# Rendiconti del Seminario Matematico della Università di Padova



John Cossey

Trevor Hawkes


**On the largest conjugacy class size in a finite group**


*Rendiconti del Seminario Matematico della Università di Padova*, tome 103 (2000), p. 171-179

<http://www.numdam.org/item?id=RSMUP_2000__103__171_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 103 (2000)

# On the Largest Conjugacy Class Size in a Finite Group.

JOHN COSSEY (*) - TREVOR HAWKES (**)

We set

$$\operatorname{lcs}(G) = \max\{\,|G : C_G(g)| : g \in G\},$$

the largest conjugacy class size of $G$. Denoting by $\sigma(G)$ the set of prime divisors of $|G|$ and by $G_p$ a Sylow $p$-subgroup of $G$, we will prove the following theorem.

THEOREM. *Let G be an abelian-by-nilpotent finite group. Then*

$$(\alpha) \qquad \operatorname{lcs}(G) \geqslant \prod_{p \in \sigma(G)} \operatorname{lcs}(G_p).$$

Our theorem fails for soluble groups in general: for a given $\varepsilon > 0$ we will show how to construct a group G of derived length 3 for which

$$\operatorname{lcs}(G) < \varepsilon\Big(\prod_{p \in \sigma(G)} \operatorname{lcs}(G_p)\Big).$$

We begin by stating and proving three elementary lemmas for use in the proof of the Theorem.

(*) Indirizzo dell'A.: Mathematics Department, School of Mathematical Sciences, The Australian National University, Canberra, 0200, Australia.
(**) Indirizzo dell'A.: Mathematics Institute, University of Warwick, Coventry CV4 7AL, England.
1991 Mathematics Subject Classification: 20 D 10.
The first-named author gratefully acknowledges the Science and Engineering Research Council Visiting Fellowship held at the University of Warwick while this work was undertaken.

LEMMA 1. *Let* $H = AB$ *with* $A \trianglelefteq H$ *and* $A \cap B = 1$. *Let* $x \in B$. *Then*

$$C_H(x) = C_A(x)\, C_B(x).$$

PROOF. Let $h \in C_H(x)$, and let

$$h = ab$$

be the unique decomposition with $a \in A$ and $b \in B$. Then

$$ab = (ab)^x = a^x b^x,$$

and since $a^x \in A$ and $b^x \in B$, it follows that $a = a^x$ and $b = b^x$. Thus $a \in C_A(x)$ and $b \in C_B(x)$, and the result is clear.

LEMMA 2. *Let* $G$ *be a group of* $\pi$*-length one for some set* $\pi$ *of primes. If* $b$ *is an element of a Hall* $\pi$*-subgroup* $B$ *of* $G$, *then* $C_B(b)$ *is a Hall* $\pi$*-subgroup of* $C_G(b)$.

PROOF. If $T = O^{\pi'}(G)$, the $\pi'$-residual of $G$, then $C_T(b)$ clearly contains a Hall $\pi$-subgroup of $C_G(b)$. Thus we can assume that $T = G$ and hence by hypothesis that $G = AB$, where $A(= O_{\pi'}(G))$ is the normal Hall $\pi'$-subgroup of $G$. But then by Lemma 1 we have

$$C_G(b) = C_A(b)\, C_B(b),$$

and the desired conclusion follows.

LEMMA 3. *Let* $A \trianglelefteq G = AB$ *with* $A \cap B = 1$. *If* $g \in G$ *and* $g = ab$ *with* $a \in A$ *and* $b \in B$, *then*

$$(\beta) \qquad |G : C_G(g)| \leqslant |A|\, |B : C_B(b)|.$$

*In particular,*

$$\operatorname{lcs}(G) \leqslant \operatorname{lcs}(B)\, |A|.$$

PROOF. Let $h = vu$ be an element of $G$ with $u \in A$ and $v \in B$. Then $g^h = a^h b^{vu} = a^h [u, b^{-v}] b^v$. Since $a^h [u, b^{-v}] \in A$, every conjugate of $g$ can be written as a $B$-conjugate of $b$ times an element of $A$. The inequality labelled $(\beta)$ now follows and the rest is clear.

