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Partial Hölder Continuity Results for Solutions of non Linear non Variational Elliptic Systems with Strictly Controlled Growth.

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SUNTO - In un aperto limitato Ω si considera il sistema non lineare non variazionale

$$(1) \quad a(x, u, Du, H(u)) = b(x, u, Du)$$

dove $a(x, u, \mu, \xi)$ e $b(x, u, \mu)$ sono vettori di \mathbb{R}^N , $N \geq 1$ misurabili in x e continui nelle altre variabili. Si dimostra che se $u \in H^2(\Omega)$ è soluzione in Ω del sistema (1), se $b(x, u, \mu)$ ha andamenti strettamente controllati, se $a(x, u, \mu, \xi)$ è di classe C^1 in ξ e verifica la condizione (A) e, unitamente a $\partial a / \partial \xi$, certe condizioni di continuità, allora il vettore Du è parzialmente holderiano in Ω con ogni esponente $\alpha < 1 - n/p$.

1. Introduction.

Let Ω be a bounded open subset of \mathbb{R}^n , $n > 4$ of class C^2 and let $x = (x_1, x_2, \dots, x_n)$ be a generic point in it.

The symbols $(\cdot | \cdot)_k$ and $\|\cdot\|_k$ denote the scalar product and the norm in \mathbb{R}^k , respectively. We shall omit the index k wherever there is no ambiguity.

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In what follows we denote by $\mu = (\mu_1, \mu_2, \dots, \mu_n)$, $\mu_i \in \mathbb{R}^N$, N integer ≥ 1 , a generic vector of \mathbb{R}^{nN} and by $\xi = \{\xi_{ij}\}$ $i, j = 1, 2, \dots, n$, $\xi_{ij} \in \mathbb{R}^N$ a generic element of \mathbb{R}^{n^2N} .

If $u: \Omega \rightarrow \mathbb{R}^N$ is a vector, we set

$$D_i u = \frac{\partial u}{\partial x_i}; \quad Du = (D_1 u, D_2 u, \dots, D_n u);$$

$$H(u) = \{D_i D_j u\} = \{D_{ij} u\}; \quad i, j = 1, 2, \dots, n.$$

We consider in Ω the following second order non linear non variational system

$$(1.1) \quad a(x, u, Du, H(u)) = b(x, u, Du)$$

where $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ are vectors in \mathbb{R}^N , measurable in x , continuous in (u, μ, ξ) and (u, μ) respectively, satisfying the conditions:

$$(1.2) \quad a(x, u, \mu, 0) = 0.$$

(A) *there exist three positive constants $\bar{\alpha}, \bar{\gamma}$ and $\bar{\delta}$ with $\bar{\gamma} + \bar{\delta} < 1$, such that, $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$, $\forall \tau, \eta \in \mathbb{R}^{n^2N}$ and for almost every $x \in \Omega$, we have⁽¹⁾*

$$\left\| \sum_{i=1}^n \tau_{ii} - \bar{\alpha}[a(x, u, \mu, \tau + \eta) - a(x, u, \mu, \eta)] \right\|^2 \leq \bar{\gamma} \|\tau\|^2 + \bar{\delta} \left\| \sum_{i=1}^n \tau_{ii} \right\|^2$$

(1.3) *there exists a constant c such that, $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$ and for almost every $x \in \Omega$ we have*

$$\|b(x, u, \mu)\| \leq c(f(x) + \|u\|^\alpha + \|\mu\|^\beta)$$

⁽¹⁾ From condition (A), assuming $\eta = 0$, it follows, $\forall u \in \mathbb{R}^N$, $\forall \mu \in \mathbb{R}^{nN}$, $\forall \tau \in \mathbb{R}^{n^2N}$ and for almost all $x \in \Omega$

$$\|a(x, u, \mu, \tau)\| \leq c\|\tau\|.$$

Moreover one can show that, if the vector $a(x, u, \mu, \xi)$ is of class C^1 with respect to ξ , with derivatives $\partial a / \partial \xi_{ij}$ bounded, then the operator $a(x, u, \mu, \xi)$ is elliptic (see [7]).

Sufficient conditions that ensure the hypothesis (A) are stated in [4] and [6].

with $f \in L^2(\Omega)$ and with

$$1 \leq \alpha < \frac{n}{n-4}, \quad 1 \leq \beta < \frac{n}{n-2}.$$

We shall denote by $H^{s,p}(\Omega, \mathbb{R}^N)$, $H_0^{s,p}(\Omega, \mathbb{R}^N)$, s integer ≥ 0 , $p \in [1, +\infty)$, the usual Sobolev spaces⁽²⁾

By a solution to the system (1.1) we mean a vector $u \in H^2(\Omega, \mathbb{R}^N)$ satisfying (1.1) for almost all $x \in \Omega$. In this work we obtain a partial Hölder continuity result for the gradient of these solutions; this result is similar to the one proved in the case of «linear growth» by S. Campanato in [5] for the elliptic systems and by M. Marino - A. Maugeri in [8] in the parabolic case.

