathcal{U}(M) \supset \{E : E \text{ has finite perimeter in } \mathbf{R}^n\}. \quad \blacksquare$$

3. Blow-up of minimizing sets.

We now consider Caccioppoli sets verifying the following condition:

$$\psi(E, r) = \varepsilon(r)\, r^{n-1}, \tag{3.1}$$

with $\varepsilon(r)$ infinitesimal as $r \to 0^+$; every such set E will be said *weakly-minimizing* (at the origin: actually, we are assuming that 0 lies on the measure-theoretical boundary of E). As we shall see later, this assumption leads to some interesting properties concerning the blow-up set $\mathcal{B}\mathcal{U}(E)$ (see especially Theorem 3.5) and, moreover, to some «density estimates» (see Remark 3.8). For a better understanding of this condition, it may be useful to recall the notion of *(boundaries of) sets with prescribed mean curvature*, i.e. minimizers of the functional

$$\mathcal{F}_H(G) = P(G) + \int_G H(x)\, dx\,, \tag{3.2}$$

where G is a set of finite perimeter in $\boldsymbol{R}^n$ and H is a given integrable function on $\boldsymbol{R}^n$. By using Hölder's inequality with $p > 1$, one has that

$$\psi(E, B_r(x)) \leq \int_{B_r(x)} |H|\, dy \leq \omega_n^{1-1/p} \|H\|_{L^p(B_r(x))} \cdot r^{n(1-1/p)} \tag{3.3}$$

holds true whenever E has prescribed mean curvature H (that is, E is a minimizer of (3.2)), for all $r > 0$ and $x \in \boldsymbol{R}^n$.

Different situations are determined, depending on the relation between p and n. Thus, given $R > 0$, $z \in \boldsymbol{R}^n$ and $H \in L^p_{loc}(\boldsymbol{R}^n)$, if $p > n$ then (3.3) becomes

$$\psi(E, B_r(x)) \leq C \cdot r^{n-1+2\alpha} \tag{3.4}$$

for all $0 < r < R$ and $x \in B_R(z)$, with C independent of x and r, and $\alpha = (p-n)/2p$. It turns out that (3.4) is satisfied by the solutions to a large class of least-area type problems, subject to various constraints and boundary conditions. By extending the original work of U. Massari ([11], [12]), it has been proved in [16], [17] that this last condition (stronger than (3.1)) implies the $C^{1,\alpha}$ regularity of the *reduced boundary* of E (denoted by $\partial^* E$) and the estimate of the Hausdorff dimension of the closed *singular set* $\partial E \backslash \partial^* E$ (which does not exceed $n-8$). In this case moreover, it is well known (see again [16] and [17]) that every blow-up of E is an area-minimizing cone, thanks to a monotonicity formula that will be discussed later on (Remark 3.6).

On the other side, when $1 < p < n$, singular points of ∂E can appear even in low dimension; as for the limit case $p = 1$, it has been proved that *every* set of finite perimeter in $\boldsymbol{R}^n$, $n \geq 2$, has prescribed mean curvature in $L^1(\boldsymbol{R}^n)$ (see [4]). At the same time, non-conical blow-ups can be found in $\mathcal{B}\mathcal{U}(E)$ even when E has mean curvature in L^q for *all* $q < n$: this is precisely the case of the two self-homothetic sets described in Section 3, example (ii), as it can be shown by a calibration argument (see [10]). Again, $p=1$ is a limit case: just remember that the universal generator of Proposition 2.7 is a bounded set with finite perimeter, therefore it has mean curvature in L^1!

The case $p=n$ is special and its study is, for several aspects, still open (see however the recent contribution by L.Ambrosio and E.Paolini in [3]). In 1993, E. Gonzalez, U. Massari and I. Tamanini ([9]) gave a two-dimensional example of a set E with prescribed mean curvature in L^2, with 0 as a singular boundary point and for which $\mathcal{BU}(E)$ consists of all half-planes through the origin: E is «trapped» between two bilogarithmic spirals parameterized by $\gamma_i(t)=t(\cos[\theta_i(t)], \sin[\theta_i(t)])$, where $i=1,2$, $0<t<1$ and $\theta_i(t)=\log(1-\log t)+(i-1)\pi$. In general, given R, z and H as before, when $p=n$ one gets from (3.3)

$$\psi(E, B_r(x)) = \varepsilon(r)\, r^{n-1} \tag{3.5}$$

for all $0<r<R$ and $x\in B_R(z)$, with $\varepsilon(r)$ infinitesimal as $r\to 0^+$. Again, this uniform condition is stronger than (3.1).

