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Some Remarks on G-Actions with Sections.

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ABSTRACT - This note is aimed at studying complete Riemannian manifolds acted on by a Lie group of isometries, in the case the action admits sections. The action is proved to be variationally complete and a Morse series is explicitly computed (in the case of non-positive curvature). The Weyl chambers of a section are studied.

1. Introduction.

In this note we will consider a complete Riemannian manifold (M, g), which is acted on isometrically by a closed Lie subgroup G of the full isometry group $I(M, g)$. In particular, we will restrict ourselves to the case when the G-action admits sections, namely when there exists a closed connected submanifold Σ which meets *every* G-orbit *orthogonally*.

The existence of a section is an important condition on the action; the regular points of a section in fact parametrize, modulo coverings, the orbit space M/G. Such a condition on the G-action has been considered in some basic papers by Conlon [5] and Palais and Terng [9] and more recently it has been studied in interesting papers on hyperpolar and asystatic actions (see among others [1] and [7]). In § 2 we prove some interesting properties of sections (in the case they actually exist). These properties have been used in [5], [9] and [10], but in the literature we could not find a complete proof of the following well-known fact: regular

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points of a section Σ are dense in Σ. Here we give an elementary proof of this fact (Theorem 2.2), and are thus able to prove Conlon's results in the general case.

Bott [2] was the first to apply Morse theory to a particular manifold admitting sections: he considered a compact connected Lie group acting on itself by adjoint action. We here take his procedure under consideration, applying it to the case of G-manifolds admitting sections with non-positive sectional curvature. A first result is the variational completeness of the action, i. e. we prove that Jacobi fields tangent to the orbits in two points are induced by G, thus generalizing a similar result by Conlon [5].

Let $G(x)$ be a principal orbit. Fix a regular point p in $M\backslash G(x)$. Bott defines a Morse series $\mathcal{M}(t)$, depending on M, $G(x)$ and p, in the following way. Let $S = S(M, G(x), p)$ be the set of geodesic segments of $(M, G(x))$, i. e. geodesics starting in $G(x)$ with orthogonal speed, ending at p and parametrized with arc length. Define the *index* of a geodesic $\sigma \in S(M, G(x), p)$ to be the sum over t of the dimension of the Jacobi fields induced by variations preserving $S = S(M, G(x), p)$ and vanishing at $\sigma(t)$. Then this index is proven to be finite and the *Morse series relative to* $(M, G(x), p)$ is given by

$$\mathcal{M}(t) = \sum_{\sigma \in S} t^{\lambda(\sigma)}$$

where $\lambda(\sigma)$ is the index of σ.

When the action is variationally complete, the index can be rewritten as

$$\lambda(\sigma) = \sum_{0 < t \leq a} \delta(\sigma(t))$$

where $\sigma = \gamma|_{[0,a]}$ and $\delta(\sigma(t)) = \dim G(p) - \dim G(\sigma(t))$.

Using a theorem due to Szenthe [11], and under a mild assumption on the Weyl group of the section, we prove that the Morse series has an explicit expression of the following form.

THEOREM 1. *Let (M, g) be a Riemannian G-manifold admitting sections. Assume the manifold (M, g) has non-positive sectional curvature and that the Weyl group W_Σ is finite. Let $G(x)$ be a principal orbit and $p \in M\backslash G(x)$ a regular point. Then the Morse series exists and we have:*

$$\mathcal{M}(t) = \sum_{\sigma \in S} t^{r_\sigma - k}$$

where k is the cohomogeneity of the action and r_σ is the codimension of the singular non-exceptional orbit of the first focal point of $G(x)$ along σ.

We then continue the study of the action by looking at the action of the Weyl group on a given section Σ and dropping the assumption on the curvature of the manifold. We generalize the concept of «walls» introduced by Bott and prove the following:

THEOREM 2. *Let (M, g) be a Riemannian G-manifold admitting sections. Suppose the singular points of the action are all non-exceptional. Given a wall S, there exists a non-trivial involutive isometry $g \in W_\Sigma$ fixing S pointwise.*

The set of walls is proven to be discrete. The components of $\Sigma \setminus \{walls\}$ are called the *Weyl chambers* and the Weyl group induces a permutation of them.