To achieve the partial Hölder regularity of the gradient we need to preliminary obtain the local L^q -regularity of the derivatives $D_{ij}u$. Section n° 2 is devoted to the above study, which in itself is of interest.

2. Local L^q -regularity of the matrix $H(u)$.

Let $u \in H^2(\Omega, \mathbb{R}^N)$ be a solution in Ω to the non variational system

$$(2.1) \quad a(x, u, Du, H(u)) = b(x, u, Du)$$

being $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ vectors of \mathbb{R}^N satisfying assumptions (1.2), (1.3) and (A).

Proceeding with the same technique used in [5] to obtain the estimate

⁽²⁾ If s, j are integers ≥ 0 and $p \in [1, +\infty)$,

$$|u|_{j,p,\Omega} = \left[\int_{\Omega} \left(\sum_{|\alpha|=j} \|D^\alpha u\|^2 \right)^{p/2} dx \right]^{1/p};$$

$$\|u\|_{s,p,\Omega} = \left\{ \sum_{j=0}^s |u|_{j,p,\Omega}^p \right\}^{1/p}$$

where $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}, \dots, D_n^{\alpha_n}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$, α_j integer ≥ 0 .

Particularly $|u|_{0,p,\Omega} = \|u\|_{0,p,\Omega} = \|u\|_{L^p(\Omega, \mathbb{R}^N)}$.

(3.2) of lemma (3.1), we have:

$$(2.2) \quad \int_{B(x^0, \sigma)} \|H(u)\|^2 dx \leq c\sigma^{-2} \int_{B(x^0, 2\sigma)} \|D(u) - (Du)_{2\sigma}\|^2 dx + \\ + c \int_{B(x^0, 2\sigma)} \|b(x, u, Du)\|^2 dx \text{ }^{(3)}\text{ }^{(4)}\text{ }^{(5)}.$$

To prove the L^q -local regularity result of the matrix $H(u)$, we suppose that, in the assumption (1.3), we have

$$(2.3) \quad f \in L^{q_0}(\Omega) \quad \text{with } 2 < q_0 \leq 2^*.$$

Moreover by the theorem of Poincaré, we have

$$(2.4) \quad \int_{B(x^0, 2\sigma)} \|Du - (Du)_{2\sigma}\|^2 dx \leq c \left(\int_{B(x^0, 2\sigma)} \|H(u)\|^{2n/(n+2)} dx \right)^{(n+2)/n}$$

where c does not depend on σ .

Now from (2.3) and from the assumption (1.3) we will also have $b(x, u, Du) \in L^{\bar{q}}_{loc}(\Omega)$ with

$$2 < \bar{q} \leq q_0 \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta}.$$

If we set

$$F(x) = \|H(u)\|^{2n/(n+2)}, \quad G(x) = \|b(x, u, Du)\|^{2n/(n+2)}.$$

The estimate (2.2) by means of the (2.4) can be written in the following manner:

$$\int_{B(x^0, \sigma)} F^{(n+2)/n} dx \leq \left(\int_{B(x^0, 2\sigma)} F dx \right)^{(n+2)/n} + c \left(\int_{B(x^0, 2\sigma)} G^{(n+2)/n} dx \right).$$

From this and by a well known lemma of Gehring - Giaquinta - G. Modica (see lemma 10.1, page 100 of [2]) written for $r = (n+2)/n$,

⁽³⁾ $B(x^0, \sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\}$.

⁽⁴⁾ If $E \subset \mathbb{R}^k$ is a measurable set with positive measure and $f \in L^1(E, \mathbb{R}^k)$, we set

$$f_E = \int_E f dx = \frac{\int_E f dx}{\text{mis } E}$$

⁽⁵⁾ $u_\sigma = u_{B(x^0, \sigma)}, (Du)_\sigma = (Du)_{B(x^0, \sigma)}, (H(u))_\sigma = (H(u))_{B(x^0, \sigma)}$.

$s = (\bar{q}(n+2))/2n$, we deduce that there exists $\varepsilon \in (0, s-r]$ such that

$$F \in L_{loc}^t(\Omega) \quad \forall t \in \left[\frac{n+2}{n}, \frac{n+2}{n} + \varepsilon \right)$$

and, $\forall B(x^0, 2\sigma) \subset\subset \Omega$, with $\sigma < 1$, one has

$$(2.5) \quad \left(\int_{B(x^0, \sigma)} F^t dx \right)^{1/t} \leq K \left\{ \left(\int_{B(x^0, 2\sigma)} F^r dx \right)^{1/r} + \left(\int_{B(x^0, 2\sigma)} G^t dx \right)^{1/t} \right\}.$$

From (2.5) written for $t = (q(n+2))/2n$ with $q \in [2, \bar{q})$ we achieve the following

THEOREM 2.1. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system (2.1) and if the assumptions (1.2), (1.3) with $f \in L^{q_0}(\Omega)$, $q_0 \leq 2^*$, and (A) hold, then there exists \bar{q}*

$$2 < \bar{q} \leq q_0 \wedge \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta},$$

such that $\forall q \in [2, \bar{q})$, $u \in H_{loc}^{2,q}(\Omega, \mathbb{R}^N)$ and $\forall B(x^0, 2\sigma) \subset\subset \Omega$, with $\sigma < 1$, one has:

$$(2.6) \quad \left(\int_{B(x^0, \sigma)} \|H(u)\|^q dx \right)^{1/q} \leq K \left\{ \left(\int_{B(x^0, 2\sigma)} \|H(u)\|^2 dx \right)^{1/2} + \left(\int_{B(x^0, 2\sigma)} \|b\|^q dx \right)^{1/q} \right\}$$

where K does not depend on σ .

