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Periodic solutions for Sellers type diffusive energy balance model in climatology

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Periodic Solutions for a Sellers Type Diffusive Energy Balance Model in Climatology.

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1. Introduction.

In this paper we consider the mathematical treatment of a time evolution model of the temperature on the Earth surface, obtained by an energy balance model. Climate models were independently introduced in 1969 by Budyko [1] and Sellers [7]. These models have a global character i.e. refer to all Earth and involves a relatively long-time scales with respect to the predition time.

We want to study the existence of periodic solutions for the nonlinear parabolic problem

$$\text{(P)} \quad \begin{cases} u_t - (\varrho(x)\,|u_x|^{p-2}u_x)_x = R_a(x,t,u) - R_e(x,u)\,, & \text{in } Q := (-1,1)\times \mathbb{R}_+ \\ (\varrho(x)\,|u_x|^{p-2}u_x)(\pm 1, t) = 0\,, \quad \forall t>0\,,\ p \geq 2 \end{cases}$$

where

$$\text{(1)} \qquad \varrho(x) := k(1-x^2)\,, \qquad \forall x \in [-1, 1]\,, \qquad k>0\,;$$

$$\text{(2)} \quad \begin{cases} R_a(x,t,u) := Q(x,t)\,\beta(u)\,, \quad \text{where } Q(x,t) \geq 0\,,\ Q \in C([-1,1]\times\mathbb{R}_+)\,, \\ Q(x,\cdot) \text{ is 1-periodic } \forall x \in [-1,1] \text{ and} \\ \beta \text{ is a nonnegative,bounded nondecreasing function for any } u \in \mathbb{R} \end{cases}$$

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$$
(3)\quad \begin{cases} R_e \in C([-1,1]\times\mathbb{R}), \quad R_e(x,\cdot) \text{ is a strictly increasing odd} \\ \text{function for } x\in[-1,1], \quad R_e(x,0)=0\,, \quad R_e(x,s)\geqslant Bs-A \\ \text{for any } x\in[-1,1], \quad \forall u\geqslant 0 \text{ and } B, A \text{ positive constants} \end{cases}
$$

In (2), $Q(x, t)$ describes the incoming solar radiation flux and the assumption $Q(x, t) \geqslant 0$, allows to consider also the polar night phenomena. Function R_a represents the fraction of the solar energy absorbed by the Earth, clearly it depends on the albedo or reflexivity of the Earth surface.

The albedo function $\alpha(u)$ is usually taken such that $0 < \alpha(u) < 1$, thus the coalbedo function $\beta(u) := 1 - \alpha(u)$, represents the fraction of the absorbed light.

In (3), function R_e represents the emitted energy by the Earth to the outer space. In the balance of energy models, one considers a rapid variation of the coalbedo function near to the critic temperature $u = -10°$ C. In this paper, we want to study the existence of periodic solutions for the Sellers model. For his model, Sellers proposed as coalbedo a function allowing a partially ice-free zone, $u_i < u < u_w$. An example of such function is

$$
\beta(u) = \begin{cases} a_w, & \text{if } u_w < u \\ a_i + ((u-u_i)/(u_w-u_i))(a_w-a_i)\,, & \text{if } u_i \leqslant u \leqslant u_w \\ a_i, & \text{if } u < u_i \end{cases}
$$

where a_i is the «ice» coalbedo (~ 0.38), a_w is the «ice-free» coalbedo (~ 0.71), u_i and u_w are fixed temperatures very close to $-10°$ C and R_e is taken of the form $R_e(x, u) = B|u|^3 u$. Our interest in the periodic forcing term is motivated by the seasonal variation of the incoming solar radiation flux during one year. As usual, $u(x, t)$ represents the mean annual temperature averaging on the latitude circles around the Earth (denoted by $x = \sin\phi$, where ϕ is the latitude).

The diffusion coefficient ϱ in (P), degenerates at $x = \pm 1$ and for $p > 2$ the equation in (P) degenerates also on the set of points where $u_x = 0$.