These results generalize similar ones by Conlon, who assumes the manifold to be simply connected and the sections to be flat.

Szenthe considered also a similar situation ([11]). In his work, the set of singular non-exceptional points of a section is investigated and the group generated by reflections around this set is taken under consideration. The relations between this group and the Weyl group are then studied, and some sufficient conditions for them to coincide are found (the sections are assumed to be simply connected and there are conditions on the intersection points of walls). Our results somehow generalize these ones, using quite a different approach in dealing with the Weyl group.

2. Some properties of sections.

Throughout the following, we will assume that the G-action on M admits sections. The aim of this section is to collect some results about sections, giving detailed proofs of them; we refer the reader to [4], [5], [9] for standard notations and results.

Conlon in [5] proves most of the following results, but with a more restrictive definition of sections (which are called G-transversal domains): he assumes sections to be flat and totally geodesic submanifolds. Here we consider the general case and we prove that all sections are totally geodesic submanifolds. First of all, we prove:

PROPOSITION 2.1. *Let Σ be a section for M. The dimension of Σ equals the cohomogeneity of the action of G on M.*

PROOF. It follows immediately from the very definition of section that $\dim \Sigma \leqslant \mathrm{chm}(G, M)$, where $\mathrm{chm}(G, M)$ denotes the cohomogeneity of the G-action.

Let $p \in \Sigma$ be a regular point and $\mathcal{T}$ a tubular neighborhood around the orbit. It is well known (see e.g. [4]) that $\mathcal{T}$ is diffeomorphic to the product $G(p) \times B_r$, where B_r is a ball of suitably chosen radius r in $T_p G(p)^{\perp}$. The submanifold $\Sigma \cap \mathcal{T}$ meets all orbits of $\mathcal{T}$ and consequently the restriction to $\Sigma \cap \mathcal{T}$ of the projection on the second factor $\Pi \colon \Sigma \cap \mathcal{T} \to B_r$ is onto. By Sard's lemma it then follows that $\dim \Sigma \geqslant \dim B_r = \mathrm{chm}(G, M)$. ∎

The content of the following basic theorem is well known in the literature, but we were not able to find a complete proof of it.

THEOREM 2.2. *The set Σ_{reg} of the regular points in Σ is open and dense in Σ.*

PROOF. We shall prove that in Σ there can not be an open set wholly made of singular points, that is, denoting by $\Sigma_{sing} = \Sigma \cap M_{sing}$ the set of singular points in Σ, that $\dim \Sigma_{sing} < \dim \Sigma$. Obviously, if $\mathfrak{T}_{sing}$ denotes the set of singular orbit types and H is any isotropy subgroup, we have

$$\Sigma_{sing} = \bigcup_{(H) \in \mathfrak{T}_{sing}} \Sigma \cap M_{(H)}$$

where the union is countable and $M_{(H)}$ denotes the set of points whose isotropy subgroups are conjugate to H.

It is then sufficient to prove that, for all H singular isotropy subgroup, we have

$$(*) \qquad \dim \Sigma \cap M_{(H)} < \dim \Sigma .$$

Note that G acts on $M_{(H)}$ and that $\Sigma \cap M_{(H)}$ is a section for this action. The previous proposition then makes $(*)$ equivalent to

$$\mathrm{chm}(G, M_{(H)}) < \mathrm{chm}(G, M) .$$

Let G/H in M_{sing} be an orbit, $\mathcal{T}$ be a tubular neighborhood and $T = G \times \times_H V$ the associated linear tube. Denoting by V^H the subspace of V given by the points of V which are fixed by H, we get

$$M_{(H)} \cap \mathcal{C} \cong G/H \times V^H.$$

It then follows $\operatorname{chm}(G, M_{(H)}) = \dim V^H$.

Write V as $V = V^H \oplus W$, so that W is a non-trivial H-invariant subspace of V. H being compact, the action of H on W is necessarily not transitive and so $\operatorname{chm}(H, W) > 0$.

Clearly we then get

$$\operatorname{chm}(H, V) \geqslant \operatorname{chm}(H, V^H) + \operatorname{chm}(H, W) > \operatorname{chm}(H, V^H) = \dim V^H.$$

On the other hand, dimension counting gives

$$\operatorname{chm}(G, M) = \operatorname{chm}(H, V)$$

and thus

$$\operatorname{chm}(G, M_{(H)}) < \operatorname{chm}(G, M). \quad \blacksquare$$

This theorem enables us to prove another interesting property of sections, which shows that the assumption made by Conlon [5] for a section to be totally geodesic is unnecessary.