Now we can show the following

LEMMA 2.1. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system (2.1) and if the assumptions (1.2), (1.3) with $f \in L^p(\Omega)$, $p > n$, and (A) hold, then there exists \bar{q} with*

$$2 < \bar{q} \leq \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta},$$

such that $\forall q \in [2, \bar{q})$, $\forall B(x^0, 2\sigma) \subset \Omega$, with $\sigma < 1$, it results:

$$(2.7) \quad \left(\int_{B(x^0, \sigma)} \|H(u)\|^q dx \right)^{1/q} \leq c \sigma^{n(1/q - 1/2)} [\phi(u, x^0, 2\sigma)]^{1/2}$$

where c does not depend on σ and

$$(2.8) \quad \phi(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx + \\ + \int_{B(x^0, \sigma)} \|D(u)\|^{2n/(n+2)} dx + \int_{B(x^0, \sigma)} \|H(u)\|^2 dx$$

with

$$\xi = n \left(1 - \frac{2}{p} \right).$$

PROOF. From (2.6) we have $\forall B(x^0, 2\sigma) \subset \Omega$, with $\sigma < 1$

$$(2.9) \quad \left(\int_{B(x^0, \sigma)} \|H(u)\|^q dx \right)^{1/q} \leq \\ \leq K \sigma^{n(1/q - 1/2)} [\phi(u, x^0, 2\sigma)]^{1/2} + \left(\int_{B(x^0, 2\sigma)} \|b\|^q dx \right)^{1/q}.$$

Now we can evaluate the last term of the right hand side of (2.9).

From assumption (1.3) it follows:

$$(2.10) \quad \|b(x, u, Du)\|^q \leq c \{ |f(x)|^q + \|u\|^{\alpha q} + \|Du\|^{\beta q} \}.$$

On the other hand, one gets

$$(2.11) \quad \left(\int_{B(x^0, 2\sigma)} |f|^q dx \right)^{1/q} \leq c \left(\int_{B(x^0, 2\sigma)} |f|^p dx \right)^{1/p} \sigma^{n(1 - q/p)(1/q)} \leq \\ \leq c \|f\|_{L^p(\Omega, \mathbb{R}^N)} \sigma^{n(1/q - 1/2)} [\phi(u, x^0, 2\sigma)]^{1/2};$$

$$\begin{aligned}
 (2.12) \quad & \left(\int_{B(x^0, 2\sigma)} \|u\|^{\alpha q} dx \right)^{1/q} \leq \\
 & \leq c \left(\int_{B(x^0, 2\sigma)} \|u\|^{2n/(n-4)} dx \right)^{\alpha(n-4)/2n} \cdot \sigma^{n(1-\alpha q(n-4)/2n)(1/q)} = \\
 & = c \sigma^{n/q} \left(\frac{\int_{B(x^0, 2\sigma)} \|u\|^{2n/(n-4)} dx}{\sigma^n} \right)^{\alpha(n-4)/2n} \leq \\
 & \leq \sigma^{n/q} \left(1 + \frac{\int_{B(x^0, 2\sigma)} \|u\|^{2n/(n-4)} dx}{\sigma^n} \right)^{1/2} \leq c \sigma^{n(1/q-1/2)} \left(\sigma^\xi + \int_{B(x^0, 2\sigma)} \|u\|^{2n/(n-4)} dx \right)^{1/2} \leq \\
 & \leq c \sigma^{n(1/q-1/2)} [\phi(u, x^0, 2\sigma)]^{1/2};
 \end{aligned}$$

$$\begin{aligned}
 (2.13) \quad & \left(\int_{B(x^0, 2\sigma)} \|Du\|^{\beta q} dx \right)^{1/q} \leq \\
 & \leq c \left(\int_{B(x^0, 2\sigma)} \|Du\|^{2n/(n-2)} dx \right)^{\beta(n-2)/2n} \cdot \sigma^{n(1-\beta q(n-2)/2n)(1/q)} \leq \\
 & \leq c \sigma^{n/q} \left(1 + \int_{B(x^0, 2\sigma)} \|Du\|^{2n/(n-2)} dx \right)^{1/2} \leq c \sigma^{n(1/q-1/2)} [\phi(u, x^0, 2\sigma)]^{1/2}.
 \end{aligned}$$

