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Decay of Solutions to the Mixed Problem for the Linearized Boltzmann Equation with a Potential Term in a Polyhedral Bounded Domain.

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ABSTRACT - We study decay of solutions to the mixed problem with the perfectly reflective boundary condition for the linearized Boltzmann equation with an external-force potential in a polyhedral bounded domain, i.e., in a bounded domain whose boundary is a 2-dimensional piecewise linear manifold. We do not assume that the domain is convex. The purpose of this paper is to prove that the solutions of the mixed problem decay exponentially in time.

1. Introduction.

The nonlinear Boltzmann equation describes the evolution of the density of rarefied gas. If an external conservative force acts on the gas particles, then the equation has the form,

$$\text{(NBE)} \qquad \partial f/\partial t + \Lambda f = Q(f, f),$$

where $Q(\cdot, \cdot)$ denotes the nonlinear collision operator (see [5], pp. 30-31),

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and Λ is a differential operator defined as follows:

$$\Lambda \equiv \xi \cdot \nabla_x - \nabla_x \phi(x) \cdot \nabla_\xi . \tag{1.1}$$

We denote the time variable, the space variable, and the velocity variable by t, x, and ξ respectively. We denote by $\phi = \phi(x)$ a potential of the conservative external force. We denote by $f = f(t, x, \xi)$ the unknown function which represents the density of gas particles that have a velocity $\xi \in \mathbb{R}^3$ at time $t \geqq 0$ and at a point $x \in \Omega$, where Ω is a domain of $\mathbb{R}^3$ (i.e., a connected open subset of $\mathbb{R}^3$). We assume that the gas particles are confined in Ω by being reflected perfectly from the boundary $\partial\Omega$.

This paper uses the assumption of cut-off hard potentials in the sense of Grad (see [5-6]). We linearize (NBE) around the equilibrium state, $M \equiv \exp(-\phi(x) - |\xi|^2/2)$, under this assumption. Substituting $f = M + M^{1/2}u$ in (NBE), and dropping the nonlinear term, we obtain *the linearized Boltzmann equation with a potential term*,

$$\partial u/\partial t = Bu , \tag{LBE}$$

$$B \equiv -\Lambda + (\exp(-\phi))(-\nu + K), \tag{1.2}$$

where $\nu = \nu(\xi)$ is a multiplication operator, and K is an integral operator. These operators act only on the velocity variable ξ, and have the following properties (see [5-6] for the proof):

LEMMA 1.1. (i) There exist positive constants ν_j, $j = 0, 1$, such that $\nu_0 \leqq \nu(\xi) \leqq \nu_1(1 + |\xi|)$ for each $\xi \in \mathbb{R}^3$.

(ii) K is a self-adjoint compact operator in $L^2(\mathbb{R}^3_\xi)$.

(iii) $(-\nu + K)$ is a nonpositive operator in $L^2(\mathbb{R}^3_\xi)$ whose null space is spanned by $\xi_j \exp(-|\xi|^2/4)$, $j = 1, 2, 3$, $\exp(-|\xi|^2/4)$, and $|\xi|^2 \exp(-|\xi|^2/4)$, where we denote the j-th component of ξ by ξ_j, $j = 1, 2, 3$, i.e., $\xi = (\xi_1, \xi_2, \xi_3)$.

We write (MP) as the mixed problem for (LBE) with the perfectly reflective boundary condition. The purpose of the present paper is to study decay of solutions to (MP). If we try to study this subject, then we easily find the need to investigate the structure of the spectrum of the operator B. However, inspecting the form of B, we can say that the structure of the spectrum of B is greatly influenced by the operator Λ. Moreover, prolonging solutions of the following system of ordinary differential equations by making use of the law of perfect reflection on the boundary

$\partial\Omega$ (see § 4 for the details), we can construct the characteristic curves of Λ:

$$\text{(SODE)} \qquad dx/dt = \xi\,, \qquad d\xi/dt = -\nabla\phi(x)\,.$$

Therefore we can conclude that the structure of the spectrum of B must be closely connected to the behavior of the prolonged solutions of (SODE). For this reason, in order to achieve the purpose of the present paper, we need to fully investigate the behavior of the prolonged solutions of (SODE).

The behavior of the prolonged solutions of (SODE) is complicated in general, and it is difficult to inspect the behavior of the prolonged solutions of (SODE) globally in time. However, it must be noted that the complexity of the behavior of the prolonged solutions of (SODE) depends largely on geometry of the boundary surface $\partial\Omega$. For example, as the geometry of $\partial\Omega$ becomes more complex, the solutions of (SODE) are prolonged by being reflected from $\partial\Omega$ in a more complicated manner. Hence the behavior of the prolonged solutions is also more complex. Conversely, as the geometry of $\partial\Omega$ becomes simpler, the solutions of (SODE) are prolonged in a simpler manner. Hence the behavior of the prolonged solutions of (SODE) is also simplified. Taking these facts into account, and recognizing the difficulty caused by the complexity of the behavior of the prolonged solutions of (SODE), we find the need to simplify the behavior of the prolonged solutions of (SODE) by imposing some assumption on geometry of $\partial\Omega$.

For this reason, in this paper, we will assume that Ω is a polyhedral bounded domain, i.e., that Ω is a bounded domain whose boundary is a 2-dimensional piecewise linear manifold. By virtue of this assumption, we can simplify the behavior of the prolonged solutions. Hence we can fully investigate the behavior of the prolonged solutions of (SODE).

Under the spatial periodicity condition, in [12] (in [16], respectively) we investigate decay of solutions of (LBE) (the structure of the spectrum of the linear transport operator with a potential term, respectively). Hence, it seems to be promising to attempt to apply the methods in [12] and [16] also to the problem treated in this paper. However, if we do so, then we immediately encounter the difficulty which is caused not only by the fact that we do not impose the spatial periodicity condition in the present paper but also by the fact that we do not assume that the domain Ω is convex. In particular, the second fact raises the difficulty such that there is a possibility that some characteristic curves of Λ tend to follow $\partial\Omega$. For these reasons, we cannot apply the methods in [12], § 7-8 and those in [16], § 6 to our problem (we will discuss this difficulty in Remarks 7.2-3).

We will overcome these difficulties by improving the results of [16], § 6 greatly. We will make that great improvement, by proving inequalities similar to those in [16], Lemma 6.1 also for such erratic characteristic curves as those which tend to follow the boundary $\partial\Omega$. In proving those inequalities, we cannot apply the methods in [16], § 6 at all; we have to perform calculations (on Jacobian matrices) which are more complicated than and quite different from those done in [16], § 6. In these calculations, we make use of the assumption that Ω is a polyhedral bounded domain. The main result of this paper is the Main Theorem demonstrated in § 3, which is as follows: the solutions to (MP) decay exponentially in time.

This paper has 7 sections in addition to the present section. § 2 presents preliminaries. In § 3, we will prove the Main Theorem, by making use of Lemmas 3.1-2. In particular, Lemma 3.2 is a key lemma which plays an essential role in this proof. In § 4, by making use of the perfectly reflective boundary condition, we prolong the solutions of (SODE) globally in time, in the same way as [14], § 5. In § 5 we prove Lemma 3.1. In § 6 we seek estimates which imply Lemma 3.2. In § 7, by making use of the assumption that Ω is a polyhedral bounded domain, we greatly improve the inequalities of [16], Lemma 6.1, as mentioned above. In § 8, combining the result of § 7 and the methods in [16], § 7-8, we will prove the estimates sought in § 6.

REMARK 1.2. (i) The subject of the present paper is closely related to the theory of dynamical systems. Cf. [12], § 1, [13], pp. 742-746 and pp. 754-756, and [16], § 1.

(ii) We can simplify the behavior of the solutions of (SODE), also by assuming that the external potential is spherically symmetric. See [13].

(iii) For studies already made on (NBE) and (LBE), see, e.g., [1-4], [7-11], and [17].

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2. Preliminaries.

(1). *Assumption.* As already mentioned in § 1, we impose the following assumption on Ω:

ASSUMPTION Ω. (i) Ω is a bounded connected open subset of $\mathbb{R}^3$.

(ii) $\partial\Omega$ is a 2-dimensional piecewise linear manifold.

We easily see that if Ω is a bounded connected open subset of $\mathbb{R}^3$ whose boundary $\partial\Omega$ is homeomorphic to a torus or to a sphere and if $\partial\Omega$ is represented as the union of finite triangles, then Ω satisfies Assumption Ω.