THE PROOF OF THE THEOREM. We argue by induction on the number of primes in $\sigma(G)$. If $|\sigma(G)| = 1$, then $G$ is a $p$-group and it is clear that

$(\alpha)$ holds. Therefore suppose that $|\sigma(G)| \geqslant 2$, and let

$$\sigma(G) = \pi_1 \dot{\cup} \pi_2$$

be a non-trivial partition of $\sigma(G)$.

Let $R$ denote the nilpotent residual of $G$ and note that, since $R$ is abelian by hypothesis, a system normalizer $D$ of $G$ is a complement to $R$ in $G$ (cf. Doerk and Hawkes [1], Theorem IV, 5.18). Since $D$ is nilpotent, we can write

$$D = D_1 \times D_2,$$

with $D_i \in \mathrm{Hall}_{\pi_i}(D)$; also

$$R = R_1 \times R_2,$$

with $R_i \in \mathrm{Hall}_{\pi_i}(R)$. For $i = 1, 2$ we set

$$H_i = R_i D_i$$

and observe that $H_i \in \mathrm{Hall}_{\pi_i}(G)$. Let $x_i$ be an element of $H_i$ belonging to a conjugacy class of largest size in $H_i$ (thus $|H_i : C_{H_i}(x_i)| = \mathrm{lcs}(H_i)$ for $i = 1, 2$), and write

$$(\gamma) \qquad x_i = r_i d_i$$

with $r_i \in R_i$ and $d_i \in D_i$. Let $\{i, j\} = \{1, 2\}$, and consider the action of $D_i$ on $R_j$. Since $(o(d_i), |R_j|) = 1$ and $R_j$ is abelian, by Proposition A, 12.5 of Doerk and Hawkes [1] we have

$$(\delta) \qquad R_j = [R_j, d_i] \times C_{R_j}(d_i),$$

and because $D_j$ centralizes $d_i$, the two subgroups $[R_j, d_i]$ and $C_{R_j}(d_i)$ are $D_j$-invariant and are therefore normal in $H_j$. We set

$$A_j = [R_j, d_i] \quad \text{and} \quad B_j = C_{R_j}(d_i)\, D_j.$$

[Note for use below that $A_j \trianglelefteq A_j B_j = H_j$, that $A_j \cap B_j = 1$, and that $A_j$ is a normal subgroup of each conjugate of $H_j$.] According to Equation $(\delta)$ we can write $r_j = a_j c_j$ with $a_j \in A_j$ and $c_j \in C_{R_j}(d_i)$, and then we obtain

$$x_j = a_j b_j$$

with $b_j = c_j d_j \in B_j$. Since $[d_i, c_j] = [d_i, d_j] = [c_i, c_j] = 1$, it follows that $b_i$

commutes with $b_j$. We aim to show that the element $g = b_i b_j$ satisfies

$$(\varepsilon) \qquad |G : C_G(g)| \geqslant \operatorname{lcs}(H_1)\operatorname{lcs}(H_2).$$

For by induction we have

$$\operatorname{lcs}(H_i) \geqslant \prod_{p \in \sigma(H_i)} \operatorname{lcs}(G_p),$$

and since $\sigma(H_1) \cup \sigma(H_2) = \sigma(G)$, the conclusion of the Theorem will then follow.

As before, let $\{i, j\} = \{1, 2\}$. As our first step in justifying the inequality labelled $(\varepsilon)$, we choose a conjugate $H$ of $H_i$ so that $H \cap C_G(b_i b_j)$ is a Hall $\pi_i$-subgroup of $C_G(b_i b_j)$. Since $b_i$ is a $\pi_i$-element of the centre of $C_G(b_i b_j)$, evidently $b_i \in H$. Because $b_i$ and $b_j$ have relatively prime orders and commute, we have

$$C_H(b_i b_j) = C_H(b_i) \cap C_H(b_j).$$

Now $b_j$ acts fixed-point-freely on $A_i = [R_i, b_j]$, and so $C_H(b_i b_j) \cap A_i \leqslant$ $\leqslant C_H(b_j) \cap A_i = 1$. Hence

$$(\zeta) \qquad |C_H(b_i b_j)| = |C_H(b_i b_j)A_i|/|A_i| \leqslant |C_H(b_i)A_i|/|A_i|.$$