Then, from the estimates (2.10), (2.11), (2.12), (2.13), we deduce

$$(2.14) \quad \left(\int_{B(x^0, 2\sigma)} \|b\|^q dx \right)^{1/q} \leq K \sigma^{n(1/q-1/2)} [\phi(u, x^0, 2\sigma)]^{1/2}.$$

The estimate (2.7) easily follows from (2.9) and (2.14). \blacksquare

Moreover we can deduce the following

LEMMA 2.2. *If $u \in H^2(B(x^0, \sigma), \mathbb{R}^N)$ with $\sigma \in (0, 1)$ is a solution to the system (2.1) and if the assumptions (1.2), (1.3) with $f \in L^p(\Omega)$, $p > n$*

and (A) hold, then there exists \bar{q}

$$2 < \bar{q} \leq \frac{2n}{\alpha(n-4)} \wedge \frac{2^*}{\beta},$$

such that $\forall q \in [2, \bar{q})$, $\forall \tau \in (0, 1)$, one has

$$(2.15) \quad \left(\int_{B(x^0, \tau\sigma)} \|u\|^{2n/(n-4)} + \|Du\|^{2n/(n-2)} \right) dx \leq \\ \leq c\tau^n \left(\int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx + \int_{B(x^0, \sigma)} \|Du\|^{2n/(n-2)} dx \right) + \\ + c\sigma^{2(1-2/q)} \int_{B(x^0, \sigma)} \|H(u)\|^2 dx$$

where c does not depend on σ and τ .

PROOF. Let $P = (P_1, P_2, \dots, P_N)$ be the polynomial vector of degree ≤ 1 such that

$$(2.16) \quad \int_{B(x^0, \sigma)} D^\alpha(u - P) = 0 \quad \forall \alpha, |\alpha| \leq 1.$$

We obtain that

$$(2.17) \quad \int_{B(x^0, \tau\sigma)} \|u\|^{2n/(n-4)} dx \leq c(\tau\sigma)^n \|P\|^{2n/(n-4)} + \int_{B(x^0, \tau\sigma)} \|u - P\|^{2n/(n-4)} dx$$

and hence taking into account (2.16) we get

$$(2.18) \quad (\tau\sigma)^n \|P\|^{2n/(n-4)} \leq \tau^n (\|u\|_{L^{2n/(n-4)}(B(x^0, \sigma))})^{2n/(n-4)} + c(u) \tau^n \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx.$$

Now, by the theorem of Poincaré, we have

$$(2.19) \quad \left(\int_{B(x^0, \tau\sigma)} \|u - P\|^{2n/(n-4)} dx \right)^{(n-4)/2n} \leq c \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx \right)^{1/2}$$

and then from the estimates (2.17), (2.18), (2.19) we obtain

$$(2.20) \quad \int_{B(x^0, \tau\sigma)} \|u\|^{2n/(n-4)} dx \leq c\tau^n \left(\int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx \right) + \\ + c(u) \tau^n \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx + c \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx \right)^{n/(n-4)};$$

Since

$$(2.21) \quad \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx \right)^{n/(n-4)} \leq \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx \right) \sigma^{n(1-2/q)4/(n-4)}. \\ \cdot \left[\int_{B(x^0, d_x^0)} \|H(u)\|^q dx \right]^{8/q(n-4)} \leq c(u, q) \sigma^{2(1-2/q)} \int_{B(x^0, \sigma)} \|H(u)\|^2 dx,$$

from (2.20) and (2.21), we have

$$(2.22) \quad \int_{B(x^0, \tau\sigma)} \|u\|^{2n/(n-4)} dx \leq c\tau^n \left(\int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx \right) + \\ + c(u) \tau^n \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx + c(u, q) \sigma^{2(1-2/q)} \int_{B(x^0, \sigma)} \|H(u)\|^2 dx.$$

Moreover being

$$(2.23) \quad \int_{B(x^0, \tau\sigma)} \|Du\|^{2^*} dx \leq \\ \leq c \left(\int_{B(x^0, \tau\sigma)} \|Du - (Du)_\sigma\|^{2^*} dx + \int_{B(x^0, \tau\sigma)} \|(Du)_\sigma\|^{2^*} dx \right),$$

and taking into account that

$$(2.24) \quad \int_{B(x^0, \tau\sigma)} \|(Du)_\sigma\|^{2^*} dx \leq \tau^n \|Du\|_{L^{2^*}(B(x^0, \sigma))}^{2^*}$$

and that, in virtue of (3.19) of [2],

$$\begin{aligned}
 (2.25) \quad & \int_{B(x^0, \sigma)} \|Du - (Du)_\sigma\|^{2^*} dx \leq c \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx \right)^{2^*/2} \leq \\
 & \leq c \left(\int_{B(x^0, \sigma)} \|H(u)\|^2 dx \right) \sigma^{n(1-2/q)(2/(n-2))} \left[\int_{B(x^0, d_x^0)} \|H(u)\|^q dx \right]^{4/q(n-2)} \leq \\
 & \leq c(u, q) \sigma^{2(1-2/q)} \int_{B(x^0, \sigma)} \|H(u)\|^2 dx,
 \end{aligned}$$

we obtain

$$\begin{aligned}
 (2.26) \quad & \int_{B(x^0, \tau\sigma)} \|Du\|^{2^*} dx \leq c\tau^n \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx + \\
 & + c(u, q) \sigma^{2(1-2/q)} \int_{B(x^0, \sigma)} \|H(u)\|^2 dx.
 \end{aligned}$$