THEOREM 2.3. *Each section Σ is a totally geodesic submanifold.*

PROOF. Since Σ_{reg} is dense in Σ, it is sufficient to prove the theorem for geodesics starting at regular points.

Follow now the proof given in [9]. $\blacksquare$

The previous theorem implies immediately that there exists only one section passing through a regular point.

The following lemma has useful consequences. One can easily see that the proof given in [5] works without the assumption of flatness.

LEMMA 2.4 ([5]). *Let $q \in M$ and S be a G_q-slice, where G_q is the isotropy of q. Then for all $p \in S$ and for all section Σ through q, we have:*

$$G_q(p) \cap \Sigma \cap S \neq \emptyset$$

As an easy corollary we obtain the following interesting proposition.

Proposition 2.5 ([5]).

(i) *The isotropy of a point* $q \in M$ *acts transitively on the set of sections through* q.

(ii) *Let* $q \in M$, S *be a* G_q*-slice and* $p \in S$. *Then there exists a section through* p *and* q.

The following useful property now follows easily.

Proposition 2.6. *Let* $q \in M$. *For all* $v \in T_q G(q)^{\perp}$ *there exists a section* Σ *through* q *to which* v *is tangent.*

Proof. Use Proposition 2.5 (ii) and the following simple remark, which replaces the role of flatness of a section:

If p, q are two points of a section Σ chosen to be sufficiently close, with $p = exp_q(v)$, then the geodesic $\gamma(t) = exp_q(tv)$ lies in Σ, since Σ is totally geodesic. ■

3. Non positive sectional curvature, variational completeness and morse theory.

We start with the following definiton.

Definition 3.1 ([3]). A geodesic $\gamma : [0, 1] \to M$ is called *transversal* if it meets the G-orbits orthogonally. A Jacobi field is called *transversal* if it is induced by a variation made of transversal geodesics.

Fix an orbit $G(z)$ and a transversal geodesic γ starting at $G(z)$. Denote by $\mathfrak{J}(G(z), \gamma)$ the space of transversal Jacobi fields along the transversal geodesic γ.

The restriction to a transversal geodesic γ of an infinitesimal motion X gives rise to a transversal Jacobi field (induced by G). It is very natural to ask when the converse is true; this leads to the following:

Definition 3.2 ([3]). The action of G on M is called *variationally complete* if a transversal Jacobi field that is tangent to the G-orbits in two different points of the underlying transversal geodesic γ is induced by G, i. e., it is the restriction to γ of an infinitesimal motion.

We give a sufficient condition for the action to be variationally complete.

THEOREM 3.3. *Let M be a G-manifold with non-positive sectional curvature. If there exist sections for M, then the action is variationally complete.*

Conlon in [5] proves an analogous theorem, under stronger hypotheses: he assumes the group G to be compact and the sections to be flat. However, we can easily get rid of these assumptions, in the following way. For as much as concerns the hypothesis on G, we just note that an isometric action is in particular a proper action and it is well known that this condition is a natural substitute for compactness.

By looking closely at Conlon's proof, and applying the properties of sections proved in § 2, it is easy to see that the crucial point is the non-existence of conjugated points on a given section and this is excluded by the assumption of non-positive sectional curvature.

4. Focal points and singular orbits.

We now show how the notion of focal point proves to be a very useful tool in investigating the set of singular points of the G-action. In order to determine the so-called «walls» of a section, we are now interested in determining the set of points where a minimal geodesic meets orbits of lower dimension. The focal locus of a principal orbit $G(z)$ turns out to be of interest in this search.

Szenthe in [12] introduces the following description of focal points which gives rise to a necessary and sufficient condition for an orbit to have lower dimension.