From Assumption Ω we easily see that we can regard $\partial\Omega$ as a polyhedron. We call an edge of this polyhedron an edge of the 2-dimensional piecewise linear manifold $\partial\Omega$. We denote $\{x \in \partial\Omega;\, x \text{ is contained in an edge of } \partial\Omega\}$ by $E(\partial\Omega)$. By $\boldsymbol{n} = \boldsymbol{n}(x)$ we denote the outer unit normal of $\partial\Omega$ at $x \in F(\partial\Omega)$, where $F(\partial\Omega) \equiv \partial\Omega \backslash E(\partial\Omega)$.

We impose the following assumption on $\phi = \phi(x)$:

ASSUMPTION ϕ. (i) $\phi = \phi(x)$ is a real-valued function defined in Ω, and have continuous partial derivatives of order up to and including 2.

(ii) $\sup_{x \in \Omega} |\partial^2 \phi(x)/\partial x_i \partial x_j| < +\infty$, $i, j = 1, 2, 3$, where we denote by x_i the i-th component of x, $i = 1, 2, 3$, i.e., $x = (x_1, x_2, x_3)$.

(iii) $\boldsymbol{n}(x) \cdot \lim_{y \to x,\, y \in \Omega} \nabla\phi(y) = 0$, for each $x \in F(\partial\Omega)$.

REMARK 2.1. (i) From Assumption ϕ, (i-ii), we see that there exists $\lim_{y \to x,\, y \in \Omega} \nabla\phi(y)$ for each $x \in \partial\Omega$. Hence, we can make Assumption ϕ, (iii).

(ii) From Assumption ϕ, (i-ii) and Assumption Ω we easily see that $\phi = \phi(x)$ is uniformly bounded in Ω. Hence, for simplicity, we will assume that $\phi = \phi(x)$ is positive-valued; there is no loss of generality.

(iii) We will make use of Assumption ϕ, (iii), only to prove (7.20). In Remark 7.6, we will discuss a role played by Assumption ϕ, (iii).

(iv) From Assumption ϕ, (i), we see that if $X \in \Omega$ and $\Xi \in \mathbb{R}^3$, then the Cauchy problem for (SODE) with the initial data $(x, \xi)(0) = (X, \Xi)$ has a unique solution, which can be prolonged until $x = x(t)$ collides with $\partial\Omega$. Moreover, it follows from Assumption Ω, (ii), and Assumption ϕ,(i-ii), that even if $X \in F(\partial\Omega)$, $\boldsymbol{n}(X) \cdot \Xi < 0$, and $\Xi \in \mathbb{R}^3$, then the Cauchy problem has a unique solution, which also can be prolonged until $x = x(t)$ collides with $\partial\Omega$. In § 4, by making use of the perfectly reflective boundary condition, we will prolong these solutions globally in time.

(v) In (ii) and (iv) of the present remark, we make use of Assumption Ω, (ii). However, Assumption Ω, (ii) does not play an essential role there. In fact, in (ii) and (iv) we have only to assume that $\partial\Omega$ is piecewise sufficiently smooth. However, in § 7, Assumption Ω, (ii), will play an essential role.

(2). *Function spaces.* By $\mathbb{B}(X, Y)$ ($\mathbb{C}(X, Y)$, respectively) we denote the set of all bounded (compact, respectively) linear operators from a Banach space X to a Banach space Y. For simplicity, we write $\mathbb{B}(X)$ and $\mathbb{C}(X)$ as $\mathbb{B}(X, X)$ and $\mathbb{C}(X, X)$ respectively. By E_α, $\alpha \geqq 0$, we denote a Hilbert space of complex-valued functions of $(x, \xi) \in \Omega \times \mathrm{R}^3$ with the following inner product (recall Remark 2.1,(ii)):

$$(u, v)_\alpha \equiv \int_{\Omega \times \mathrm{R}^3} u(x, \xi)\, \overline{v(x, \xi)}(1 + E(x, \xi))^\alpha\, dx\, d\xi \,,$$

$$E(x, \xi) \equiv \phi(x) + |\xi|^2/2\,. \tag{2.1}$$

Define $\|u\|_\alpha \equiv ((u, u)_\alpha)^{1/2}$. Write $|||\cdot|||_\alpha$ as the norm of operators of $\mathbb{B}(E_\alpha)$. Write $\|\cdot\|$, $|||\cdot|||$ as $\|\cdot\|_0$, $|||\cdot|||_0$, respectively for simplicity.

(3). *The domains of operators.* We denote the domain of an operator L by $D(L)$. Let us define the domain of Λ (see (1.1) for Λ) as follows: $D(\Lambda) \equiv \{u = u(x, \xi) \in E_\alpha;\ \Lambda u \in E_\alpha$, and $u = u(x, \xi)$ satisfies the boundary condition,

$$(\gamma_+ u(\cdot, \cdot))(X, \Xi) = (\gamma_- u(\cdot, \cdot))(X, \Xi - 2(\boldsymbol{n}(X)\cdot\Xi)\boldsymbol{n}(X))\,, \tag{PRBC}$$

for a.e. $(X, \Xi) \in \boldsymbol{F}_+ \}$, where we denote by $\gamma_\pm$ the trace operators along the characteristic curves of Λ onto

$$\boldsymbol{F}_\pm \equiv \{(X, \Xi) \in F(\partial\Omega) \times \mathrm{R}^3;\ \pm\boldsymbol{n}(X)\cdot\Xi > 0\}\,. \tag{2.2}$$

We similarly define the domain of the operator,

$$A \equiv -\Lambda + (\exp(-\phi))(-\nu)\,, \tag{2.3}$$

as follows: $D(A) \equiv \{u = u(x, \xi) \in E_\alpha;\ Au \in E_\alpha$, and $u = u(x, \xi)$ satisfies

(PRBC) for a.e. $(X, \Xi) \in \boldsymbol{F}_+\}$. Applying Remark 2.1, (ii), we deduce that

$$\boldsymbol{K} \equiv (\exp(-\phi))K \in \mathbb{B}(E_\alpha, E_{\alpha+1}) \quad \text{for each } \alpha \geqq 0\,, \tag{2.4}$$

in the same way as [12], Lemma 2.1, (iv). Hence, noting that $E_{\alpha+1} \subset E_\alpha$ and that $B = A + \boldsymbol{K}$, we can define $D(B) \equiv D(A)$ (see (1.2)).

REMARK 2.2. (i) The boundary condition (PRBC) is the perfectly reflective boundary condition. If $u = u(x, \xi) \in D(\Lambda)$ or if $u = u(x, \xi) \in$ $\in D(A)$, then $u = u(x, \xi)$ is absolutely continuous along the characteristic curves of Λ. See [15], p. 33.

(ii) By $Q(\mathbb{R}^3_\xi)$ we denote the set of all the one-rank operators of the form, $(\mathrm{ku}(\cdot, \cdot))(x, \xi) = (u(x, \cdot), f(\cdot))g(\xi)$, where the brackets $(\cdot, \cdot)$ denote the inner product in $L^2(\mathbb{R}^3_\xi)$. $f = f(\xi)$ and $g = g(\xi)$ are infinitely partially differentiable functions of $\xi \in \mathbb{R}^3$ which have compact supports. Making use of Lemma 1.1, (ii), and performing calculations similar to, but easier than, those in [16, Lemma 2.2], we can deduce that the operator K can be approximated in $\mathbb{B}(E_0)$ with a finite sum of operators of $Q(\mathbb{R}^3_\xi)$.

(4). *The purely imaginary point spectrum of* B. Performing calculations similar to those in [14], Theorem 4.1, (i-ii) and [15], Theorem 4.1, (i-ii), we can obtain the purely imaginary point spectrum of B. Let l be a straight line $\subset \mathbb{R}^3$. From Assumption Ω, we see that $\partial\Omega$ is represented as the union of finite triangles. Making use of this result, we see that if $\varepsilon >$ > 0 is sufficiently small, then the ε-degree rotation upon l cannot map Ω into itself. Therefore Ω has no axis of rotation, whereas if the θ-degree rotation upon a line maps Ω into itself for each $\theta > 0$, then we say that the line is an axis of rotation of Ω. Making use of this result, and recalling [14], (3.9) and [15], pp. 33-34, we can greatly simplify the calculations performed in [14], Theorem 4.1, (i-ii) and [15], Theorem 4.1, (i-ii), and we can obtain the following lemma (see (2.1) for $E(x, \xi)$):

LEMMA 2.3. *The intersection of* $\{\mu \in \mathbb{C}; \operatorname{Re}\mu \geqq 0\}$ *and the point spectrum of* B *is equal to* $\{0\}$. *The null space of* B *is spanned by* $\mathrm{e}^{-E(x,\xi)/2}$ *and* $E(x, \xi)\,\mathrm{e}^{-E(x,\xi)/2}$.