Therefore from (2.22) and (2.26) we deduce (2.15). ■

3. Partial Hölder continuity of the vector Du .

Let $u \in H^2(\Omega, \mathbb{R}^N)$ be a solution to the non variational system

$$(3.1) \quad a(x, u, Du, H(u)) = b(x, u, Du)$$

where $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ are vectors of \mathbb{R}^N with the following properties:

(3.2) $b(x, u, \mu)$ is measurable in x , continuous in (u, μ) and such that $\forall u \in \mathbb{R}^N, \forall \mu \in \mathbb{R}^{nN}$ and for x a.e. in Ω

$$\|b(x, u, \mu)\| \leq c\{f(x) + \|u\|^\alpha + \|\mu\|^\beta\},$$

with $f \in L^p(\Omega, \mathbb{R}^n)$, $p > n$ and with $1 \leq \alpha < n/(n-4)$, $1 \leq \beta < n/(n-2)$;

(3.3) $a(x, u, \mu, \xi)$ is continuous in (x, u, μ) , of class C^1 in ξ , with derivatives $\partial a / \partial \xi_{ij}$ ⁽⁶⁾ uniformly continuous and bounded in $\Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \times \mathbb{R}^{n^2N}$ and such that

$$a(x, u, \mu, 0) = 0,$$

(A) there exist three positive constants $\bar{\alpha}, \bar{\gamma}$ and $\bar{\delta}$ with $\bar{\gamma} + \bar{\delta} < 1$, such that, $\forall u \in \mathbb{R}^N, \forall \mu \in \mathbb{R}^{nN}, \forall \tau, \eta \in \mathbb{R}^{n^2N}$ and for almost every $x \in \Omega$

$$\begin{aligned} \left\| \sum_{i=1}^n \tau_{ii} - \bar{\alpha}[a(x, u, \mu, \tau + \eta) - a(x, u, \mu, \eta)] \right\|^2 &\leq \\ &\leq \bar{\gamma} \|\tau\|^2 + \bar{\delta} \left\| \sum_{i=1}^n \tau_{ii} \right\|^2, \end{aligned}$$

(B) there exists a non negative function $\omega(t)$, defined for $t \geq 0$, continuous, bounded, concave, non decreasing with $\omega(0) = 0$ such that $\forall x, y \in \Omega, \forall u, v \in \mathbb{R}^N, \forall \mu, \bar{\mu} \in \mathbb{R}^{nN}$ and $\forall \xi, \tau \in \mathbb{R}^{n^2N}$:

$$\|a(x, u, \mu, \xi) - a(y, v, \bar{\mu}, \xi)\| \leq \omega(d^2(x, y) + \|u - v\|^2 + \|\mu - \bar{\mu}\|^2) \cdot \|\xi\|,$$

$$\left\| \frac{\partial a(x, u, \mu, \xi)}{\partial \xi} - \frac{\partial a(x, u, \mu, \tau)}{\partial \xi} \right\| \leq \omega(\|\xi - \tau\|^2) \quad (7).$$

Let us start by showing the following

LEMMA 3.1. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system (3.1) and if the assumptions (3.2) and (3.3) hold, then $\forall B(x^0, \sigma) \subset \subset \Omega$, with $\sigma < 2$, $\forall \tau \in (0, 1)$ and $\forall \varepsilon \in (0, n - 2 - 2(1/p + 1/q)]$, where $q \in (2, \bar{q})$ ⁽⁸⁾, it results:*

$$(3.4) \quad \begin{aligned} \phi(u, x^0, \tau\sigma) &\leq A\phi(u, x^0, \sigma) \{ \tau^\lambda + \sigma^{2(1-2/q)} + \\ &+ [\omega(c\sigma^{2-n}\phi(u, x^0, \sigma))]^{1-2/q} + \left[\omega \int_{B(x^0, \sigma)} \|H(u) - (H(u))_\sigma\|^2 d\sigma \right]^{1-2/q}, \end{aligned}$$

$$(6) \quad \frac{\partial a(x, u, \mu, \xi)}{\partial \xi_{ij}} = \left\{ \frac{\partial a^h(x, u, \mu, \xi)}{\partial \xi_{ij}^k} \right\}, \quad h, k = 1, 2, \dots, N.$$

$$(7) \quad \frac{\partial a(x, u, \mu, \eta)}{\partial \xi} = \left\{ \frac{\partial a(x, u, \mu, \eta)}{\partial \xi_{ij}} \right\}, \quad i, j = 1, 2, \dots, n.$$

(8) \bar{q} is the constant (> 2) that appears in theorem 2.1.