To prove the existence of periodic solutions for (P), we consider an initial-boundary problem associated to (P)

$$
(\mathrm{P}_1)\quad \begin{cases} u_t - (\varrho(x)|u_x|^{p-2}u_x)_x = R_a(x,t,u) - R_e(x,u)\,, & \text{in } [-1,1]\times(0,T) \\ (\varrho(x)|u_x|^{p-2}u_x)(\pm 1, t) = 0\,, & t\in(0,T) \\ u(x,0) = u_0(x)\,, & x\in(-1,1) \end{cases}
$$

with $T \geqslant 1$ arbitrary and

$$u_0 \in L^{\infty}(-1, 1). \tag{4}$$

The problem (P_1) is a model used in climatology to describe the climate energy balance models. Since (P_1) degenerates at $x = \pm 1$ and where $u_x = 0$, we cannot expect that (P_1) has classical solutions (see [3] for $\varrho = 1$ and $R_a = 0$), thus we shall deal with a weak solution for (P_1).

It was proved in [2] that if $u_0 \in L^{\infty}(-1, 1)$ there exists at least one bounded weak solution for (P_1).

The assumption

$$\begin{cases} \text{There exists a constant } L > 0 \text{ such that} \\ s \to R_a(x, t, s) - R_e(x, s) - Ls \text{ is nonincreasing} \end{cases} \tag{5}$$

shall be utilized to prove the uniqueness of the bounded weak solution for (P_1). Because of the degenerate diffusion coefficient $\varrho(x)$, the natural energy space associated to (P_1), is the one defined by

$$V := \{w \in L^2(-1, 1) : w_x \in L^p(-1, 1; \varrho)\}$$

where

$$L^p(-1, 1; \varrho) := \left\{ v : \|v\|_{L^p(-1,1;\varrho)} := \left(\int_{-1}^{1} \varrho(x) \, |v(x)|^p \, dx \right)^{1/p} < +\infty \right\}.$$

V is a separable and reflexive Banach space with the norm

$$\|v\|_v := \|v\|_{L^2(-1,1)} + \|v_x\|_{L^p(-1,1;\varrho)}.$$

To prove the existence of periodic solutions of the problem (P), we construct a subsolution $\underline{v}(x)$ and a supersolution $\overline{u}(x)$ of (P_1).

Then, we consider the Poincaré map F associated to (P_1) i.e. the operator assigning to every initial data of the ordered interval $[\underline{v}(x), \overline{u}(x)]$ the solution of (P_1) after 1-period. One proves that F is continuous, compact and pointwise increasing. By the Schauder fixed point theorem, there exists at least a fixed point for F.

This fixed point is a periodic solution for the problem (P).

Finally, one shows that (P) has a smallest and a greatest periodic solution.

The existence of periodic solutions for (P) both on a Riemannian manifold without boundary and for the Budyko type mode, (β is a bounded maximal monotone graph of $\mathbb{R}^2$ and $R_e(x, u) = Bu + A$, $B > 0$, $A > 0$), shall be the argument of a forthcoming paper.

In the nondegenerate case i.e. $p = 2$, the study of the periodic case for the climate energy balance models has been carried out by [4-5].

2. Existence and uniqueness of the solution.

DEFINITION 1. *For a bounded weak solution to* (P_1) *we mean a function* $u \in C([0, T]; L^2(-1,1)) \cap L^\infty(Q_T) \cap L^p(0, T; V)$, ($Q_T := (-1, 1) \times (0, T)$) *such that*

$$\int_{-1}^{1} u(x, T)\, v(x, T)\, dx - \int_0^T \int_{-1}^{1} u(x, t)\, v_t(x, t)\, dx\, dt +$$

$$+ \int_0^T \int_{-1}^{1} \varrho(x) \,|u_x(x, t)|^{p-2} u_x(x, t)\, v_x(x, t)\, dx\, dt =$$

$$= \int_0^T \int_{-1}^{1} (Q(x, t)\, \beta(u(x, t)) - R_e(x, u(x, t))\, v(x, t)\, dx\, dt +$$

$$+ \int_{-1}^{1} u_0(x)\, v(x, 0)\, dx$$

$\forall v \in L^p(0, T; V) \cap L^\infty(Q_T)$ *such that* $v_t \in L^{p'}(0, T; V')$.