After this preliminary discussion of our main assumption (3.1), we start deriving from it the following *upper area-density estimate*, which implies, in particular, that $\mathcal{BU}(E)$ is not empty (see Remark 2.5). Lower area-density and volume-density estimates will be discussed in Remark 3.8.

PROPOSITION 3.1. *If E is weakly-minimizing, then*

$$\limsup_{r\to 0^+} \alpha_E(r) \leqslant \frac{n\omega_n}{2}. \tag{3.6}$$

PROOF. Fix $\eta>1$, then for every F such that $F\,\triangle\, E \subset\subset B_{\eta r}$ it holds that

$$P(E,\eta r) - P(F,\eta r) \leqslant \psi(E,\eta r) = \varepsilon(\eta r)\,\eta^{n-1} r^{n-1}.$$

Now, take $F=E\cup B_r$, then $F=E\setminus B_r$ and finally, summing the two corresponding inequalities, one obtains

$$P(E,r) \leqslant \left(\frac{n\omega_n}{2} + \varepsilon(\eta r)\eta^{n-1}\right) r^{n-1},$$

and the conclusion follows at once. ∎

We proceed with some preliminary lemmas, the first of which is a uniform convergence result – an easy consequence of pointwise convergence combined with monotonicity and the continuity of the limit.

LEMMA 3.2. *Let $I \subset \boldsymbol{R}$ be an open interval and let f_h be a sequence of non-decreasing (non-increasing) functions, pointwise converging on I to a continuous function g. Then g is non-decreasing (non-increasing) and f_h converges to g uniformly on every compact subinterval $[a, b]$ of I.*

LEMMA 3.3. *Let us consider a sequence E_h converging to F, such that for all $r > 0$ one has*

$$\lim_h \psi(E_h, r) = 0 .$$

Then, for all $r > 0$,

(i) $\psi(F, r) = 0$,

(ii) $\lim_h P(E_h, r) = P(F, r)$,

(iii) *$P(F, r)$ and $\alpha_F(r)$ are continuous and non-decreasing on $(0, \infty)$.*

PROOF. (i) follows from Property (P.2), (ii) follows from (P.3) taking account of (i) and (P.10), while (iii) follows from (i), (P.8) and (P.10). ■

LEMMA 3.4. *Suppose that E is weakly-minimizing. Then, for each $F \in \mathcal{B}\mathfrak{U}(E)$ and for every sequence $E_h = \lambda_h E$ converging to F, we have $\psi(F, r) = 0$ for all $r > 0$, α_F continuous and non-decreasing on $(0, \infty)$ and moreover*

$$\alpha_{E_h}(r) \underset{h}{\to} \alpha_F(r)$$

uniformly on every compact subset of $(0, \infty)$.

PROOF. Fix $F \in \mathcal{B}\mathfrak{U}(E)$ and consider a sequence $E_h = \lambda_h E$ converging to F. Given $r > 0$ we have (recall Property (P.4))

$$\psi(E_h, r) = \lambda_h^{n-1} \psi(E, r\lambda_h^{-1}) = \varepsilon(r\lambda_h^{-1})\, r^{n-1}.$$

From Lemma 3.3 we obtain $\psi(F, r) = 0$ and the convergence of $P(E_h, r)$ toward $P(F, r)$, for all $r > 0$, with $P(F, r)$ continuous on $(0, \infty)$; then, by using Lemma 3.2 we deduce that $P(E_h, r)$ converges to $P(F, r)$ (hence

α_{E_h} converges to α_F) uniformly on every compact subset of $(0, \infty)$. The remaining statements follow directly from Lemma 3.3. ■

We are now in a position to prove the following result:

THEOREM 3.5. *Let E be a weakly-minimizing set, i.e. satisfying* (3.1), *and define*

$$l = \liminf_{r \to 0^+} \alpha_E(r) \quad \text{and} \quad L = \limsup_{r \to 0^+} \alpha_E(r),$$

so that $0 \leq l \leq L \leq n\omega_n/2$ (recall Proposition 3.1). Then we have the following alternative:

(a) *if $l = L$ then every $F \in \mathcal{BU}(E)$ is an area-minimizing cone with surface measure $\alpha_F = l$;*

(b) *if $l < L$ then, for each $\lambda \in [l, L]$, there exists an area-minimizing cone $C_\lambda \in \mathcal{BU}(E)$ with surface measure $\alpha_{C_\lambda} = \lambda$.*

PROOF.