where

$$\lambda = n \left(1 - \frac{2}{p} \right) - \varepsilon ,$$

and

$$(3.5) \quad \phi(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0, \sigma)} [\|u\|^{2n/(n-4)} + \|D(u)\|^{2^*} + \|H(u)\|^2] dx$$

with

$$\xi = n \left(1 - \frac{2}{p} \right) .$$

PROOF. Let us fix the ball $B(x^0, \sigma)$, with $\sigma < 1$, such that $B(x^0, 2\sigma) \subset\subset \Omega$ and let us set:

$$\frac{\partial \tilde{a}(x, u, \mu, \eta)}{\partial \xi_{ij}} = \left\{ \int_0^1 \frac{\partial a^h(x, u, \mu, t\eta)}{\partial \xi_{ij}^k} dt \right\} \quad h, k = 1, 2, \dots, N$$

$$\frac{\partial \tilde{a}(x, u, \mu, \eta)}{\partial \xi} = \left\{ \frac{\partial \tilde{a}(x, u, \mu, \eta)}{\partial \xi_{ij}} \right\} \quad i, j = 1, 2, \dots, n .$$

In $B(x^0, \sigma)$ the system (3.1) can also be written in the following form:

$$(3.6) \quad a(x^0, u_\sigma, (Du)_\sigma, H(u)) =$$

$$= [-a(x, u, Du, H(u)) + a(x^0, u_\sigma, (Du)_\sigma, H(u))] + b(x, u, Du) =$$

$$= B_1 + b(x, u, Du) .$$

On the other hand, denoting by $a^h(x^0, u_\sigma, (Du)_\sigma, \eta)$, $h = 1, 2, \dots, N$, the h -th component of the vector $a(x^0, u_\sigma, (Du)_\sigma, \eta)$ one gets:

$$a^h(x^0, u_\sigma, (Du)_\sigma, \eta) = a^h(x^0, u_\sigma, (Du)_\sigma, \eta) - a^h(x^0, u_\sigma, (Du)_\sigma, 0) =$$

$$= \sum_{i,j=1}^n \sum_{k=1}^N \int_0^1 \frac{\partial a^h(x^0, u_\sigma, (Du)_\sigma, t\eta)}{\partial \xi_{ij}^k} dt \eta_{ij}^k \quad h = 1, 2, \dots, N$$

from which

$$a(x^0, u_\sigma, (Du)_\sigma, \eta) = \sum_{i,j=1}^n \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, \eta)}{\partial \xi_{ij}} \eta_{ij}.$$

Hence, from (3.6) the system (3.1) can be written in the following form:

$$\sum_{i,j=1}^n \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, H(u))}{\partial \xi_{ij}} D_{ij} u = B_1 + b(x, u, Du)$$

or, equivalently

$$\begin{aligned} (3.7) \quad & \sum_{i,j=1}^n \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} u = \\ & = \sum_{i,j=1}^n \left(- \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, H(u)_\sigma)}{\partial \xi_{ij}} + \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} \right) D_{ij} u + \\ & \quad + B_1 + b(x, u, Du) = B_2 + B_1 + b(x, u, Du). \end{aligned}$$

Letting w to be solution in $B(x^0, \sigma)$ to the elliptic Dirichlet problem

$$(3.8) \quad \begin{cases} w \in H_0^2(B(x^0, \sigma), \mathbb{R}^N) \\ \sum_{i,j=1}^n \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} w = B_2 + B_1 \end{cases}$$

it results, in $B(x^0, \sigma)$, $u = w + v$ where $v \in H^2(B(x^0, \sigma), \mathbb{R}^N)$ is solution to the linear system

$$(3.9) \quad \sum_{i,j=1}^n \frac{\partial \tilde{a}(x^0, u_\sigma, (Du)_\sigma, (H(u))_\sigma)}{\partial \xi_{ij}} D_{ij} v = b(x, u, Du).$$

It is known (see [1]) that for v we have the following estimate

$$\begin{aligned} \int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx &\leq c\tau^n \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + \\ &\quad + c \int_{B(x^0, \sigma)} \|b(x, u, Du)\|^2 dx, \quad \forall \tau \in (0, 1) \end{aligned}$$

from which and in virtue of assumption (3.2), it follows

$$(3.10) \quad \int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c\tau^n \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + \\ + c \int_{B(x^0, \sigma)} |f|^2 dx + \int_{B(x^0, \sigma)} \|u\|^{2\alpha} dx + \int_{B(x^0, \sigma)} \|Du\|^{2\beta} dx$$

where c does not depend on x^0 , τ and σ .