[Here we have used the fact that $H$ normalizes $A_i$.] Next we observe that

$$|C_H(b_i)A_i| = |C_H(b_i)|\,|A_i|/|C_H(b_i) \cap A_i|.$$

Since metanilpotent groups have $\pi$-length one for all sets $\pi$ of primes, we can twice apply Lemma 2 (with $H$ and then $H_i$ in place of $B$) to conclude that

$$(\eta) \qquad |C_H(b_i)| = |C_{H_i}(b_i)|.$$

Since $b_i \in B_i$ and $H_i = A_i B_i$ is a semidirect product of $A_i$ by $B_i$, it follows from Lemma 1 that

$$(\theta) \qquad |C_{H_i}(b_i)| = |C_{A_i}(b_i)|\,|C_{B_i}(b_i)|.$$

Hence from $(\zeta)$ we obtain

$$\begin{aligned} |C_H(b_i b_j| &\leqslant |C_H(b_i)|/|C_{A_i}(b_i)| \\ (\iota) \qquad &= |C_{H_i}(b_i)|/|C_{A_i}(b_i)| \\ &= |C_{B_i}(b_i)| \qquad \text{(by } (\theta)). \end{aligned}$$

Therefore, we have

$$\begin{aligned}
|H : C_H(b_i b_j)| &= |H|/|C_H(b_i b_j)| \\
&\geqslant |H|/|C_{B_i}(b_i)| && \text{(by } (\iota)) \\
&= |A_i||B_i : C_{B_i}(b_i)| \\
&\geqslant |H_i : C_{H_i}(x_i)| && \text{(by Lemma 3)} \\
&= \operatorname{lcs}(H_i).
\end{aligned}$$

If $\widetilde{H}$ is a conjugate of $H_j$ with the property that $\widetilde{H} \cap C_G(b_i b_j)$ is a Hall $\pi_j$-subgroup of $C_G(b_i b_j)$, we similarly obtain

$$|\widetilde{H} : C_{\widetilde{H}}(b_i b_j)| \geqslant \operatorname{lcs}(H_j).$$

Thus, finally, we can deduce that

$$\begin{aligned}
|C : C_G(b_i b_j)| &= |H : C_H(b_i b_j)|\,|\widetilde{H} : C_{\widetilde{H}}(b_i b_j)| \\
&\geqslant \operatorname{lcs}(H_i)\operatorname{lcs}(H_j).
\end{aligned}$$

We have now justified the inequality labelled $(\varepsilon)$ and the Theorem is proved.

Before we move to the promised construction of examples, we prove another elementary lemma.

Lemma 4. *Let $A$ be an abelian normal subgroup of prime index $p$ in a group $G$. If $x$ and $y$ are elements of $G$ not in $A$, then*

$$|x^G| = |y^G|.$$

Proof. Let $C = C_G(x)$. Since $x \in C$ and $\langle x, A\rangle = G$, we have $G = CA$ and therefore

$$|x^G| = |CA : C| = |A : A \cap C| = |A : C_A(x)|.$$

If $y \in G \backslash A$, then $y = x^i a$ for some $a \in A$ and $i \in \{1, \dots, p-1\}$. Then we have

$$C_A(y) = C_A(x^i) = C_A(\langle x^i \rangle) = C_A(\langle x \rangle) = C_A(x),$$

and the conclusion of the lemma follows.

A FAMILY OF EXAMPLES. Let $p$ be a prime, $p \geqslant 5$, and let $q$ be a prime dividing $p-1$. Let $E$ be a non-abelian group of order $pq$. Thus $E = PQ$, where $Z_p \cong P = \boldsymbol{O}_p(E) = \boldsymbol{F}(E)$, the Fitting subgroup of E, and $Z_q \cong Q \in \in \mathrm{Syl}_q(E)$; moreover, the non-trivial elements of $P$ fall into $(p-1)/q$ orbits of length $q$ under the action by conjugation of Q. We now define two abelian groups $A$ and $B$ on which $E$ acts as an operator group.