DEFINITION 2. *For an* 1*-periodic bounded weak solution to* (P), *we mean a function* $u \in C(\mathbb{R}_+; L^2(-1, 1)) \cap L^\infty(Q)$ *such that* $u \in L^p_{loc}(\mathbb{R}_+; V)$, $u(x, t+1) = u(x, t)$, $u_t \in L^{p'}_{loc}(\mathbb{R}_+; V')$ *and satisfies* $\forall I := [t_0, t_1]$ *the following equality*

$$\int_I \langle u_t, z\rangle\, dt + \int_I \int_{-1}^{1} (\varrho(x)\, |u_x|^{p-2} u_x)\, z_x\, dx\, dt -$$

$$- \int_I \int_{-1}^{1} (R_a(x, t, u) - R_e(x, u))\, z(x, t)\, dx\, dt = 0$$

$$\forall z \in L^p(I; V) \cap L^\infty((-1, 1) \times I)\,.$$

In [2] has been proved, by means of a regularization argument, the existence of solutions to (P_1). This method consists to replace $\varrho(x)$ by

$$\varrho_\varepsilon(x) = \varrho(x) + \varepsilon \,. \tag{6}$$

In order to approximate u by classical solutions of a related problem to (P_1), we replace the data u_0, β, Q and R_e by C^∞ functions $u_{0,m}$, β_ε, Q_n, $R_{e,k}$ such that $u_0(\pm 1) = 0$, $\|u_{0,m}\|_{L^\infty(-1,1)} \leq \|u_0\|_{L^\infty(-1,1)}$ and $u_{0,m} \to u_0$ in $L^2(-1,1)$ as $m \to \infty$, $Q_n \to Q$ in $C(\overline{Q}_T)$, Q_n 1-periodic in t.

$R_{e,k}$ satisfies (3), $R_{e,k}(\cdot, u) \to R_e(\cdot, u)$ in $C([-1,1])$ for any fixed $u \in \mathbb{R}$.

Then, given ε, m, n and k positive constants, we consider the approximating problem for $T \geq 1$

$$(P_\varepsilon)\quad \begin{cases} u_t - (\varrho_\varepsilon(x)|u_x|^{p-2}u_x)_x - \varepsilon u_{xx} = Q_n(x,t)\beta_\varepsilon(u) - R_{e,k}(x,u)\,, & \text{in } Q_T \\ \varrho_\varepsilon(x)(|u_x|^{p-2}u_x + \varepsilon u_x)(\pm 1, t) = 0\,, & \text{in } (0,T) \\ u(x,0) = u_{0,m}(x)\,, & \text{in } (-1,1)\,. \end{cases}$$

The problem (P_ε) is now uniformly parabolic and by well-known results (see [6]) has a unique classic solution $u_{\varepsilon,m,n,k}$.

Moreover, it has been proved in [2] that

$$\|u_{\varepsilon,m,n,k}\|_{L^\infty(Q_T)} \leq C \tag{7}$$

$$\|\varrho_\varepsilon(u_{\varepsilon,m,n,k})\|_{L^p(0,T;L^p(-1,1))} \leq C \tag{8}$$

where C is a positive constant, independent of ε, m, n, k.

Using the a priori estimates, we can pass to the limit as ε goes to zero and $m, n, k \to \infty$ and we get

THEOREM 1 ([2]). *With assumptions* (1)-(3) *for any* $u_0 \in L^\infty(-1,1)$, *there exists at least a bounded weak solution to* (P_1).