(*a*) Let us consider a sequence $E_h = \lambda_h E$ converging to F and observe that, for all $r > 0$,

$$\alpha_F(r) = \lim_h \alpha_{E_h}(r) = \lim_h \alpha_E(r\lambda_h^{-1}) = l$$

by virtue of Lemma 3.4 and Property (P.7). Finally, from minimality of F (see again Lemma 3.4) and from (P.9), we immediately deduce that F is a cone.

(*b*) Fix two sequences a_i, z_i of positive real numbers, decreasing toward 0 and such that

$$\lim_i \alpha_E(a_i) = L, \qquad \lim_i \alpha_E(z_i) = l, \qquad z_i \in (a_{i+1}, a_i).$$

For $r \in (0, 1)$ define (keeping in mind (P.6))

$$\eta(r) = \sup_{\varrho \leq r} \varrho^{1-n} P(E, \partial B_\varrho) = \sup_{\varrho \leq r} [\alpha_E(\varrho^+) - \alpha_E(\varrho)]$$

and observe that $\eta(r) = 0$ if and only if α_E is continuous on $(0, r]$. We claim that

$$\lim_{r \to 0^+} \eta(r) = \limsup_{r \to 0^+} \frac{P(E, \partial B_r)}{r^{n-1}} = 0; \tag{3.7}$$

to see this we argue by contradiction: if (3.7) did not hold, there would exist a sequence $\varrho_h \downarrow 0$ and a real number $\varepsilon > 0$ such that $\alpha_E(\varrho_h^+) - - \alpha_E(\varrho_h) \geqslant \varepsilon$, that is

$$\alpha_{\varrho_h^{-1}E}(1^+) - \alpha_{\varrho_h^{-1}E}(1) \geqslant \varepsilon > 0 . \tag{3.8}$$

Now, up to subsequences, $E_h = \varrho_h^{-1} E$ must converge to an area-minimizing set M (from Lemma 3.4), with uniform convergence of α_{E_h} to α_M in a neighborhood of $r = 1$. This contradicts (3.8), because of the continuity of α_M, thus proving our claim.

Fix now $\lambda \in (l, L)$ and consider the interval $I(r) = [\lambda - \eta(r), \lambda + + \eta(r)]$. For sufficiently large i we have

$$I(a_{i-1}) \subset (l, L) , \tag{3.9}$$

$$\alpha_E(z_{i-1}) < \lambda - \eta(a_{i-1}) , \tag{3.10}$$

$$\alpha_E(a_i) > \lambda + \eta(a_{i-1}) . \tag{3.11}$$

Define

$$b_i = \inf \{ b \in (a_i, z_{i-1}) : \alpha_E(b) \in I(a_{i-1}) \} .$$

where the previous set is certainly not empty by virtue of (3.9), (3.10) and (3.11). Observe that the infimum b_i is actually a minimum, thanks to (P.5), hence for all $r \in [a_i, b_i)$ we get

$$\alpha_E(r) > \alpha_E(b_i) = \lambda + \eta(a_{i-1}) \tag{3.12}$$

with the help of (P.6). At this point, we claim that

$$\lim_i \frac{a_i}{b_i} = 0 . \tag{3.13}$$

Indeed, if this were false, we would have, passing possibly to a subsequence,

$$\lim_i \frac{a_i}{b_i} = k \in (0, 1] .$$

By setting $E_i = b_i^{-1} E$ and using formula (P.7) we would also obtain

$$\alpha_{E_i}(1) = \alpha_E(b_i) \quad \text{and} \quad \alpha_{E_i}\left(\frac{a_i}{b_i} \right) = \alpha_E(a_i) ,$$

while, up to subsequences, E_i would converge to an area-minimizing set

M (Lemma 3.4). From the uniform convergence of α_{E_i} to α_M, together with the monotonicity and continuity of α_M, it would follow

$$L = \lim_i \alpha_E(a_i) = \lim_i \alpha_{E_i}\left(\frac{a_i}{b_i}\right) = \alpha_M(k) \leqslant$$

$$\alpha_M(1) = \lim_i \alpha_{E_i}(1) = \lim_i \alpha_E(b_i) = \lambda\,,$$

thanks to (3.12) and (3.7). This contradicts our assumption $\lambda < L$ and proves (3.13).