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We denote by $\boldsymbol{P}$ the projection operator (in E_0) upon the null space of B. It follows from Remark 2.1, (ii), and Lemma 2.3 that

$$\boldsymbol{P} \in \mathrm{C}(E_\alpha, E_\beta) \quad \text{for each } \alpha, \beta \geqq 0 . \tag{2.5}$$

3. Main Theorem.

Write (MP) as the mixed problem for (LBE) with the boundary condition (PRBC) and with the initial data,

$$u|_{t=0} = u_0 \in D(B) \cap E_{\alpha, \perp}, \tag{ID}$$

where we denote by $E_{\alpha,\perp}$ the set of all functions of E_α which are perpendicular (in E_0) to the null space of B. In what follows throughout the paper, we denote some positive constant by c, and we use the letter c as a *generic* constant replacing any other constants (such as c^3 or $c^{1/2}$) by c. Hence they are not the same at each occurrence. The following theorem is the main result of this paper:

MAIN THEOREM. For each $\alpha \geqq 0$, the mixed problem (MP) has a unique solution $u = u(t) \in C^1((0, +\infty); E_\alpha)$ which satisfies the following (3.1-2):

$$u(t) \in E_{\alpha,\perp}, \tag{3.1}$$

$$\|u(t)\|_\alpha \leqq c\|u_0\|_\alpha \exp(-c_{3.0}t), \tag{3.2}$$

for each $t \geqq 0$, where $c_{3.0}$ is a positive constant independent of t and u_0.

In order to prove the Main Theorem we will apply the following lemma, which deals with the semigroup generated by A (see (2.3)) and with the resolvent operator of A:

LEMMA 3.1. (i) For any $\alpha \geqq 0$, the operator A generates a strongly continuous semigroup in E_α, which satisfies the following inequality:

$$|||e^{tA}|||_\alpha \leqq \exp(-c_{3.1.1}t), \qquad \text{for each } t \geqq 0,$$

where $c_{3.1.1} \equiv \nu_0 \exp\left(-\sup_{x\in\Omega}\phi(x)\right)$, (see Lemma 1.1, (i) for ν_0).

(ii) Let α, $c_{3.1.2}$ and C be constants such that $\alpha \geqq 0$, $c_{3.1.1} > c_{3.1.2} > 0$, and $C > 0$. If $f = f(t)$ is a continuous function from $[0, +\infty)_t$ to E_α such that (see (2.4) for $\boldsymbol{K}$)

$$\|f(t)\|_\alpha \leqq C\exp(-c_{3.1.2}t), \qquad \text{for each } t \geqq 0,$$

then

$$\left\| \int_0^t e^{(t-s)A}\boldsymbol{K}f(s)\,ds \right\|_{\alpha+1} \leqq cC\exp(-c_{3.1.2}t), \qquad \text{for each } t \geqq 0.$$

(iii) Write $\mu = \gamma + i\delta$, γ, $\delta \in \mathbb{R}$. Define $A^* \equiv \Lambda + (\exp(-\phi))(-\nu)$. If $\beta \equiv \gamma + c_{3.1.1} > 0$ and $f \in E_0$, then,

$$\int_{-\infty}^{+\infty} \|(\mu - A)^{-1}f\|^2 d\delta, \qquad \int_{-\infty}^{+\infty} \|(\mu - A^*)^{-1}f\|^2 d\delta \leqq c\|f\|^2/\beta.$$

This lemma plays the same role as [12], Lemma 3.1 and Lemma 3.3. In addition to this lemma, in order to prove the Main Theorem we need the following lemma, which plays the same role as [16], Lemma 3.2:

LEMMA 3.2. $\mathbb{L} = \mathbb{L}(\mu) \equiv (\boldsymbol{K} - \boldsymbol{P})(\mu - A)^{-1}$ is an analytic operator-valued function of $\mu \in \mathbb{D} \equiv \{\mu \in \mathbb{C};\ \mathrm{Re}\,\mu > -c_{3.1.1}\}$, and satisfies the following (i-ii), (see (2.4-5) for $\boldsymbol{K}$ and $\boldsymbol{P}$): (i) $\mathbb{L}^4(\mu) \in \mathrm{C}(E_0)$ for each $\mu \in \mathbb{D}$, (ii) $|||\mathbb{L}^4(\mu)||| \to 0$ as $|\mu| \to +\infty$, $\mu \in \mathbb{D}$.

PROOF OF THE MAIN THEOREM. It follows from Lemma 3.1, (i), that $(\mu - A)^{-1}$ is an analytic operator-valued function of $\mu \in \mathbb{D}$. Hence we can set the resolvent equation,

$$(3.3) \qquad (\mu - \mathbb{B})^{-1} = (\mu - A)^{-1} + (\mu - A)^{-1}(1 - \mathbb{L}(\mu))^{-1}\mathbb{L}(\mu),$$

where $\mu \in \mathbb{D}$ and $\mathbb{B} \equiv B - \boldsymbol{P}$. We consider the operator $\mathbb{B}$ in place of B, in order to remove the null space of B (cf. [12], p. 1833). By (2.4-5) and Lemma 3.1, (i), we easily see that $\mathbb{B}$ generates a strongly continuous semigroup in E_α, which is represented in terms of the inverse Laplace transformation of (3.3). Applying Lemmas 3.1-2 to that inverse Laplace transformation in the same way as [12], pp. 1833-1834, we can prove (3.2). By Lemma 1.1, (iii), and Lemma 2.3, we can obtain (3.1).

REMARK 3.3. (i) In [12], pp. 1833-1834 we do not need to directly apply the spatial periodicity condition. Hence, without the aid of that restrictive condition, we can apply Lemmas 3.1-2 in the proof above.

(ii) In [12] and [16], we obtain [12], Lemma 3.1 and Lemma 3.3 and [16], Lemma 3.2 under the spatial periodicity condition, but in the present paper we prove Lemmas 3.1-2 by Assumption Ω and Assumption ϕ. Lemma 3.1 will be proved in § 5. The proof of Lemma 3.2 is entirely different from and more complicated than that of [16], Lemma 3.2. For the same reason as [16], § 5, we consider the 4-th power $\mathbb{L}^4(\mu)$ in Lemma 3.2.

(iii) If Lemmas 3.1-2 are proved, then we can complete the proof of the Main Theorem. We can decompose $\mathbb{L}(\mu)$ as follows: $\mathbb{L}(\mu) = L_{\boldsymbol{K}}(\mu) + L_{\boldsymbol{P}}(\mu)$, where $L_{\boldsymbol{K}}(\mu) \equiv \boldsymbol{K}(\mu - A)^{-1}$ and $L_{\boldsymbol{P}}(\mu) \equiv -\boldsymbol{P}(\mu - A)^{-1}$. We can easily derive Lemma 3.2 from the following (3.4-7):

(3.4) $$L_{\boldsymbol{K}}^4(\mu) \in \mathrm{C}(E_0) \quad \text{for each } \mu \in \mathbb{D},$$

(3.5) $$|||L_{\boldsymbol{K}}^4(\mu)||| \to 0 \quad \text{as } |\mu| \to +\infty, \quad \mu \in \mathbb{D},$$

(3.6) $$L_{\boldsymbol{P}}(\mu) \in \mathrm{C}(E_0) \quad \text{for each } \mu \in \mathbb{D},$$

(3.7) $$|||L_{\boldsymbol{P}}(\mu)||| \to 0 \quad as |\mu| \to +\infty, \quad \mu \in \mathbb{D}.$$

Making use of (2.5) and Lemma 2.3, and performing calculations similar to, but much easier than, those in proving (3.4-5), we can obtain (3.6-7). Hence we will prove (3.4-5) only. (3.4-5) will be proved in § 6-8.