Moreover, setting

$$F(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx + \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx,$$

being

$$(3.11) \quad \int_{B(x^0, \sigma)} |f|^2 dx \leq c\sigma^\xi \|f\|_{0, p, \Omega}^2 \leq \bar{c}\sigma^\xi$$

$$(3.12) \quad \int_{B(x^0, \sigma)} \|u\|^{2\alpha} dx \leq \sigma^{n-(n-4)\alpha} \left(\int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} dx \right)^{(n-4)(\alpha/n)} \leq \\ \leq \sigma^\xi + \int_{B(x^0, \sigma)} \|u\|^{2n/(n-4)} d\sigma$$

$$(3.13) \quad \int_{B(x^0, \sigma)} \|Du\|^{2\beta} dx \leq \sigma^{n-\beta(n-2)} \left(\int_{B(x^0, \sigma)} \|Du\|^{2^*} dx \right)^{2\beta/2^*} \leq \\ \leq \sigma^\xi + \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx,$$

the estimate (3.10) can be written in the following form

$$(3.14) \quad \int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c\tau^\xi \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + cF(u, x^0, \sigma).$$

On the other hand from (2.15) we obtain, $\forall \tau \in (0, 1)$:

$$(3.15) \quad F(u, x^0, \tau\sigma) \leq \tau^\xi F(u, x^0, \sigma) + c(u, q) \sigma^{2(1-2/q)} \phi(u, x^0, \sigma).$$

In virtue of the estimates (3.14) and (3.15), using the lemma 1, II of

Chap. I of [2], we get

$$\forall \varepsilon \in \left(0, (n-2) - 2\left(\frac{n}{p} - \frac{2}{q}\right)\right], \quad \forall \tau \in (0, 1), \forall \sigma \in (0, 1)$$

$$\int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c \left\{ \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + F(u, x^0, \sigma) \right\} \cdot \tau^{n(1-2/p)-\varepsilon} + K(\tau\sigma)^{2(1-2/q)} \phi(u, x^0, \sigma),$$

from which we obtain $\forall \lambda \in [2(1-2/q), \xi)$, $\forall \tau \in (0, 1)$ and $\forall \sigma \in (0, 1)$

$$(3.16) \quad \int_{B(x^0, \tau\sigma)} \|H(v)\|^2 dx \leq c\tau^\lambda \int_{B(x^0, \sigma)} \|H(v)\|^2 dx + c(\tau^\lambda + \sigma^{2(1-2/q)})\phi(u, x^0, \sigma).$$

The function w satisfies the following estimate:

$$(3.17) \quad \int_{B(x^0, \sigma)} \|H(w)\|^2 dx \leq c \left(\int_{B(x^0, \sigma)} \|\mathcal{B}_1\|^2 dx + \int_{B(x^0, \sigma)} \|\mathcal{B}_2\|^2 dx \right).$$

Now let us estimate the integrals in the right hand side of (3.17).

From assumption (3.2), (3.3) and (2.6) we have

$$(3.18) \quad \int_{B(x^0, \sigma)} \|\mathcal{B}_1\|^2 dx \leq \int_{B(x^0, \sigma)} \omega^2 (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) \cdot \|H(u)\|^2 dx \leq c\sigma^n \left(\int_{B(x^0, \sigma)} \|H(u)\|^q dx \right)^{2/q} \cdot \left(\int_{B(x^0, \sigma)} \omega (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) dx \right)^{1-2/q} \leq \left[\omega \left(\int_{B(x^0, \sigma)} (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) dx \right) \right]^{1-2/q} \cdot \left[c \left(\sigma^n \int_{B(x^0, 2\sigma)} \|H(u)\|^2 + \sigma^n \left(\int_{B(x^0, 2\sigma)} \|b\|^q dx \right)^{2/q} \right) \right].$$

Now we observe that:

$$(3.19) \quad \int_{B(x^0, \sigma)} \|u - u_\sigma\|^2 dx \leq \\ \leq \sigma^4 \left(\frac{\int_{B(x^0, \sigma)} \|Du\|^{2^*} dx}{\sigma^{2^*}} \right)^{2/2^*} \leq \sigma^2 \left(1 + \int_{B(x^0, \sigma)} \|Du\|^{2^*} dx \right)$$

$$(3.20) \quad \int_{B(x^0, \sigma)} \|Du - (Du)_\sigma\|^2 dx \leq c\sigma^2 \int_{B(x^0, \sigma)} \|H(u)\|^2 dx.$$

Hence

$$(3.21) \quad \omega \left[\int_{B(x^0, \sigma)} (\sigma^2 + \|u - u_\sigma\|^2 + \|Du - (Du)_\sigma\|^2) dx \right] \leq \omega(c\sigma^{2-n} \phi(u, x^0, \sigma)).$$

Moreover, from (2.13), we have

$$(3.22) \quad \sigma^n \left(\int_{B(x^0, 2\sigma)} \|b\|^q dx \right)^{2/q} \leq c(u, f) \phi(u, x^0, 2\sigma).$$

Therefore, from (3.18)-(3.22) we deduce

$$(3.23) \quad \int_{B(x^0, \sigma)} \|\mathcal{B}_1\|^2 dx \leq [\omega(c\sigma^{2-n} \phi(u, x^0, 2\sigma))]^{1-2/q} \cdot c\phi(u, x^0, 2\sigma).$$