(A) If $q = 2$, let $A$ be a cyclic group of order $2^n$ $(n \geqslant 3)$, and let $E$ act on $A$ with $P$ as the kernel of the action so that the elements of the non-identity coset of $P$ in $E$ act on $A$ by inversion, sending each $a \in A$ to its inverse.

If $q > 2$, let $U$ be the trivial simple $P$-module over the field $\mathbb{F}_q$ of $q$-elements and let $A = U^E$; thus $A$ is isomorphic with the regular $\mathbb{F}_q(E/P)$-module, and, in particular, $A_Q \cong \mathbb{F}_q Q$, the regular $Q$-module.

(B) Next let $V$ be the trivial simple $Q$-module over $\mathbb{F}_p$ and let $B = V^E$. By easy applications of Mackey's theorem for induced representations we have:

(i) $B_P \cong \mathbb{F}_p P$ and, in particular, $|C_B(x)| = p$ for all $1 \neq x \in P$;

(ii) $B_Q \cong V \oplus r\mathbb{F}_p Q$, where $r = (p-1)/q$.

Let $G$ be the semidirect product

$$G = [A \oplus B]\, E\,,$$

where the action of $E$ as a group of operators on $A \oplus B$ is determined by the action of $E$ on $A$ and $B$ described above. In what follows we will use multiplicative notation for $A \oplus B$ when it is regarded as a subgroup $AB$ of $G$. Evidently $BP$ is a Sylow $P$-subgroup of $G$ and $AQ$ is a Sylow $q$-subgroup of $G$. Set

$$M = \begin{cases} \min\{2^{n-2}, p^{r-1}\} & \text{if } q = 2, \ \textit{and} \\ \min\{q^{q-2}, p^{r-1}\} & \text{if } q > 2 \end{cases}$$

Since

$$r = (p-1)/q$$

and $p \geqslant 5$, it follows that $r \geqslant 2$ and hence that $M \geqslant 2$. In fact, it is easy to see that by judicious choice of $p$ and $q$ we can make $M$ arbitrarily large. We will show that

$$(\kappa) \qquad \operatorname{lcs}(BP)\operatorname{lcs}(AQ) \geqslant M \operatorname{lcs}(G).$$

*Step* 1: We assert that

$$(\lambda) \qquad \operatorname{lcs}(BP) = p^{p-1}.$$

Since $B_P$ is a regular $\mathbb{F}_p P$-module, the group $BP$ is isomorphic with $Z_p \wr_{\mathrm{reg}} Z_p$ and $|C_B(P)| = p$. The conjugacy classes of $BP$ contained in $B$ obviously have lengths 1 or $p$, while elements $x$ in $BP \backslash B$ belong to classes of length $|B : C_B(P)| = p^{p-1}$ by Lemma 4. Thus Assertion ($\lambda$) is justified.

*Step* 2: We now assert that

$$(\mu) \qquad \operatorname{lcs}(AQ) = \begin{cases} 2^{n-1} & \text{if } q = 2, \ \textit{and} \\ q^{q-1} & \text{if } q > 2\,. \end{cases}$$

The conjugacy classes of $AQ$ contained in the abelian normal subgroup $A$ obviously have length 1 or $q$. In the case $q = 2$, as well as in the case $q > 2$, it is easy to see from the action of $Q$ on $A$ that $|C_A(Q)| = q$. Hence, if $x \in AQ \backslash A$, it follows from Lemma 4 that

$$(\nu) \qquad |x^{AQ}| = |A : C_A(Q)| = \begin{cases} 2^{n-1} & \text{if } q = 2, \ \textit{and} \\ q^{q-1} & \text{if } q > 2\,. \end{cases}$$

Assertion ($\mu$) is now clear.

*Step* 3: Our next assertion is that

$$(\xi) \qquad \operatorname{lcs}(G) = \begin{cases} \max\{2p^{p-1}, 2^{n-1}p^{p-r}\} & \text{when } q = 2, \ \text{and} \\ \max\{qp^{p-1}, q^{q-1}p^{p-r}\} & \text{when } q > 2\,. \end{cases}$$

Let $x \in G$, and let $y$ be a generator of $Q$. We consider three cases.