4. Prolonged solutions of (SODE).

By (CP) we denote the Cauchy problem for (SODE) with the initial data,

(4.1) $$(x, \xi)(0) = (X, \Xi) \in (\Omega \times \mathbb{R}^3) \cup \boldsymbol{F}_-.$$

See (2.2) for $\boldsymbol{F}_-$. Recall Remark 2.1,(iv). We denote by $(x, \xi) = (x(t), \xi(t))$ the solution to (CP). In the same way as [14], pp. 1284-1285, we will prolong the solution of (CP) globally in $t \in \mathbb{R}$ by the law of perfect reflection,

(4.2.s) $$(x(s+0), \xi(s+0)) = (x(s-0), \xi(s-0) - 2(\boldsymbol{n}(x(s-0)) \cdot \xi(s-0))\boldsymbol{n}(x(s-0))),$$

where $x(s \pm 0) \in F(\partial\Omega)$. Hereafter, we denote the solution of (CP) thus prolonged in $t \in \mathbb{R}$ by

(4.3) $$(x, \xi) = (x(t, X, \Xi), \xi(t, X, \Xi)).$$

We can decompose $\partial\Omega \times \mathbb{R}^3$ into four disjoint subsets as follows: $\partial\Omega \times \mathbb{R}^3 = \boldsymbol{E} \cup \boldsymbol{F}_0 \cup \boldsymbol{F}_+ \cup \boldsymbol{F}_-$, where $\boldsymbol{E} \equiv E(\partial\Omega) \times \mathbb{R}^3$ and $\boldsymbol{F}_0 \equiv \{(x, \xi) \in F(\partial\Omega) \times \mathbb{R}^3; \boldsymbol{n}(x)\cdot\xi = 0\}$. If $(x, \xi) = (x(t, X, \Xi), \xi(t, X, \Xi))$ goes into $\boldsymbol{E} \cup \boldsymbol{F}_0$, or if $x = x(t, X, \Xi)$ collides with $F(\partial\Omega)$ an infinite number of times in a finite time interval, then we cannot prolong the solution globally in time by the method in [14], pp. 1284-1285. Conversely, if $(x, \xi) = (x(t, X, \Xi), \xi(t, X, \Xi))$ does not go into $\boldsymbol{E} \cup \boldsymbol{F}_0$, and if $x = x(t, X, \Xi)$ does not collide with $F(\partial\Omega)$ an infinite number of times in a finite time interval, then we can prolong the solution globally in time by the method in [14], pp. 1284-1285. By virtue of the following lemma, we can prolong the solution of (CP) globally in time for almost all $(X, \Xi) \in \Omega \times \mathbb{R}^3$:

LEMMA 4.1. D_k, $k = 0, \ldots, 3$, are null sets in $\Omega \times \mathbb{R}^3$, where D_0 (D_1, respectively) $\equiv \{(X, \Xi) \in \Omega \times \mathbb{R}^3; (x(t, X, \Xi), \xi(t, X, \Xi)) \in \boldsymbol{F}_0$ ($\in \boldsymbol{E}$, respectively) for some $t \in \mathbb{R}\}$; D_2 (D_3, respectively) $\equiv \{(X, \Xi) \in \Omega \times \mathbb{R}^3$; there exists a strictly-monotone-increasing, positive-valued (strictly-monotone-decreasing, negative-valued, respectively) bounded sequence $\{t^j(X, \Xi)\}_{j \in \mathbb{N}}$ such that $x(t^j(X, \Xi), X, \Xi) \in F(\partial\Omega)$ for each $j \in \mathbb{N}$ and such that $x(t, X, \Xi) \in \Omega$ for each $t \in [0, t^\infty(X, \Xi)) \backslash \{t^j(X, \Xi)\}_{j \in \mathbb{N}}$ ($t \in (t^\infty(X, \Xi), 0] \backslash \{t^j(X, \Xi)\}_{j \in \mathbb{N}}$, respectively), where $t^\infty(X, \Xi) \equiv \lim_{j \to +\infty} t^j(X, \Xi)\}$.

PROOF. We can prove the present lemma when $k = 0, 2, 3$, in exactly the same way as [14], Lemma 5.1. In [14] we assume that $\partial\Omega$ is a C^2-class surface (see [14], Assumption 2.1,(ii)), and hence $\partial\Omega \times \mathbb{R}^3$ does not have such a singular subset as $\boldsymbol{E}$ in [14]. However, making use of the fact that $\boldsymbol{E}$ is a null set in $\partial\Omega \times \mathbb{R}^3$, and performing calculations similar to, but easier than, those in proving the present lemma with $k = 0$ (see [14], pp. 1289-1290), we can deduce that D_1 is a null set.

Let $(X, \Xi) \in \Omega \times \mathbb{R}^3$. By X_j, Ξ_j we denote the j-th component of X, Ξ respectively, $j = 1, 2, 3$, i.e., $X = (X_1, X_2, X_3)$, $\Xi = (\Xi_1, \Xi_2, \Xi_3)$. We denote the (i, j) component of the Jacobian matrix,

(4.4) $$J = J(t, X, \Xi) \equiv \partial(x(-t, X, \Xi), \xi(-t, X, \Xi))/\partial(X, \Xi),$$

by $m_{ij} = m_{ij}(t, X, \Xi)$, $i, j = 1, \ldots, 6$, i.e., if $1 \leqq i, j \leqq 3$, then $m_{ij}(t, X, \Xi) \equiv \partial x_i(-t, X, \Xi)/\partial X_j$. If $1 \leqq i \leqq 3$ and $4 \leqq j \leqq 6$, then $m_{ij}(t, X, \Xi) \equiv$

$\equiv \partial x_i(-t, X, \Xi)/\partial \Xi_{j-3}$. If $4 \leqq i \leqq 6$ and $1 \leqq j \leqq 3$, then $m_{ij}(t, X, \Xi) \equiv \equiv \partial \xi_{i-3}(-t, X, \Xi)/\partial X_j$. If $4 \leqq i, j \leqq 6$, then $m_{ij}(t, X, \Xi) \equiv \partial \xi_{i-3}(-\, - t, X, \Xi)/\partial \Xi_{j-3}$. The following lemma deals with this Jacobian matrix, and will be employed in § 7-8:

LEMMA 4.2. (i) Let $(X, \Xi) \in \Omega \times \mathbb{R}^3$ be such that $-\infty < t_-(X, \Xi) \equiv \equiv \inf\{s < 0; x(-t, X, \Xi) \in \Omega$ for each $t \in (s, 0]\}$. If

$$t_-(X, \Xi) < t < t_+(X, \Xi), \tag{4.5}$$

where $t_+(X, \Xi) \equiv \sup\{s > 0; x(-t, X, \Xi) \in \Omega$ for each $t \in [0, s)\}$, then

$$\| J(t, X, \Xi) \| \leqq \| J(t_-(X, \Xi) + 0, X, \Xi) \| \exp\{c_{4.2}(t - t_-(X, \Xi))\},$$

where $c_{4.2}$ is a positive constant dependent only on $\sup_{x \in \Omega,\, i, j = 1, \dots, 3} |\partial^2 \phi(x)/\partial x_i \partial x_j|$; we denote by $\| \cdot \|$ a norm of matrices defined as follows: $\| (a_{ij})_{i, j = 1, \dots, 6} \| \equiv \left(\sum_{i, j = 1}^{6} |a_{ij}|^2 \right)^{1/2}$.

(ii) If t satisfies (4.5), then

$$\sup_{(X, \Xi) \in \Omega \times \mathbb{R}^3} \| J(t, X, \Xi) - I \| \leqq c|t| \exp(c_{4.2}|t|),$$

where I denotes the 6×6 identity matrix.

(iii) Let $T > 0$ be a constant. Let $(X, \Xi) \in \Omega \times \mathbb{R}^3$ satisfy the following (4.6):

(4.6) $(x, \xi) = (x(-s, X, \Xi), \xi(-s, X, \Xi))$ does not go into $\boldsymbol{E} \cup \boldsymbol{F}_0$ when $s \in [0, T]$, and $x = x(-s, X, \Xi)$ does not collide with $F(\partial\Omega)$ an infinite number of times when $s \in [0, T]$.

If $t \in [0, T]$ satisfies the following (4.7):

$$x(-t, X, \Xi) \notin \partial\Omega, \tag{4.7}$$

then

$$|\det(J(t, X, \Xi))| = 1,$$

where we denote the determinant of a square matrix M by $\det(M)$.

(iv) If t, X, and Ξ satisfy (4.6-7), then $\partial x(-t, X, \Xi)/\partial t = -\, - \Lambda(X, \Xi)\, x(-t, X, \Xi)$, where $\Lambda(X, \Xi) \equiv \Xi \cdot \nabla_X - \nabla_X \phi(X) \cdot \nabla_\Xi$. Cf. (1.1).