The uniqueness of the bounded weak solution for (P_1), is obtained using the assumption (5). In fact

THEOREM 2. *If* (1)-(3) *and* (5) *hold, for any* $u_0 \in L^\infty(-1,1)$ *there exists a unique bounded weak solution for the problem* (P_1).

PROOF. If by contradiction there exist two solutions u_1 and u_2 for (P_1), multiplying by $(u_1 - u_2)^+ \in L^p(0, T; V)$

$$u_{1t} - u_{2t} - (\varrho(x) |u_{1x}|^{p-2} u_{1x})_x + (\varrho(x) |u_{2x}|^{p-2} u_{2x})_x =$$
$$= R_a(x, t, u_1) - R_e(x, u_1) - R_a(x, t, u_2) + R_e(x, u_2)$$

and integrating on $(-1, 1)$, since $(u_1 - u_2)_t \in L^{p'}(0, T; V')$ (see [2]), one has

$$(9) \quad (1/2)\, d/dt \int_{-1}^{1} (u_1 - u_2)^{+2}\, dx =$$
$$= -\int_{-1}^{1} (\varrho(x) |u_{1x}|^{p-2} u_{1x} - \varrho(x) |u_{2x}|^{p-2} u_{2x})(u_1 - u_2)_x^+\, dx +$$
$$+ \int_{-1}^{1} (R_a(x, t, u_1) - R_e(x, u_1) - R_a(x, t, u_2) + R_e(x, u_2))(u_1 - u_2)^+\, dx\,.$$

Since the operator $A(u_x) := \varrho(x) |u_x|^{p-2} u_x$ is nondecreasing, by (5) and integrating on $(0, t)$ we have

$$(10) \quad \int_{-1}^{1} (u_1(x, t) - u_2(x, t))^{+2}\, dx \leq \int_{-1}^{1} (u_{01}(x) - u_{02}(x))^{+2}\, dx +$$
$$+ 2L \int_0^t \int_{-1}^{1} (u_1(x, s) - u_2(x, s))^{+2}\, dx\, ds\,.$$

By the Gronwall lemma, it follows the uniqueness of the solution.

3. Subsolutions-supersolutions.

We assume that

$$(11) \quad Q_1(x) \leq Q(x, t) \leq Q_2(x), \quad \text{with } Q_1, Q_2 \in C([-1, 1]),\ Q_1 \geq 0 \text{ and } Q_2 > 0\,.$$

We consider the stationary problems

$$(PS)_1 \quad \begin{cases} -(\varrho(x) |v_x|^{p-2} v_x)_x = Q_1(x) \beta(v) - R_e(x, v), & \text{in } (-1, 1) \\ (\varrho(x) |v_x|^{p-2} v_x)(\pm 1) = 0 \end{cases}$$

$$(\text{PS})_2 \qquad \begin{cases} -(\varrho(x)\,|u_x|^{p-2}u_x)_x = Q_2(x)\,\beta(u) - R_e(x,u), & \text{in } (-1,1) \\ (\varrho(x)\,|u_x|^{p-2}u_x)_x(\pm 1) = 0 \end{cases}$$

A subsolution for $(\text{PS})_1$ is given by the function

$$\underline{v}(x) = -10 - a|x|^{p^*} - b, \quad \forall x \in [-1,1]$$

with $a<0$, $b>0$, $10<a+b$, suitable constants to be chosen later with $1/p + 1/p^* = 1$.

In fact

$$|\underline{v}_x(x)|^{p-2}\underline{v}_x(x) = (|a|p^*)^{p-1}|x|\,\text{sgn}\,x, \quad ((p^*-1)(p-1)=1).$$

Hence,

$$-(k(1-x^2)\,|\underline{v}_x|^{p-2}\underline{v}_x)_x = -k(|a|p^*)^{p-1}(1-3x^2).$$

We want that

$$-k(|a|p^*)^{p-1}(1-3x^2) \leqslant Q_1(x)\,\beta(\underline{v}) - R_e(x,\underline{v}).$$