Fix a principal orbit $G(z)$ and a transversal geodesic γ. The Jacobi fields belonging to $\mathfrak{J}(G(z), \gamma)$ can be characterized by the conditions ([6]):

(i) $J(0) \in T_z G(z)$;

(ii) $\nabla_{\gamma'(0)} J(0) - \sigma_{\gamma'(0)} J(0) \in N_z G(z)$ (where $\sigma_{\gamma'(0)}$ denotes the Weingarten operator). Among these, it is natural to consider the Jacobi fields for which

(a) $J(0) = 0$ and those verifying

(b) $\nabla_{\gamma'(0)} J(0) - \sigma_{\gamma'(0)} J(0) = 0$.

The space $\mathfrak{J}(G(z), \gamma)$ then naturally splits as the direct sum of two subspaces: the union of the conjugate loci of points belonging to $G(z)$, given by Jacobi fields verifying (a) and denoted by $\mathfrak{J}_0(G(z), \gamma)$, and the so-called *strong Jacobi fields*, given by Jacobi fields verifying (b) and denoted by $\mathfrak{J}_s(G(z), \gamma)$:

$$\mathfrak{J}(G(z), \gamma) = \mathfrak{J}_0(G(z), \gamma) \oplus \mathfrak{J}_s(G(z), \gamma) .$$

It is easily seen that $\dim \mathfrak{J}_s(G(z), \gamma) = \dim G(z)$.

Write a transversal Jacobi field J as $J = J_0 + J_s$. The existence of sections guarantees that if at a point $t_0 \in \mathbb{R}$ J vanishes, then either $J_0(t_0) = 0$ or $J_s(t_0) = 0$ (see [12]).

REMARK 4.1 ([12]). Strong Jacobi fields verify a sort of a variational completeness.

In fact, note that if $X \in \mathfrak{g}$,then the restriction of X to γ gives rise to a strong Jacobi field. In order to see this, we must show that $\nabla_{\gamma'(0)} X(0) \in$ $\in T_z G(z)$. Since the action admits sections, the distribution (defined on M_{reg}) given by $p \mapsto T_p G(p)^\perp$ is integrable. According to Frobenius' theorem, it must be also involutive, so we get that for all $U, V \in N_z G(z)$ and for all $X \in \mathfrak{g}$

$$0 = \langle X, [U, V] \rangle = \langle X, \nabla_U V - \nabla_V U \rangle =$$

$$= \langle A_X U, V \rangle - \langle A_X V, U \rangle = 2\langle A_X U, V \rangle$$

thanks to the well-known antisimmetry property of A_X (see e. g. [8]). It follows that $A_X U = \nabla_U X \in T_z G(z)$ for all $U \in N_z G(z)$. By applying this fact to $U = \gamma'(0)$ the assertion follows.

In order to conclude, use the fact that the space of Killing fields along γ and the space of strong Jacobi fields both have dimension equal to $\dim G(z)$.

DEFINITION 4.2 ([12]). A point $x = \gamma(\tau) \in M$ is a *strong focal point* if there exists a non-indentically zero strong Jacobi field J along γ such that $J(\tau) = 0$.

The following theorem justifies the introduction of strong Jacobi fields.

THEOREM 4.3. *Suppose the action admits sections. Let $G(z)$ be a principal orbit. Then the orbit of a point $x \in M$ is singular non-exceptional if and only if x is a strong focal point for $G(z)$.*

PROOF. Assume first that x has a singular non-exceptional orbit. According to Proposition 2.6, there exists a section Σ through z and x.

Then, since the isotropy subgroup of a regular point fixes Σ pointwise, we must have

$$G_z \subset G_x$$

(proper inclusion). Consequently, there is an $X \in \mathfrak{g}$ such that $X(z) \neq 0$ and $X(x) = 0$. Then the restriction of X to γ gives rise to the required strong Jacobi field.

Assume now that x is a strong focal point. Then there exists a Jacobi field $J \in \mathfrak{J}_s(G(z), \gamma)$ along $\gamma(t) = Exp(tv)$ and a $t_0 \in \mathbb{R}$ with $x = Exp(t_0 v)$ and $J(t_0) = 0$. By applying now Remark 4.1 the assertion follows. ■

The case of non-positive curvature

In this case there are no conjugate points, so all Jacobi fields are strong Jacobi fields. Therefore we once again get that the action is variationally complete.