*Case* 1: We have $x \notin ABP$. Since $G/AB (\cong E)$ is a Frobenius group, $AB\langle x \rangle$ is conjugate to $ABQ$, and therefore in calculating $|x^G|$, we can suppose without loss of generality that $x \in ABQ \backslash AB$. Since $x$, like $y$, acts fixed-point-freely on $P$, we have $C_G(x) \leqslant AB\langle x \rangle = ABQ$, and so $|x^G| = |P|\,|x^{ABQ}|$. By Lemma 4 we have

$$|x^{ABQ}| = |y^{ABQ}| = |AB : C_{AB}(Q)| = |A : C_A(Q)|\,|B : C_B(Q)|\,.$$

Since the restriction $B_Q$ of $B$ to $Q$ is the sum of a trivial module and $r$ regular modules, we have $|C_B(Q)| = p^{r+1}$; hence from ($\nu$) we conclude that

$$(\pi) \qquad |x^G| = \begin{cases} 2^{n-1}p^{p-r} & \text{if } q=2, \text{ and} \\ q^{q-1}p^{p-r} & \text{if } q>2 . \end{cases}$$

We note that $pq$ divides $|x^G|$ in this case because $p-r>(q-1)r \geqslant r \geqslant$ $\geqslant 2$ and by assumption $n \geqslant 3$.

*Case* 2: We have $x \in ABP \backslash AB$. Since $ABP/AB$ is self-centralizing in $G/AB$, it follows that $C_G(x) \leqslant ABP$ and hence from Lemma 4 that $|x^G| = |Q|\,|x^{ABP}| = |AB : C_{AB}(P)|$. Now $P$ centralizes $A$ and $B_P$ is a regular module, and therefore

$$|AB : C_{AB}(P)| = p^{p-1}.$$

Hence

$$(\varrho) \qquad |x^G| = qp^{p-1}$$

in this case, and again $|x^G|$ is divisible by $pq$.

*Case* 3: We have $x \in AB$. Since $AB$ is abelian, we have $|x^G| =$ $= |E : C_E(x)|$, which is a divisor of $pq$. In this case $|x^G|$ is smaller than the values obtained for it is Cases 1 and 2.

Assertion ($\xi$) now follows from ($\pi$) and ($\varrho$).

To justify the inequality labelled ($\kappa$), we deduce from ($\lambda$), ($\mu$), and ($\xi$) that for $q=2$

$$\begin{aligned} \frac{\operatorname{lcs}(BP)\operatorname{lcs}(QA)}{\operatorname{lcs}(G)} &= \frac{2^{n-1}p^{p-1}}{\max\{2p^{p-1},\, 2^{n-1}p^{p-r}\}} \\ &\geqslant \min\left\{\frac{2^{n-1}p^{p-1}}{2p^{p-1}},\, \frac{2^{n-1}p^{p-1}}{2^{n-1}p^{p-r}}\right\} \\ &= \min\{2^{n-2},\, p^{r-1}\} \\ &= M . \end{aligned}$$

Similarly, for $q > 2$, we obtain

$$\frac{\operatorname{lcs}(BP)\operatorname{lcs}(QA)}{\operatorname{lcs}(G)} \geqslant \min\left\{\frac{q^{q-1}p^{p-1}}{qp^{p-1}}, \frac{q^{q-1}p^{p-1}}{q^{q-1}p^{p-r}}\right\}$$

$$= \{q^{q-2}, p^{r-1}\}$$

$$= M.$$

Thus we have shown that $(\kappa)$ holds for all values of $q$. Given $\varepsilon > 0$, it is easy to find primes $p$ and $q$ so that $1/M < \varepsilon$. Thus, as promised at the outset, we have shown the existence of $\{p, q\}$-groups of derived length 3 satisfying

$$\operatorname{lcs}(G) < \varepsilon\Big(\prod_{p \in \sigma(G)} \operatorname{lcs}(G_p)\Big).$$

## REFERENCES

[1] K. Doerk - T. O. Hawkes, *Finite Soluble Groups*, Walter de Gruyter, Berlin-New York, 1992.

Manoscritto pervenuto in redazione l'1 settembre 1998.