Now, choose $r \in (0, 1)$. Then there exists i_r such that $\frac{a_i}{b_i} < r$ for every $i \geqslant i_r$. Taking $E_i = b_i^{-1} E$ and M as before and observing that $rb_i \in (a_i, b_i)$ whenever $i \geqslant i_r$, we get, as a consequence of (3.12),

$$\alpha_M(r) = \lim_i \alpha_{E_i}(r) = \lim_i \alpha_E(rb_i) \geqslant \lim_i \alpha_E(b_i) = \alpha_M(1)\,,$$

and the monotonicity of α_M yields, for all $r < 1$,

$$\alpha_M(r) = \alpha_M(1) = \lambda\,.$$

From (P.9) we deduce that (up to a negligible set) M coincides, in B_1, with an area-minimizing cone, hence, by analytic continuation $C_\lambda = M$ must be a cone in $\boldsymbol{R}^n$ with surface measure $\alpha_{C_\lambda} = \lambda$.

When $\lambda = l$ (respectively, $\lambda = L$) the argument is virtually the same, but calculations are much simpler: the sequence $z_i^{-1} E$ (resp., $a_i^{-1} E$) produces a limit cone of least area, with surface measure l (resp., L). ■

REMARK 3.6. The case (a) of Theorem 3.5 applies, in particular, to boundaries with prescribed mean curvature in L^p, $p > n$: indeed, from (3.4) and (P.8) one deduces that the function

$$\alpha_E(r) + \frac{(n-1)\,C}{2\alpha} r^{2\alpha}$$

is non-decreasing in r, thus admitting a limit as $r \to 0^+$, hence $l = L$. More generally, this holds if merely $\int_0^R r^{-n} \psi(E, r)\, dr < \infty$, which is indeed the case of the «bilogarithmic spiral» quoted above (see especially [9], Remark 2.3, or [10]).

REMARK 3.7. As shown above, if $E \subset \boldsymbol{R}^n$ has prescribed mean curvature in L^n, then it is a weakly-minimizing set (actually, in a «uniform way»). Therefore, the alternative of Theorem 3.5 applies as well (for example, the bilogarithmic spiral falls within the case (a) of that theorem). On the other hand, it is well known that, in dimension $n \leq 7$, area-minimizing cones are either half-spaces (see [7], [13]) or trivial cones ($\boldsymbol{R}^n$ and $\emptyset$), hence in this case $l = L$. In higher dimension, the existence of families of area-minimizing cones with surface measures filling a continuum represents an open problem which seems quite interesting in itself. However, in the special case of dimension $n = 8$ we conjecture that such a family cannot exist and will investigate this fact in a future work.

REMARK 3.8. Area-density and volume-density estimates for sets of *least perimeter* are well known in the literature: if E is any such set, then one has $\omega_{n-1} \leq \alpha_E(x, r) \leq n\omega_n/2$ for all $r > 0$ and $x \in \partial E$ (see, for instance, [6], pp. 52 and 55). We have seen (Proposition 3.1) that weak-minimality is sufficient to give the (asymptotic) upper area-density estimate

$$\limsup_{r \to 0^+} \alpha_E(r) \leq n\omega_n/2 ,$$

however we cannot expect an analogous estimate from below: for example, if 0 is a cuspidal point of a Caccioppoli set $E \subset \boldsymbol{R}^3$ such that

$$\lim_{r \to 0^+} \alpha_E(r) = \lim_{r \to 0^+} \frac{|E \cap B_r|}{r^n} = 0 ,$$

then (3.1) is clearly true, because $\psi(E, r) \leq P(E, r) = \alpha_E(r)\, r^{n-1}$.

A straightforward consequence of Theorem 3.5 is, again, the following alternative: keeping the same notation, we have that either $L = 0$, in which case $r^{-n}\min(|E \cap B_r|, |E^c \cap B_r|) \to 0$ as $r \to 0^+$ owing to the isoperimetric inequality on balls, or $l \geq \omega_{n-1}$, in which case one gets by integration (see [10]) the *volume-density estimate*

$$\liminf_r r^{-n}\min(|E \cap B_r|, |E^c \cap B_r|) \geq \omega_{n-1}/n .$$

To see this latter fact, suppose $L > 0$, then L must be the surface measure of a non trivial area-minimizing cone C_L, whence $\omega_{n-1} \leq L \leq$ $\leq n\omega_n/2$; if $l = L$ the assertion follows, otherwise we still have that λ is the surface measure of some non-trivial area-minimizing cone C_λ for every $l < \lambda \leq L$, hence $\lambda \geq \omega_{n-1}$ and therefore $l \geq \omega_{n-1}$ as well. Actually, recalling that ω_{n-1} is an isolated value for the surface measure of area-minimizing cones (see [8]), we would have in this last case

$$\omega_{n-1} < l < L \leqslant n\omega_n/2\,,$$

that is, every C_λ is a *singular* cone. Lower area-density estimates are well known for boundaries with prescribed mean curvature in L^p, $p \geqslant n$. More precisely, if $p > n$ one obtains $l \geqslant \omega_{n-1}$ by means of monotonicity (Remark 3.6) combined with a density argument (see [12], [16]); this fact still remains true when $p = n$, because in this case one firstly obtains $l > 0$ (see, e.g., [9], [10]) so that, by the previous argument, l must actually be greater than or equal to ω_{n-1}.