Since, $\underline{v}(x) \leqslant -10-b-a$, we have by (3) that

$$Q_1(x)\beta(\underline{v}) - R_e(x,\underline{v}) \geqslant -R_e(x,-10-b-a) \geqslant$$

$$\geqslant R_e(x, 10+b+a) \geqslant B(10+b+a) - A.$$

Moreover, $-k(|a|p^*)^{p-1}(1-3x^2) \leqslant 2k(|a|p^*)^{p-1}$, therefore we choose a, b such that

$$2k(|a|p^*)^{p-1} \leqslant (10+b+a)\,B - A, \quad \text{with } (10+b+a)\,B > A.$$

A supersolution for $(\text{PS})_2$ is given by the function

$$\overline{u}(x) = -10 + a|x|^{p^*} + b, \quad \forall x \in [-1,1]$$

with a, b suitable constants, with $a<0$, $b>0$, $10<a+b$ as before, $1/p + 1/p^* = 1$.

In fact

$$|\overline{u}_x(x)|^{p-2}\overline{u}_x(x) = -(|a|p^*)^{p-1}|x|^{(p^*-1)(p-1)}\text{sgn}\,x = -(|a|p^*)^{p-1}|x|\,\text{sgn}\,\text{x}.$$

Hence,

$$-(k(1-x^2)\,|\overline{u}_x|^{p-2}\overline{u}_x)_x = k(|a|p^*)^{p-1}(1-3x^2).$$

We require that

$$k(|a|p^*)^{p-1}(1-3x^2) \geqslant Q_2(x)\,\beta(\overline{u}) - R_e(x, \overline{u}).$$

Since,

$$\overline{u}(x) \geqslant -10 + a + b\,,$$

we have

$$Q_2(x)\beta(\overline{u}) - R_e(x, \overline{u}) \leqslant$$

$$\leqslant \tilde{Q}_2 M - R_e(x, -10 + a + b) \leqslant \tilde{Q}_2 M - (-10 + a + b)\,B + A\,,$$

because of (3), where $\tilde{Q}_2 := \max\{Q_2(x), x \in [-1, 1]\}$ and M is such that $\beta(u) \leqslant M$, for any $u \in \mathbb{R}$.

Moreover, $k(|a|p^*)^{p-1}(1-3x^2) \geqslant -2k(|a|p^*)^{p-1}$, therefore we want that a, b verify

$$2k(|a|p^*)^{p-1} \leqslant (-10 + a + b)\,B - (A + \tilde{Q}_2 M),$$

$$\text{with } (-10 + a + b)\,B > A + \tilde{Q}_2 M\,.$$

Now, it is possible to prove the following result

THEOREM 3. *If* (1)-(5) *and* (11), *hold the solution* u *of* (P_1) *with* $u_0 \in \in [\underline{v}(x), \overline{u}(x)]$ *verifies*

$$\underline{v}(x) \leqslant u(x, t) \leqslant \overline{u}(x)\,, \quad \forall (x, t) \in Q_T$$

PROOF. Multiplying by $(u - \overline{u})^+$ and integrating on $(-1, 1)$, we obtain

$$(12) \quad (1/2)\, d/dt \int_{-1}^{1} (u - \overline{u})^{+2}\, dx \leqslant$$

$$\leqslant -\int_{-1}^{1} (\varrho(x)\,|u_x|^{p-2} u_x - \varrho(x)\,|\overline{u}_x|^{p-2}\overline{u}_x)(u - \overline{u})_x^+\, dx +$$

$$+ \int_{-1}^{1} (R_a(x, t, u) - R_e(x, u) - R_a(x, \overline{u}) + R_e(x, \overline{u}))(u - \overline{u})^+\, dx\,,$$

where $R_a(x, \overline{u}) = Q_2(x)\,\beta(\overline{u})$.