Moreover, in this case Theorem 4.3 can be restated as:

THEOREM 4.4. *Suppose that M has non-positive sectional curvature and that the action admits sections. Let $G(z)$ be a principal orbit. Then the orbit of a point $x \in M$ is singular non-exceptional if and only if x is a first focal point for $G(x)$.*

PROOF OF THEOREM 1. In order to prove Theorem 1, we now need to find a sufficient condition for the existence of the Morse series, i. e. for it to have only a finite number of terms. Let $p \in M \backslash G(x)$ be a regular point. In order to calculate the Morse series $\mathcal{M}(t)$ relative to $G(x)$ and p, we need to know:

(a) the geodesic segments σ in $S = S(M, G(x), p)$

(b) the points where these meet orbits of lower dimension.

Let us start from (a). Suppose the Weyl group is finite. Let $\sigma \in S(M, G(x), p)$. Since $\sigma'(0) \in T_{\sigma(0)} G(x)^{\perp}$, there exists a section Σ to which σ is tangent and so σ lies in Σ. In particular, $p \in \Sigma$ and so, by the regularity of p, <u>all</u> the geodesics of $S(M, G(x), p)$ belong to Σ. The finiteness of the Weyl group guarantees then the finiteness of S.

Consider now (b). Recall from §1 the expression for the Morse series for variationally complete actions.

Theorem 4.3 tells us exactly which points are such that $\delta(\sigma(t)) \neq 0$ and allows us to complete the proof of Theorem 1.

5. Weyl chambers.

We now want to analyze the action induced by G on a section Σ by means of the Weyl group. More precisely, we want to generalize the concepts of «walls» and of « Weyl chambers», which naturally arise in the theory of compact Lie groups (see Bott [2]) and which were further investigated by Conlon [5] and Szenthe [12] in the more general setting of variationally complete actions.

We begin by individuating the natural candidates to be the walls of the section.

LEMMA 5.1. *Let* $p \in \Sigma$ *and let* $S_p = \{x \in \Sigma : G_p \subseteq G_x\}$. *Each component of* S_p *is a closed, totally geodesic submanifold of* Σ.

PROOF. S_p is obviously closed in Σ.

Let $x \in S_p$. Consider

$$V = \{v \in T_x \Sigma : G_p v = v\} \subset N_x(G(x)) .$$

Clearly $Exp(V) \subseteq S_p$ and, in a sufficiently small neighborhood U of $x \in \Sigma$, $U \cap S_p = U \cap Exp(V)$. ■

Let K be the isotropy of any regular point in Σ. Suppose that the action does not have exceptional singular points, i. e. singular points whose isotropy subgroup has the same dimension as K. Then the set Σ_{sing} coincides with the set of points $p \in \Sigma$ such that dim $G_p >$ dim K or, equivalently, with the set of points $p \in \Sigma$ such that $S_p \neq \Sigma$.

DEFINITION 5.2 ([5]). A subset S of Σ is called a *singular submanifold* if it is a component of S_p for $p \in \Sigma_{sing}$ but it is not a proper subset of any S_q (for $q \in \Sigma_{sing}$).

LEMMA 5.3. *If* $S \subset \Sigma$ *is a singular submanifold, there exist a point* $p \in S$ *and a neighborhood* U *of* p *in* Σ *such that* $S \cap U = \Sigma_{sing} \cap U$.

PROOF. Follow Conlon's proof to find a point $p \in S$ and a neighborhood U of p in Σ such that $G_y \subseteq G_p$ for all $y \in U$ and $G_y = G_p$ for all $y \in S \cap \cap U$. Note that S is then a component of S_p.

Restrict, if necessary, U to a normal neighborhood of p. If $x \in \Sigma_{sing} \cap \cap U$, then $G_x \subseteq G_p$ and so G_x fixes the only minimal geodesic joining x to p pointwise. In particular γ is included in $U \cap S_x \subseteq \Sigma_{sing} \cap U$ and so, if $x \notin S \cap U$, S would be properly included in S_x, against its definition. It then follows $\Sigma_{sing} \cap U \subseteq S \cap U$ and the reverse inclusion is obvious. ■

The following proposition is of significant interest and allows us to call the singular submanifolds «walls» of the section.