Similarly we obtain

$$(3.24) \quad \int_{B(x^0, \sigma)} \|\mathcal{B}_2\|^2 dx \leq \\ \leq c \left[\omega \left(\int_{B(x^0, 2\sigma)} (\|H(u) - (H(u))_{2\sigma}\|^2 dx) \right) \right]^{1-2/q} \phi(u, x^0, 2\sigma).$$

Finally, from (3.17) (3.23) and (3.24) we get

$$(3.25) \quad \int_{B(x^0, \sigma)} \|H(w)\|^2 dx \leq c\phi(u, x^0, 2\sigma) \cdot \left\{ [\omega(c\sigma^{2-n}\phi(u, x^0, 2\sigma))]^{1-2/q} + \left[\omega \left(\int_{B(x^0, 2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2 dx \right) \right]^{1-2/q} \right\}.$$

Since $u = v + w$ it follows, from (3.16) and (3.25) that $\forall \tau \in (0, 1)$ and $\forall \lambda \in [2(1 - 2/q), \xi]$ we have

$$(3.26) \quad \int_{B(x^0, \tau\sigma)} \|H(u)\|^2 dx \leq c\phi(u, x^0, 2\sigma) \cdot \left\{ \tau^\lambda + \sigma^{2(1-2/q)} + [\omega(c\sigma^{2-n}\phi(u, x^0, 2\sigma))]^{1-2/q} + \left[\omega \left(\int_{B(x^0, 2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2 dx \right) \right]^{1-2/q} \right\}.$$

From the estimates (3.15) and (3.26) we achieve $\forall \tau \in (0, 1)$, $\forall \sigma \in (0, 1)$ and $\forall \lambda \in [2(1 - 2/q), \xi]$

$$(3.27) \quad \phi(u, x^0, \tau\sigma) \leq c\phi(u, x^0, 2\sigma) \cdot \left\{ \tau^\lambda + \sigma^{2(1-2/q)} + [\omega(c\sigma^{2-n}\phi(u, x^0, 2\sigma))]^{1-2/q} + \left[\omega \left(\int_{B(x^0, 2\sigma)} \|H(u) - (H(u))_{2\sigma}\|^2 dx \right) \right]^{1-2/q} \right\}.$$

This estimate is easily true for $\tau \in [1, 2)$ and thus the lemma is proved. ■

Let us set:

$$\mathcal{B}_1 = \left\{ x \in \Omega : \lim'' \int_{B(x, \sigma)} \|H(u) - (H(u))_\sigma\|^2 dy > 0 \right\}$$

$$\mathcal{B}_2 = \left\{ x \in \Omega : \lim' \sigma^{2-n} \phi(u, x, \sigma) > 0 \right\}.$$

By a well known property of the Lebesgue integral we have

$$m \mathcal{B}_1 = 0$$

and taking into account a well known theorem of Giusti [2], we obtain

$$\mathcal{H}_{n-2}(\mathcal{B}_2) = 0$$

where \mathcal{H}_γ is the γ -dimensional Hausdorff measure.

Hence the set $\mathcal{B}_1 \cup \mathcal{B}_2$ has measure zero.

Now, reasoning exactly as in theorem 5.1 of [3], it is easy to prove

LEMMA 3.2. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system (3.1) if the assumptions (3.2) and (3.3) hold, then, for every fixed $\varepsilon \in (0, 1 - n/p)$, it is possible to associate to every $x^0 \in \Omega \setminus \mathcal{B}_1 \cup \mathcal{B}_2$ a ball $B(x^0, R_{x^0}) \subset \subset \Omega \setminus \mathcal{B}_2$ and a positive number σ_ε such that, $\forall t \in (0, 1)$ and $\forall y \in B(x^0, R_{x^0})$*

$$\phi(u, y, t\sigma_\varepsilon) \leq (1 + A)t^{n(1-2/p)-2\varepsilon} \phi(u, y, \sigma_\varepsilon)$$

and hence (see [2])

$$H(u) \in L^{2, n(1-2/p)-2\varepsilon}(B(x^0, R_{x^0}), \mathbb{R}^{n^2N})$$

$$D(u) \in \mathcal{L}^{2, n(1-2/p)-2\varepsilon+2}(B(x^0, R_{x^0}), \mathbb{R}^{nN}).$$

From lemma (3.2) the following result of partial Hölder continuity for Du easily follows.

THEOREM 3.1. *If $u \in H^2(\Omega, \mathbb{R}^N)$ is a solution to the system (3.1) and if the hypotheses (3.2) (3.3) are fulfilled, then there exists a set \mathcal{B}_0 , closed in $\mathcal{B}^{(9)}$ with*

$$\mathcal{B}_2 \subset \mathcal{B}_0 \subset \mathcal{B}_1 \cup \mathcal{B}_2$$

such that

$$Du \in C^{0, \alpha}(\Omega \setminus \mathcal{B}_0, \mathbb{R}^{nN}), \quad \forall \alpha < 1 - \frac{n}{p}.$$

⁽⁹⁾ In particular $m(\mathcal{B}_0) = 0$.

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