PROOF. If t satisfies (4.5), then the particle $x = x(-t, X, \Xi)$ does not collide with $\partial\Omega$. Hence we do not need to take the boundary condition into account. We see that $(x, \xi) = (x(-t, X, \Xi), \xi(-t, X, \Xi))$ satisfies the Cauchy problem for the following system of ordinary differential equations with the initial condition (4.1) (cf. (SODE)):

$$dx/dt = -\xi, \qquad d\xi/dt = \nabla\phi(x). \tag{4.8}$$

Differentiating both sides of (4.8) with respect to (X, Ξ), we obtain an ordinary differential equation satisfied by $J = J(t, X, \Xi)$. We write (4.9) as that equation. Applying Assumption ϕ,(i-ii), to (4.9), we can obtain (i-ii). Performing calculations similar to, but much easier than, those in Proof of Lemma 7.1, we can prove (iii) (hence we omit the details). (iv) can be proved in the same way as [16], (7.1).

REMARK 4.3. (i) If we do not assume (4.6), then there is a possibility that we cannot define $(x, \xi) = (x(-t, X, \Xi), \xi(-t, X, \Xi))$ for some $t \in [0, T]$. (4.8-9) do not hold at the time when $x = x(-t, X, \Xi)$ collides with $\partial\Omega$. Hence we need to impose (4.7).

(ii) We easily see that $\{(X, \Xi) \in \Omega \times \mathbb{R}^3; (X, \Xi)$ does not satisfy (4.6)$\}$ is a null set and that $\{t \in [0, T]; x(-t, X, \Xi) \in \partial\Omega\}$ is a finite set for a.e. $(X, \Xi) \in \Omega \times \mathbb{R}^3$ (see Lemma 4.1). Therefore Lemma 4.2, (iii), (iv) can play the same roles as [16], (2.1), (7.1), respectively.

5. The operator A.

PROOF OF LEMMA 3.1. We easily see that A (see (2.3)) generates a strongly continuous semigroup in E_α for each $\alpha \geqq 0$. The semigroup e^{tA} has the form,

$$(e^{tA}f(\cdot, \cdot))(X, \Xi) = f(x(-t, X, \Xi), \xi(-t, X, \Xi))\, e(t, X, \Xi), \tag{5.1}$$

where $e(t, X, \Xi) \equiv \exp\left(-\int_0^t e^{-\phi(x(-s, X, \Xi))}\nu(\xi(-s, X, \Xi))\, ds\right)$. Making use of Lemma 1.1, (i), and Remark 2.1, (ii), we deduce that

$$e(t, X, \Xi) \leqq \exp(-c_{3.1.1}t), \qquad \text{for each } t \geqq 0. \tag{5.2}$$

Applying (5.2), (2.4), Lemma 4.2, (iii), and the following conservation law of energy (see (2.1)) to (5.1), we can obtain Lemma 3.1:

$$E(X, \Xi) = E(x(-t, X, \Xi), \xi(-t, X, \Xi)) . \tag{5.3}$$

Let $\mu \in \mathbb{D}$. By restricting the domain of integration of the Laplace transformation of (5.1) within a Lebesgue measurable set $M \subseteq [0, +\infty)$, we define the following operator (cf. [16], (4.3-4)):

$$(\mathbb{R}(\mu, M) f(\cdot, \cdot))(X, \Xi) \equiv \tag{5.4}$$

$$\equiv \int_{t \in M} R(\mu, t, X, \Xi) f(x(-t, X, \Xi), \xi(-t, X, \Xi)) \, dt ,$$

where $R(\mu, t, X, \Xi) \equiv e(t, X, \Xi) \exp(-\mu t)$. For this operator we can obtain the following lemma in the same way as [16], Lemma 4.1 (cf. [16], Remark 4.2):

LEMMA 5.1. If $\beta \equiv \operatorname{Re}\mu + c_{3.1.1} > 0$ and $M \subseteq [0, +\infty)$, then

$$|||\mathbb{R}(\mu, M)||| \leqq \int_{t \in M} e^{-\beta t} dt .$$

REMARK 5.2. In Proofs of Lemma 3.1 and Lemma 5.1, we do not need the spatial periodicity condition at all. Recall Remark 3.3, (i-ii).

6. Discussion on (3.4-5).

We take an approach similar to that in [16], § 5. We will seek estimates which imply (3.4-5). Consider operators of the following form:

$$\mathbb{G}(\mu, T) \equiv \prod_{j=1}^{4} \{e^{-\phi(x)} k_j \mathbb{R}(\mu, [0, T])\} , \tag{6.1}$$

where $\mu \in \mathbb{D}$, $1 \leqq T < +\infty$, and $k_j \in Q(\mathbb{R}^3_\xi)$, $j = 1, \ldots, 4$. See Remark 2.2,(ii) for $Q(\mathbb{R}^3_\xi)$. See Lemma 3.2 for $\mathbb{D}$. By $\prod_{j=1}^{m} A_j$ we denote the product $A_m A_{m-1} \ldots A_2 A_1$ for the operators A_j, $j = 1, \ldots, m$. Making use of Lemma 5.1 and Remark 2.2, (ii), we can derive (3.4-5) from the following (6.2-3):

$$\mathbb{G}(\mu, T) \in \mathbb{C}(E_0) \quad \text{for each } \mu \in \mathbb{D} \text{ and } T \in [1, +\infty), \tag{6.2}$$

$$|||\mathbb{G}(\mu, T)||| \to 0 \text{ as } |\mu| \to +\infty, \quad \mu \in \mathbb{D}, \tag{6.3}$$

$$\text{for each } T \in [1, +\infty).$$

Hence, we have only to prove (6.2-3), which will be proved in § 8.

We write $(x_4, \xi_4) \in \Omega \times \mathbb{R}^3$ as the variable of $\mathrm{G}(\mu, T)u$, i.e., we write $\mathrm{G}(\mu, T)\,u = (\mathrm{G}(\mu, T)\,u(\cdot, \cdot))(x_4, \xi_4)$. In the same way as [16], (5.7-9), we can extract the integration kernel of $\mathrm{G}(\mu, T)$ as follows:

$$(6.4)\quad (\mathrm{G}(\mu, T)\,u(\cdot, \cdot))(x_4, \xi_4) = \int_{0 \leqq t_j \leqq T,\ |\eta_j| \leqq \boldsymbol{R},\, j=1,\dots,4} Gu(x_0, \xi_0)\, dt\, d\eta\,,$$

where $dt \equiv dt_1 \dots dt_4$, $d\eta \equiv d\eta_1 \dots d\eta_4$, and

$$(6.5)\quad G = G(\mu, x_4, \xi_4;\, \eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1, t_1) \equiv$$

$$\equiv \prod_{j=1}^{4} \{e^{-\phi(x_j)} k_j(\xi_j, \eta_j)\, R(\mu, t_j, x_j, \eta_j)\}\,.$$

We define

$$(6.6)\quad x_j \equiv x(-t_{j+1}, x_{j+1}, \eta_{j+1}), \qquad \xi_j \equiv \xi(-t_{j+1}, x_{j+1}, \eta_{j+1}), \qquad j = 0, \dots, 3\,.$$

Recall (4.3). By $k_j(\xi_j, \eta_j)$ we denote the integration kernel of $k_j \in Q(\mathbb{R}^3_\xi)$, $j = 1, \dots, 4$, i.e., $(k_j f(\cdot, \cdot))(x_j, \xi_j) = \int k_j(\xi_j, \eta_j)\, f(x_j, \eta_j)\, d\eta_j$, $j = 1, \dots, 4$. $\boldsymbol{R} > 0$ is a constant so large that supp $k_j(\cdot, \cdot)$, $j = 1, \dots, 4$, can be contained in $\{\xi;\ |\xi| \leqq \boldsymbol{R}\} \times \{\eta;\ |\eta| \leqq \boldsymbol{R}\}$.

7. Estimates for the Rank of $J = J(t, X, \Xi)$.

For the same reason as [16], Remark 5.1, (i), we need to fully investigate the Jacobian matrix $J = J(t, X, \Xi)$ (see (4.4)) in order to prove (6.2-3). For this purpose we will prove Lemma 7.1 in the present section. Lemma 7.1 is similar to [16], Lemma 6.1. However, the proof of Lemma 7.1 is entirely different from and more complicated than [16], Proof of Lemma 6.1.