Since $A(u_x) := -\varrho(x)\,|u_x|^{p-2}u_x$ is a nondecreasing operator, we get

$$(13)\qquad (1/2)\,d/dt\int_{-1}^{1}(u-\overline{u})^{+2}dx \leqslant \int_{-1}^{1}(R_a(x,t,u)-R_e(x,u)-R_a(x,t,\overline{u})+$$

$$+R_e(x,\overline{u})+R_a(x,t,\overline{u})-R_a(x,\overline{u}))(u-\overline{u})^{+}\,dx\,.$$

Now, (5) gives us

$$(14)\qquad (1/2)\,d/dt\int_{-1}^{1}(u-\overline{u})^{+2}\,dx \leqslant L\int_{-1}^{1}(u-\overline{u})^{+2}\,dx+$$

$$+\int_{-1}^{1}(Q(x,t)-Q_2(x))\,\beta(\overline{u})(u-\overline{u})^{+}\,dx \leqslant L\int_{-1}^{1}(u-\overline{u})^{+2}\,dx\,.$$

Integrating on $(0, t)$ and by the Gronwall lemma, one has

$$u(x,t)\leqslant \overline{u}(x),\qquad \forall (x,t)\in Q_T$$

In a similar way one proves that

$$u(x,t)\geqslant \underline{v}(x),\qquad \forall (x,t)\in Q_T.$$

If we denote with F the Poincaré map defined by

$$F(u_0(x)) = u(x,1)$$

(u is the solution of (P_1)), to apply the Schauder fixed point theorem in the space $L^{\infty}(-1,1)$, we need of a closed and convex set $K\subset L^{\infty}(-1,1)$ and to show that

i) $F(K)\subset K$;

ii) $F|_K$ is continuous;

iii) $F(K)$ is relatively compact in $L^{\infty}(-1,1)$.

Define

$$K := \{w\in L^{\infty}(-1,1)\colon \underline{v}(x)\leqslant w(x)\leqslant \overline{u}(x)\}$$

it is easy to prove that K is a closed, convex and nonempty set.

Now, i) it follows from the Theorem 3 because we have showed that $F[\underline{v}(x), \overline{u}(x)]\subset[\underline{v}(x), \overline{u}(x)]$.

LEMMA 4. *With the assumptions of the Theorem 3, let $u_{0n}, u_0 \in K$ be such that $u_{0n} \to u_0$ in $L^\infty(-1, 1)$ as $n \to \infty$. Then, if u_n (respectively) u are the solutions of* (P_1) *with initial data u_{0n} and u_0 respectively, we have that $u_n(x, t) \to u(x, t)$ in $L^\infty(-1, 1)$ as $n \to \infty$, $\forall t \in [0, T]$.*

PROOF. Subtracting member to member and multiplying by $\operatorname{sgn}^+(u_n - u) \in V$, after an integration on Q_t we have

$$(15)\qquad \int_0^t \int_{-1}^1 (u_n - u)_s \operatorname{sgn}^+(u_n - u)\, dx\, ds -$$

$$-\int_0^t \int_{-1}^1 [(\varrho(x)\,|u_{nx}|^{p-2} u_{nx})_x - (\varrho(x)\,|u_x|^{p-2} u_x)_x] \operatorname{sgn}^+(u_n - u)\, dx\, ds =$$

$$= \int_0^t \int_{-1}^1 (R_a(x, s, u_n) - R_e(x, u_n) - R_a(x, s, u) + R_e(x, u)) \operatorname{sgn}^+(u_n - u)\, dx\, ds$$

by which

$$(16)\qquad \int_{-1}^1 (u_n(t) - u(t))^+\, dx -$$

$$- \int_{-1}^1 (u_{0n}(x) - u_0(x))^+\, dx \leq L \int_0^t \int_{-1}^1 (u_n(s) - u(s))^+\, dx\, ds\,.$$

The Gronwall lemma gives us

$$(17)\qquad \int_{-1}^1 (u_n(t) - u(t))^+\, dx \leq \exp(LT) \int_{-1}^1 |u_{0n}(x) - u_0(x)|\, dx\,.$$

Changing $u(t)$ with $u_n(t)$, one has

$$(18)\qquad \int_{-1}^1 |u_n(t) - u(t)|\, dx \leq \exp(LT) \int_{-1}^1 |u_{0n}(x) - u_0(x)|\, dx\,.$$

Since u_{0n} converges in $L^\infty(-1, 1)$ to u_0 as $n \to \infty$ we have that $u_n(t)$ converges in $L^1(-1, 1)$ and a.e. to $u(t)$ as n goes to infinity.