Finally, we consider the problem of the *uniqueness* of the tangent cone to a minimizing set E: by a careful use of Property (P.9) we show that uniqueness is implied by some assumptions on the initial behavior of α_E (e.g., when α_E is Hölder-continuous near 0). The following result stands as a model in this direction (see [10] for a more detailed discussion):

THEOREM 3.9. *Let E be such that*

$$\psi(E, r) \leqslant c \cdot r^{n-1+\varepsilon}$$

for some R, c, $\varepsilon > 0$ and for all $0 < r < R$ (in particular, this holds for sets with prescribed mean curvature in L^p, $p > n$, as (3.4) says), and suppose $\alpha_E(r)$ be of class $C^{0,\beta}$ in $(0, R)$. Then $\mathcal{BU}(E)$ contains a unique area-minimizing cone.

PROOF. We know from the preceding discussion (Remark 3.6) that each member of $\mathcal{BU}(E)$ is an area-minimizing cone, so only uniqueness has to be proved. For that, it is sufficient to show

$$\lim_{r, s \to 0^+} \Delta_{r,s} = 0\,, \tag{3.14}$$

where

$$\Delta_{r,s} := \int_{\partial B_1} |\chi_E(rx) - \chi_E(sx)|\, d\mathcal{H}^{n-1}(x)\,.$$

Indeed, this condition means that the trace of tE on the boundary of B_1 has the Cauchy property (with respect to the $L^1(\partial B_1)$ norm) when $t \to \infty$, therefore it is equivalent to the uniqueness of the tangent cone.

Choose $0 < r < s < R$ with s sufficiently small, such that (recall

Proposition 3.1)

$$\alpha_E(t) \leqslant n\omega_n \qquad \text{for all } t \leqslant s\,, \tag{3.15}$$

and set $s_i = 2^{-i}s$, $i = 0, \ldots, k+1$, with $s_{k+1} \leqslant r < s_k$. By using the triangle inequality, we have

$$\Delta_{r,s} \leqslant \sum_{i=0}^{k-1} \Delta_{s_{i+1}, s_i} + \Delta_{r, s_k}. \tag{3.16}$$

and then, applying Property (P.9) together with (3.15) and the minimality assumption on E, we obtain

$$\Delta_{s_{i+1}, s_i} \leqslant \sqrt{2}\left[\alpha_E(s_i) - \alpha_E(s_{i+1}) + (n-1)\int_{s_{i+1}}^{s_i} t^{-1}\alpha_E(t)\,dt\right]^{1/2} \cdot \tag{3.17}$$

$$\cdot\left[\alpha_E(s_i) - \alpha_E(s_{i+1}) + (n-1)\int_{s_{i+1}}^{s_i} t^{-n}\psi(E, t)\,dt\right]^{1/2} \leqslant$$

$$\leqslant C_1[\alpha_E(s_i) - \alpha_E(s_{i+1}) + C_2\varepsilon^{-1}(s_i^\varepsilon - s_{i+1}^\varepsilon)]^{1/2},$$

where $C_1 = [2n\omega_n(1 + (n-1)\log 2)]^{1/2}$ and $C_2 = (n-1)\,c$. Similarly, also using the monotonicity of $\alpha_E(t) + C_2\varepsilon^{-1}t^\varepsilon$ (Remark 3.6), we deduce

$$\Delta_{r, s_k} \leqslant C_1[\alpha_E(s_k) - \alpha_E(s_{k+1}) + C_2\varepsilon^{-1}(s_k^\varepsilon - s_{k+1}^\varepsilon)]^{1/2}. \tag{3.18}$$

Then, combining (3.16), (3.17) and (3.18) with the assumption about α_E, we get

$$\Delta_{r,s} \leqslant C_1\sum_{i=0}^{\infty}[C_3 2^{-(i+1)\beta}s^\beta + C_2\varepsilon^{-1}(2^\varepsilon - 1)2^{-(i+1)\varepsilon}s^\varepsilon]^{1/2}$$

$$\leqslant C_4 s^{\beta/2} + C_5 s^{\varepsilon/2},$$

and (3.14) follows at once. ■

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