We denote the i-th row vectors of $J = J(t, X, \Xi)$ by $J_i = J_i(t, X, \Xi)$, $i = 1, \dots, 6$, i.e., we define $J_i = J_i(t, X, \Xi) \equiv (m_{i1}, \dots, m_{i6})$, $i = 1, \dots, 6$. See (4.4). Let b_j, $j = 1, \dots, N$, be linearly independent vectors in $\mathbb{R}^n$, $n, N \in \mathbb{N}$. We orthogonalize these vectors, i.e., we define $b_{j,\perp}$, $j = 1, \dots, N$, as follows (we do not normalize them):

$$b_{1,\perp} \equiv b_1, \qquad b_{m+1,\perp} \equiv b_{m+1} - \sum_{k=1}^{m} (b_{m+1} \cdot b_{k,\perp})\, b_{k,\perp} / |b_{k,\perp}|^2, m = 1, \dots, N-1\,.$$

LEMMA 7.1. Let $1 \leqq r, T < +\infty$ be constants. If $t \in [0, T]$ and $(X, \Xi) \in D(r) \equiv \{(X, \Xi) \in \Omega \times \mathbb{R}^3;\ |\Xi| \leqq r\}$ satisfy (4.6-7), then

$$6^{-5/2}\exp(-5c_{4.2}t) \leqq |J_{i,\perp}(t, X, \Xi)| \leqq$$
$$\leqq |J_i(t, X, \Xi)| \leqq 6^{1/2}\exp(c_{4.2}t), \qquad i = 1, \dots, 6.$$

REMARK 7.2. (i) See Lemma 4.2, (i) for $c_{4.2}$. We impose (4.6-7) for the same reason as Remark 4.3, (i). This lemma can play the same role as [16], Lemma 6.1 for the same reason as Remark 4.3, (ii).
(ii) As already mentioned in § 1, we do not assume that Ω is convex. Hence a characteristic curve of Λ, $(x, \xi) = (x(-t, X, \Xi), \xi(-t, X, \Xi))$, $t \in \mathbb{R}$, can behave very erratically. For example, there is a possibility that the trajectory of $x = x(-t, X, \Xi)$ tends to follow the boundary surface $\partial\Omega$. Hence we have to prove that even if a characteristic curve of Λ behaves thus erratically, then the inequalities of Lemma 7.1 still hold.

PROOF OF LEMMA 7.1. In what follows throughout the proof, we assume that (X, Ξ) satisfies (4.6). We easily see that only the following two cases exist:

(I) $x = x(-t, X, \Xi)$ does not collide with $\partial\Omega$ when $t \in [0, T]$.

(II) $x = x(-t, X, \Xi)$ collides only with $F(\partial\Omega)$ a finite number of times when $t \in [0, T]$, i.e., there exists a strictly-monotone-increasing, positive-valued, finite sequence $\{t^n(X, \Xi)\}_{n=1,\dots,m} \subset [0, T]$ which depends on (X, Ξ) and satisfies the following (1-2): (1) If $t \in [0, T] \setminus \{t^n(X, \Xi)\}_{n=1,\dots,m}$, then $x(-t, X, \Xi) \in \Omega$, (2) $x(-t^n(X, \Xi), X, \Xi) \in F(\partial\Omega)$ for $n = 1, \dots, m$, where m is a positive integer dependent on (X, Ξ).

REMARK 7.3. (i) If (I) holds, then we can obtain the present lemma in exactly the same way as [16], Proof of Lemma 6.1. Hence, hereafter we assume that (II) holds.

(ii) In Lemma 4.1 we have already proved that $\{(X, \Xi) \in D(r);\ x = x(-t, X, \Xi)$ collides with $F(\partial\Omega)$ an infinite number of times in a finite time interval$\}$ is a null set. However, the number m of (II) depends on (X, Ξ), i.e., there is a possibility that this number tends to infinity as (X, Ξ) moves. Hence we need to prove that the inequalities of Lemma 7.1 hold uniformly for m (cf. Remark 7.2,(ii)). Hence we have to carefully inspect the change of $J = J(t, X, \Xi)$ before and after the particle $x=x(-t, X, \Xi)$ collides with $F(\partial\Omega)$, i.e., the difference between $J(t^n(X, \Xi)+0, X, \Xi)$ and $J(t^n(X, \Xi)-0, X, \Xi)$, $n=1, \dots, m$.

(iii) Considering the definition of (4.3) (see [14], p.1285), we easily see that if t and s satisfy (4.7), then

$$(7.1) \qquad (x(-t, X, \Xi), \xi(-t, X, \Xi))=$$

$$=(x(-(t-s), x(-s, X, \Xi), \xi(-s, X, \Xi)), \xi(-(t-s), x(-s, X, \Xi), \xi(-s, X, \Xi))).$$

Therefore we can regard $(x, \xi)=(x(-t, X, \Xi), \xi(-t, X, \Xi))$ as function of $\tau\equiv t-s$ and $(x(-s, X, \Xi), \xi(-s, X, \Xi))$. Hence we can define the Jacobian matrix (cf. (4.4)),

$$(7.2) \qquad J(t, s, X, \Xi)\equiv\partial(x(-t, X, \Xi), \xi(-t, X, \Xi))/\partial(x(-s, X, \Xi), \xi(-s, X, \Xi)).$$

Let $t\downarrow t^n(X, \Xi)$ and $s\uparrow t^n(X, \Xi)$ in the equality $J(t, X, \Xi)=$ $=J(t, s, X, \Xi)\, J(s, X, \Xi)$. Then we have

$$(7.3) \qquad J(t^n(X, \Xi)+0, X, \Xi)=U^n(X, \Xi)\, J(t^n(X, \Xi)-0, X, \Xi), \qquad n=1, \dots, m,$$

where $U^n(X, \Xi)\equiv\lim_{t\downarrow t^n(X, \Xi),\, s\uparrow t^n(X, \Xi)} J(t, s, X, \Xi)$. We have only to investigate $U^n(X, \Xi)$, $n=1, \dots, m$. We will prove that these matrices are orthogonal.

We will first inspect $U^1(X, \Xi)$ for simplicity. We let $t\downarrow t^1(X, \Xi)$ and $s\uparrow t^1(X, \Xi)$ in (7.2). Hence we assume that t and s satisfy

$$(7.4) \qquad 0\leqq s<t^1(X, \Xi)<t<t^2(X, \Xi).$$

For simplicity, we write $(\underline{X}, \underline{\Xi})$ as $(x(-s, X, \Xi), \xi(-s, X, \Xi))$, and we write t^j and $\underline{t}^j$ as $t^j(X, \Xi)$ and $t^j(\underline{X}, \underline{\Xi})$ respectively, $j=1, 2$. We easily see that $t^j-s=\underline{t}^j$, $j=1, 2$. Hence, it follows from (7.4) that

$$(7.5) \qquad 0<\underline{t}^1<\tau<\underline{t}^2,$$

where $\tau\equiv t-s$. Moreover we deduce that if $t\downarrow t^1$ and $s\uparrow t^1$, then

$$(7.6) \qquad \tau\downarrow 0, \qquad \text{and } (\underline{X}, \underline{\Xi}) \text{ converges to } (X^1, \Xi^{1-0}) \text{ along } \Gamma^1(X, \Xi),$$

where $\Gamma^1(X, \Xi)$ is a characteristic curve of Λ which connects (X, Ξ) and $(X^1, \Xi^{1-0})\equiv(x(-t^1, X, \Xi), \xi(-(t^1-0), X, \Xi))$, i.e., $\Gamma^1(X, \Xi)\equiv$ $\equiv\{(x, \xi)=(x(-s, X, \Xi), \xi(-s, X, \Xi));\ s \text{ satisfies } (7.4)\}$.