As $u_n(x, t) \in K$, by the Lebesgue theorem, one has that $u_n(\cdot, t)$ strongly converges to $u(\cdot, t)$ in $L^p(-1, 1)$, $\forall 1 \leq p \leq +\infty$. This proves ii).

The proof that $F(u)$ is relatively compact follows by a result of [2], where it is showed that $V \subset L^\infty(-1, 1)$, with compact embedding, for $p > 2$.

Now, $F(K)$ is bounded in V and by the quoted result, it follows that $F(K)$ is relatively compact in $L^\infty(-1, 1)$. Then, by the Schauder fixed point theorem, there exists a fixed point for the Poincaré map F. This fixed point is a periodic solution for (P).

If together to (P_1), we consider the problems

$$(\underline{P}) \quad \begin{cases} z_t - (\varrho(x)\,|z_x|^{p-2} z_x)_x = R_a(x, t, z) - R_e(x, z), & \text{in } Q_T \\ (\varrho(x)\,|z_x|^{p-2} z_x)(\pm 1, t) = 0, & \text{in } (0, T) \\ z(x, 0) = \underline{v}(x) & \text{in } (-1, 1) \end{cases}$$

$$(\overline{P}) \quad \begin{cases} w_t - (\varrho(x)\,|w_x|^{p-2} w_x)_x = R_a(x, t, w) - R_e(x, w), & \text{in } Q_T \\ (\varrho(x)\,|w_x|^{p-2} w_x)(\pm 1, t) = 0, & \text{in } (0, T) \quad (\overline{P}) \\ w(x, 0) = \overline{u}(x), & \text{in } (-1, 1) \end{cases}$$

as it was proved in the Theorem 3, for the solution of $(\underline{P})$ we have $\underline{v} \leqslant F(\underline{v})$, while for the solution of $(\overline{P})$ one has $F(\overline{u}) \leqslant \overline{u}$.

If we define by recurrence the sequences

$$z_1 = F(\underline{v}),\ \ldots,\ z_n = F(z_{n-1}),\ \ldots$$

and

$$w_1 = F(\overline{u}),\ \ldots,\ w_n = F(w_{n-1}),\ \ldots$$

the Picar iterates $\{z_n\}$ and $\{w_n\}$ makes two sequences, the first one is nondecreasing, the second one nonincreasing regard to the pointwise ordering,

$$\underline{v} \leqslant z_1 \leqslant \ldots \leqslant z_n \leqslant w_n \leqslant \ldots \leqslant w_1 \leqslant \overline{u}\,,$$

with

$$\|z_n(1)\|_{L^\infty(-1,1)} \leqslant C \quad \text{and} \quad \|w_n(1)\|_{L^\infty(-1,1)} \leqslant C\,.$$

There exist the following pointwise limits

$$\lim_n z_n(x, 1) = \underline{z}(x, 1) \tag{19}$$

$$\lim_n w_n(x, 1) = \overline{w}(x, 1)\,. \tag{20}$$

By the Lebesgue theorem, the convergence in (19) and (20) is uniform.

Since F is a continuous map, $\underline{z}(x, 1) = \lim_n z_n(x, 1) = \lim_n F(z_{n-1}) = F(\underline{z}(x, 1))$ and $\overline{w}(x, 1) = F(\overline{w}(x, 1))$.

Thus, $\underline{z}(x, 1)$ and $\overline{w}(x, 1)$ are the smallest respctively greatest periodic solutions of (P) in the ordered interval $[\underline{v}(x), \overline{u}(x)]$ of $L^\infty(-1, 1)$.

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