PROPOSITION 5.4. *Suppose all the singular points of the action are non-exceptional. If $S \subset \Sigma$ is a singular submanifold, then S has codimension 1 in Σ.*

PROOF. Note that $\operatorname{codim}_\Sigma S \geq 1$. Let $p \in S$ and U as in the previous lemma and choose $x \in U \setminus S$, x is then a regular point. Consider the geodesic $\gamma : [0, \lambda] \to U \subset \Sigma$ starting at x and meeting S orthogonally at $\gamma(\lambda)$. For $t < \lambda$, γ is then minimal, while there exist two points $x, x' \in G(x)$ in U at which the distance of $G(x)$ from the point $\gamma(\lambda) \in S$ is achieved. If $\operatorname{codim}_\Sigma S > 1$, then there would exist a point $y \in U \setminus S$ which could be joined to x and x' with minimal geodesics taking values in Σ. In particular y would be a (strong) focal point for the orbit $G(x)$ and so, by Theorem 4.3, a singular point, against the construction of U. ■

PROOF OF THEOREM 2. While proving the previous proposition, a point $x' = g(x) \in G(x)$ was associated to a regular point $x \in \Sigma$. $x' = g(x)$ is a point of $G(x)$ belonging to Σ and is on the other side of the «wall» S. As x is regular, the element $g \in G$ determines a non-trivial element of the Weyl group of Σ. In order to prove Theorem 2, we wish to show that the restriction of g to Σ is an involutive isometry fixing S pointwise.

Let $U \subset \Sigma$, $x \in U$ and $g \in W_\Sigma$ defined as above. The proof is divided in steps.

(1): $g(U) = U$ and g fixes S pointwise.

We begin by showing $g(p) = p$. We have

$$d(p, g(p)) \leq d(p, g(x)) + d(g(x), g(p)) < 2r$$

where r is the radius of U. The Weyl group W_Σ being discrete, the distance $d(w(p), p)$ has a positive minimum d when w runs in the set $\{a \in W : a(p) \neq p\}$; by choosing $r < d/2$ we are able to conclude $g(p) = p$. Therefore, g is an isometry fixing the center of U, so $g(U) = U$. Moreover g fixes S pointwise, since $S = S_p$ and $g \in G_p$.

(2): g «changes sides» of $U \cap S$ in $U \subset \Sigma$.

By restricting U, we can assume U to be an open chart of Σ with coordinates $(y_1, \ldots, y_k)$ where the submanifold S is $y_k = 0$. Let $U^+ = \{y \in \in U : y_k > 0\}$ e $U^- = \{y \in U : y_k < 0\}$. $S \cap U$ separates U in the two components U^+ and U^-. Let $x \in U \backslash S$ as above, if $x \in U^+$, then $g(x)$ has negative k-th coordinate, that is, $g(U^+) \cap U^- \neq \emptyset$. It then follows, g being continuous, $g(U^+) = U^-$.

(3): $(g|_\Sigma)^2 = Id|_\Sigma$.

It is sufficient to prove that $(g|_U)^2 = Id|_U$. Let $y \in U \backslash S$ and suppose $g^2(y) \neq y$. Take U to be a chart as above. Since g switches the sides of $U \cap S$ in U, the points $g^2(y)$ and y turn out to be on the same side of $S \cap \cap U$ in U, say in U^+. Then there would exist a point $q \in U^+$ which could be joined both to y and to $g^2(y)$ with geodesics taking values in U^+ (and therefore minimal). The point y is regular by construction and the two geodesics both realize the distance of the point q from the orbit $G(y)$. The point q would then necessarily be a strong focal point (since U contains no conjugate points) which is absurd.

Theorem 2 is thus proved.

We call g the *orthogonal reflection* of Σ around S. We conclude by giving some immediate properties of walls.

PROPOSITION 5.5. *The set of singular submanifolds is discrete.*

PROOF. Suppose there exists a sequence of singular points $\{p_n\}$ with an accumulation point p (p is necessarily singular). For sufficiently large n, p_n belongs to a G_p-slice and thus $G_{p_n} \subseteq G_p$, that is $p \in S_{p_n}$. All these points must then belong to the same singular submanifold. ∎

Σ_{sing} being the union of the walls, the set Σ_{reg} is naturally decomposed in the components of $\Sigma \backslash \Sigma_{sing}$. Each of these components is called *Weyl chamber* of the section and the Weyl group W_Σ acts freely as a permutation group of the Weyl chambers.

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