Substituting $(\underline{X}, \underline{\Xi})$, (7.1), and $\tau = t - s$ in (7.2), we obtain

$$J(t, s, X, \Xi) = J(\tau, \underline{X}, \underline{\Xi}) . \tag{7.7}$$

From (7.5-6) we see that $\lim_{t \downarrow t^1, s \uparrow t^1}$ is equivalent to $\lim_{\tau \downarrow 0, \underline{t}^1 < \tau < \underline{t}^2, (\underline{X}, \underline{\Xi}) \to (X^1, \Xi^{1-0}), (\underline{X}, \underline{\Xi}) \in \Gamma^1(X, \Xi)}$. Applying this result to (7.7), we deduce that

$$U^1(X, \Xi) = \lim_{\tau \downarrow 0, \underline{t}^1 < \tau < \underline{t}^2, (\underline{X}, \underline{\Xi}) \to (X^1, \Xi^{1-0}), (\underline{X}, \underline{\Xi}) \in \Gamma^1(X, \Xi)} J(\tau, \underline{X}, \underline{\Xi}) . \tag{7.8}$$

We can obtain the right hand side of (7.8) as follows:

LEMMA 7.4. If $\tau \downarrow 0$ with (7.5), and if $(\underline{X}, \underline{\Xi})$ converges to (X^1, Ξ^{1-0}) along $\Gamma^1(X, \Xi)$, then $J(\tau, \underline{X}, \underline{\Xi})$ converges to a 6×6 matrix in such a way that, for each 3-dimensional column vector y,

$$(\partial x_i(-\tau, \underline{X}, \underline{\Xi})/\partial \underline{X}_j)_{i, j=1,2,3}\, y \to y - 2(\boldsymbol{n}^1 \cdot y)\, \boldsymbol{n}^1 , \tag{7.9.1}$$

$$(\partial x_i(-\tau, \underline{X}, \underline{\Xi})/\partial \underline{\Xi}_j)_{i, j=1,2,3} \to O , \tag{7.9.2}$$

$$(\partial \xi_i(-\tau, \underline{X}, \underline{\Xi})/\partial \underline{X}_j)_{i, j=1,2,3}\, y \to \tag{7.9.3}$$

$$\to -2(\boldsymbol{n}^1 \cdot \nabla_X \phi(X)\,|_{X=X^1}/\boldsymbol{n}^1 \cdot \Xi^{1-0})(\boldsymbol{n}^1 \cdot y)\, \boldsymbol{n}^1 ,$$

$$(\partial \xi_i(-\tau, \underline{X}, \underline{\Xi})/\partial \underline{\Xi}_j)_{i, j=1,2,3}\, y \to y - 2(\boldsymbol{n}^1 \cdot y)\, \boldsymbol{n}^1 , \tag{7.9.4}$$

where we denote the 3×3 zero matrix by O; $\boldsymbol{n}^1 \equiv \boldsymbol{n}(X^1)$. See § 2 for $\boldsymbol{n} = \boldsymbol{n}(x)$. We denote by x_j, ξ_j, $\underline{X}_j$, and $\underline{\Xi}_j$ the j-th components of x, ξ, $\underline{X}$, and $\underline{\Xi}$ respectively, $j = 1, 2, 3$.

PROOF OF LEMMA7.4 Recalling the methods employed in [14], p. 1285, we easily see that if τ satisfies (7.5), then

$$x(-\tau, \underline{X}, \underline{\Xi}) = x(-(\tau - \underline{t}^1), x(-\underline{t}^1, \underline{X}, \underline{\Xi}), \xi(-(\underline{t}^1 + 0), \underline{X}, \underline{\Xi})) , \tag{7.10.1}$$

$$\xi(-\tau, \underline{X}, \underline{\Xi}) = \xi(-(\tau - \underline{t}^1), x(-\underline{t}^1, \underline{X}, \underline{\Xi}), \xi(-(\underline{t}^1 + 0), \underline{X}, \underline{\Xi})) , \tag{7.10.2}$$

$$\xi(-(\underline{t}^1 + 0), \underline{X}, \underline{\Xi}) = \xi(-(\underline{t}^1 - 0), \underline{X}, \underline{\Xi}) - 2(\boldsymbol{n}^1 \cdot \xi(-(\underline{t}^1 - 0), \underline{X}, \underline{\Xi}))\, \boldsymbol{n}^1 , \tag{7.10.3}$$

where (7.10.3) follows from (PRBC) immediately. In order to obtain the limit in (7.8), we differentiate both sides of (7.10.1-2) with respect to $(\underline{X}, \underline{\Xi})$, let $(\underline{X}, \underline{\Xi})$ converge to (X^1, Ξ^{1-0}) along $\Gamma^1(X, \Xi)$, and let $\tau \downarrow 0$ with (7.5). Trying to perform these calculations, and noting that $\tau - \underline{t}^1 \downarrow 0$, we find the need to employ Lemma 4.2,(ii), with $t = \tau - \underline{t}^1$ and the need to obtain the following limits:

(7.11.1) $$\lim_{*}(\partial/\partial \underline{X}_j)\, \underline{t}^1, \qquad \lim_{*}(\partial/\partial \underline{\Xi}_j)\, \underline{t}^1,$$

(7.11.2) $$\lim_{*}(d/d\underline{X}_j)\, x_i(-\underline{t}^1, \underline{X}, \underline{\Xi}), \lim_{*}(d/d\underline{\Xi}_j)\, x_i(-\underline{t}^1, \underline{X}, \underline{\Xi}),$$

(7.11.3) $$\lim_{*}(d/d\underline{X}_j)\, \xi_i(-(\underline{t}^1+0), \underline{X}, \underline{\Xi}), \lim_{*}(d/d\underline{\Xi}_j)\, \xi_i(-(\underline{t}^1+0), \underline{X}, \underline{\Xi}),$$

(7.11.4) $$\lim_{**}((\partial/\partial t)\, x_i(-t, x(-\underline{t}^1, \underline{X}, \underline{\Xi}), \xi(-(\underline{t}^1+0), \underline{X}, \underline{\Xi})))\,|_{t=\tau-\underline{t}^1},$$

(7.11.5) $$\lim_{**}((\partial/\partial t)\, \xi_i(-t, x(-\underline{t}^1, \underline{X}, \underline{\Xi}), \xi(-(\underline{t}^1+0), \underline{X}, \underline{\Xi})))\,|_{t=\tau-\underline{t}^1},$$

$i, j = 1, 2, 3$, where we write $\lim\limits_{**}$, $\lim\limits_{*}$ as $\lim\limits_{\tau\downarrow 0,\, \underline{t}^1<\tau<\underline{t}^2,\, (\underline{X}, \underline{\Xi})\to(X^1, \Xi^{1-0}),\, (\underline{X}, \underline{\Xi})\in\Gamma^1(X, \Xi)}$, $\lim\limits_{(\underline{X}, \underline{\Xi})\to(X^1, \Xi^{1-0}),\, (\underline{X}, \underline{\Xi})\in\Gamma^1(X, \Xi)}$ respectively for simplicity.

Remark 7.5. (i) We easily see that $x(-\underline{t}^1, \underline{X}, \underline{\Xi}) = X^1$ and $\xi(-(\underline{t}^1-0), \underline{X}, \underline{\Xi}) = \Xi^{1-0}$. Hence, applying (4.8) and Remark 2.1,(i), we have

(7.12.1) $$((\partial/\partial t)\, x(-t, \underline{X}, \underline{\Xi}))\,|_{t=\underline{t}^1-0} = -\Xi^{1-0},$$

(7.12.2) $$((\partial/\partial t)\, \xi(-t, \underline{X}, \underline{\Xi}))\,|_{t=\underline{t}^1-0} = \nabla_X\phi(X)\,|_{X=X^1}.$$

(ii) In order to simplify the calculations in obtaining the limits (7.11.1-5), we will introduce a 3-dimensional rectangular coordinate system (x_1, x_2, x_3) in such a way that the origin coincides with X^1, that the x_2x_3 plane $(x_1=0)$ includes the face of $\partial\Omega$ which contains X^1 (recall Assumption Ω, (ii)), and that $\boldsymbol{n}^1 = (1, 0, 0)$. That face of $\partial\Omega$ is represented as follows: $x_1 = 0$. Hence, it follows from $X^1 = x(- -\underline{t}^1, \underline{X}, \underline{\Xi})$ that

(7.13) $$x_1(-\underline{t}^1, \underline{X}, \underline{\Xi}) = 0\,.$$

Differentiate both sides of (7.13) with respect to $(\underline{X}, \underline{\Xi})$, and apply $\lim\limits_{*}$. Noting that $\underline{t}^1 \downarrow 0$, and applying Lemma 4.2, (ii), and (7.12.1) to the equalities thus obtained, we can obtain (7.11.1) as follows:

(7.14.1) $$\lim_{*} \partial\underline{t}^1/\partial\underline{X}_1 = 1/\Xi_1^{1-0}, \lim_{*} \partial\underline{t}^1/\partial\underline{X}_j = 0\,, \qquad j = 2, 3\,,$$

(7.14.2) $$\lim_{*} \partial\underline{t}^1/\partial\underline{\Xi}_j = 0\,, \qquad j = 1, 2, 3\,,$$

where Ξ_j^{1-0} denotes the j-th component of Ξ^{1-0}, $j=1, 2, 3$, i.e., $\Xi^{1-0}=(\Xi_1^{1-0}, \Xi_2^{1-0}, \Xi_3^{1-0})$.

It follows from (7.10.3) that

$$\xi_i(-(\underline{t}^1+0), \underline{X}, \underline{\Xi}) = \varrho(i)\, \xi_i(-(\underline{t}^1-0), \underline{X}, \underline{\Xi}), \qquad i=1, 2, 3, \tag{7.15}$$

where $\varrho(1)\equiv -1$, and $\varrho(i)\equiv 1$ for $i=2, 3$. Making use of (7.13), (7.12.1-2), (7.14.1-2), (7.15), and Lemma 4.2, (ii), and noting that $\underline{t}^1 \downarrow 0$, we can obtain (7.11.2-3) as follows:

$$(d/d\underline{X}_j)\, x_1(-\underline{t}^1, \underline{X}, \underline{\Xi}) = 0, \qquad j=1, 2, 3, \tag{7.16.1}$$

$$\lim_{*}(d/d\underline{X}_1) x_i(-\underline{t}^1, \underline{X}, \underline{\Xi}) = -\Xi_i^{1-0}/\Xi_1^{1-0}, \qquad i=2, 3, \tag{7.16.2}$$

$$\lim_{*}(d/d\underline{X}_j)\, x_i(-\underline{t}^1, \underline{X}, \underline{\Xi}) = \delta_{ij}, \qquad i, j=2, 3, \tag{7.16.3}$$

$$\lim_{*}(d/d\underline{\Xi}_j)\, x_i(-\underline{t}^1, \underline{X}, \underline{\Xi}) = 0, \qquad i, j=1, 2, 3, \tag{7.17}$$

$$\lim_{*}(d/d\underline{X}_1)\, \xi_i(-(\underline{t}^1+0), \underline{X}, \underline{\Xi}) = \tag{7.18.1}$$

$$= \varrho(i)(\partial\phi(X)/\partial X_i)\,|_{X=X^1}/\Xi_1^{1-0}, \; i=1, 2, 3,$$

$$\lim_{*}(d/d\underline{X}_j)\, \xi_i(-(\underline{t}^1+0), \underline{X}, \underline{\Xi}) = 0, \; i=1, 2, 3, \; j=2, 3, \tag{7.18.2}$$

$$\lim_{*}(d/d\underline{\Xi}_j)\, \xi_i(-(\underline{t}^1+0), \underline{X}, \underline{\Xi}) = \varrho(i)\, \delta_{ij}, \qquad i, \; j=1, 2, 3. \tag{7.19}$$

Performing calculations similar to those in Remark 7.5, (i), with the aid of (4.8) and (7.15), and noting that $\tau-\underline{t}^1 \downarrow 0$, we see that (the limit (7.11.4)) $= -\varrho(i)\, \Xi_i^{1-0}$ and (the limit (7.11.5)) $= (\partial\phi(X)/\partial X_i)\,|_{X=X^1}$, $i= =1, 2, 3$. If we combine these results, (7.14.1-2),(7.16.1)-(7.19), and Lemma 4.2, (ii), then we can obtain the right hand side of (7.8) in terms of the rectangular coordinate system defined in Remark 7.5, (ii). Substituting the following equalities, which follow immediately from the definition of the rectangular coordinate system, in the result thus obtained, we can obtain Lemma 7.4: $\boldsymbol{n}^1=(1, 0, 0)$, $\boldsymbol{n}^1\cdot\nabla_X\phi(X)\,|_{X=X^1} = (\partial\phi(X)/\partial X_1)\,|_{X=X^1}$, $\boldsymbol{n}^1\cdot\Xi^{1-0}=\Xi_1^{1-0}$.

Let us continue proving Lemma 7.1. Noting that $\boldsymbol{n}^1\cdot\Xi^{1-0}\neq 0$, and applying Assumption ϕ,(iii) to (7.9.3), we have

$$(\partial\xi_i(-\tau, \underline{X}, \underline{\Xi})/\partial\underline{X}_j)_{i,j=1,2,3} \to O. \tag{7.20}$$

It follows from (7.9.1-2), (7.9.4), and (7.20) that $U^1(X, \Xi)$ is an orthogonal matrix.

Performing the same calculations as above for $n=2, \dots, m$, we see that $U^n(X, \Xi)$, $n=2, \dots, m$, are orthogonal matrices. Applying these results to (7.3), we have

$$(7.21) \qquad \| J(t^n(X,\Xi)+0, X, \Xi) \| = \| J(t^n(X,\Xi)-0, X, \Xi) \|, \qquad n=1,\dots,m.$$

See Lemma 4.2, (i) for $\|\cdot\|$. Combining these equalities and Lemma 4.2, (i) with $(t_-(X, \Xi), t_+(X, \Xi)) = (t^n(X, \Xi), t^{n+1}(X, \Xi))$ successively for $n = 1, \dots, m$, we have (cf. [16], (6.1))

$$(7.22) \qquad \| J(t, X, \Xi) \| \leqq \| J(0, X, \Xi) \| \exp(c_{4.2} t), \qquad \text{for each } t \geqq 0.$$

Making use of this inequality and Lemma 4.2, (iii), we can obtain Lemma 7.1 in the same way as [16], Lemma 6.1.

REMARK 7.6. (i) Even if the number m of (II) of Proof of Lemma 7.1 tends to infinity, then (7.21-22) still hold, because $U^n(X, \Xi)$, $n = 1, \dots, m$, are orthogonal.

(ii) Assume that $\partial\Omega$ is only piecewise sufficiently smooth, in place of Assumption Ω,(ii). If we perform calculations similar to, but more complicated than, those done above, we can obtain the limit in (7.8), also under such a relaxed assumption. However, the limit comes to contain the second fundamental quantities of $\partial\Omega$ in a complicated manner, and moreover, the second fundamental quantities of such a surface are not always equal to 0; the limit in (7.8) becomes very complex. Therefore we cannot obtain a simple result such as Lemma 7.4. If we impose Assumption Ω,(ii), then we can regard $\partial\Omega$ as a plane in the neighborhood of X^1. Hence all the second fundamental quantities are equal to 0 in the neighborhood of X^1. By virtue of this fact, we can greatly simplify the limit in (7.8), as already shown in Lemma 7.4. This is the reason for imposing Assumption Ω, (ii).

(iii) If we do not impose Assumption ϕ, (iii), then it does not follow from Lemma 7.4 that $U^1(X, \Xi)$ is orthogonal. Hence we cannot obtain a result such as (7.21).

(iv) From Assumption ϕ, (iii) and Assumption Ω, (ii), we see that if x is contained in an edge of $\partial\Omega$, then $\lim_{y\to x,\, y\in\Omega} \nabla\phi(y)$ is parallel to the edge, and that if x coincides with a vertex of $\partial\Omega$, then $\lim_{y\to x,\, y\in\Omega} \nabla\phi(y)=0$.

8. Proof of (6.2-3).

We can prove (6.2-3) in the same way as in [16], § 7-8, i.e., we can prove (6.2-3) by the following 5 steps:

(1) Combining Lemma 7.1 (recall Remark 7.2, (i)), [16], Lemma 6.3, (ii), and Lemma 4.2, (iv), and performing the same calculations as those in [16], Proof of Lemma 7.1, we can obtain the same estimates for $\partial(x_3)/\partial(\eta_4, t_4)$ as [16], Lemma 7.1 (recall (6.6)).

(2) Applying Lemma 7.1, [16], Lemma 6.3, (ii), Lemma 4.2, (iv), and the estimates obtained in (1), and performing the same calculations as those in [16], Proof of Lemma 7.2, we can obtain the same estimates for $\partial(x_2)/\partial(\eta_4, t_4, \eta_3, t_3)$ as [16], Lemma 7.2.

(3) Applying Lemma 7.1, [16], Lemma 6.3, (ii), Lemma 4.2, (iv), and the estimates obtained in (2), we can obtain the same estimates for $\partial(x_1)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2)$ as [16], Lemma 7.3, in the same way as [16], Proof of Lemma 7.3.

(4) By Lemma 7.1, [16], Lemma 6.3, (ii), Lemma 4.2, (iv), and the estimates obtained in (3), we can obtain the same estimates for $\partial(x_0, \xi_0)/\partial(\eta_4, t_4, \eta_3, t_3, \eta_2, t_2, \eta_1)$ as [16], Lemma 7.4, in the same way as [16], Proof of Lemma 7.4.

(5) Making use of (5.2-3), we can prove the same estimates for (6.5) as [16], (5.10-11). Combining these estimates and the estimates obtained in (4), we can prove (6.2-3) in exactly the same way as [16], § 8.

In [16], § 7-8 and Proof of Lemma 6.3, (ii), we do not need to apply the spatial periodicity condition. Hence we do not need that condition in the steps above.

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