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On some model theoretic problems concerning certain extensions of abelian groups by groups of finite exponent

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On Some Model Theoretic Problems Concerning Certain Extensions of Abelian Groups by Groups of Finite Exponent.

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1. Introduction.

For every group G, let $\mathcal{K}(G)$ denote the class of groups S admitting a normal abelian subgroup A such that the quotient S/A is elementarily equivalent to G, $S/A \equiv G$. For G abelian, let $\mathcal{K}_{ab}(G)$ denote the class of abelian groups in $\mathcal{K}(G)$.

The aim of this note is to investigate $\mathcal{K}(G)$ and, possibly, $\mathcal{K}_{ab}(G)$ for some given G. Of course, one may wonder which is the interest of this analysis. As we will see just a few lines below, the originating question was the first order axiomatizability of $\mathcal{K}(G)$ in the language for groups. But it is worth emphasizing that $\mathcal{K}(G)$ (and so $\mathcal{K}_{ab}(G)$) do admit some natural algebraic characterization, not only because the elementary equivalence between two given structures can be generally translated in algebraic terms by using ultraproducts (and the Keisler-Shelah Theorem) or, for finite languages, partial isomorphisms (and the Fraïssé Theorem), but also because we will see later that, at least for some suitable G, $\mathcal{K}(G)$ can be introduced in a genuine group theoretic way.

The starting line of the matter was the case «G finite». Here the axiomatizability problem seems solved in a positive way [O2]: for, in this case, $\mathcal{K}(G)$ is just the class of abelian-by-G groups, namely the class of the groups S admitting a normal abelian subgroup A such that S/A is isomorphic to G («$\equiv G$» means «$\simeq G$» under the finiteness assumption); and the latter class is elementary, and even finitely axiomatizable. Notice that, for G finite, the analysis benefits by (at least) two significant

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advantages. The former is that any group in $\mathcal{K}(G)$ is stable and, accordingly, enjoys some useful algebraic properties, like chain conditions, definability of centralizers, and so on. The latter advantage is related to the former one: for, A has a natural structure of $\mathbb{Z}[G]$-module and most model theory (including stability) of S -as a group- reduces to A -as a module-.

Now assume G infinite. As we will see below, both the previous advantages fail. Nevertheless some partial results were obtained in [MaT] and [T]. In particular

(a) if G is abelian of bounded exponent, then $\mathcal{K}_{ab}(G)$ is elementary (but needs infinitely many sentences to be axiomatized in the first order language of groups);

(b) if G is abelian of prime exponent, then $\mathcal{K}(G)$ is elementary [T] (by the way, there is a slight inaccuracy in the final Lemma in [T]; see Lemma 3.2 below for a sharp statement and proof);

(c) if G is abelian of unbounded exponent, then, in most cases, $\mathcal{K}_{ab}(G)$ and, consequently, $\mathcal{K}(G)$ are not elementary.

With respect to first order axiomatizability, let us also recall that, for every group G, $\mathcal{K}(G)$ -and $\mathcal{K}_{ab}(G)$ for G abelian- are always closed under ultraproducts; so, in order to test elementarity, we have to check that $\mathcal{K}(G)$, or $\mathcal{K}_{ab}(G)$, are closed under $\equiv$, hence that, if $S \in \mathcal{K}(G)$, then every model of the theory of S is in $\mathcal{K}(G)$, too. Moreover, by a Löwenheim-Skolem argument, we can assume S countable. In fact, given $S \in \mathcal{K}(G)$ and a corresponding normal abelian subgroup A with $S/A \equiv G$, a countable model (S_0, A_0) of the theory of (S, A) still satisfies $S_0 \equiv S$ and $S_0/A_0 \equiv G$.

Another useful remark is that, owing to the Oger result quoted before [O2], the groups S which are not abelian-by-finite are an elementary class (just list finite groups G and state that S is not abelian-by-G for every G).

The main purpose of this note is to deal with $\mathcal{K}(G)$ when G is infinite, of finite exponent (and possibly nonabelian). We aim to find some significant and direct group theoretic characterizations, and to discuss first order axiomatizability. More precisely, our project is to consider $G = C(p)^{(\aleph_0)} \oplus H$ where H is a finite group and the order of H is prime to p (hereafter, for every positive integer m, $C(m)$ is the multiplicative cyclic group of order m); notice that this extends the case quoted in (b) and intersects (c).

Let us summarize here the plan of the paper. In § 2, we give some algebraic characterizations of the groups $S \in \mathcal{K}(G)$, featuring the derived subgroups S' and the subgroup S^r generated by the r-th powers in S (r a positive integer); by the way, recall that both S' and S^r are fully invariant subgroups of S. Consequently in § 3 we discuss the definability of S' and S^r; this overlaps the stability question for groups in $\mathcal{K}(G)$; we will see that, even in the easiest case $H = 1$, there do exist unstable groups in $\mathcal{K}(G)$. In the final sections, we study elementarity in the «simplest» nontrivial case for H, namely when H is a simple group. In particular, in § 4, we deal with first order axiomatizability when H is simple and nonabelian. In § 5 we obtain a full positive result when H is simple abelian and $S'S^p$ is not abelian; finally we discuss the case when both H and $S'S^p$ are abelian (and the order of H is squarefree); we give a more pregnant group theoretic characterization and we prove first order axiomatizability at least for nil-2 groups S.

We refer to [H] for model theory and to [R] for group theory. L is the first order language for groups. Among groups, $\leqslant$ will denote the relation «to be a subgroup», and $<$ will mean «to be a proper subgroup». We gratefully acknowledge helpful suggestions from Prof. Guido Zappa and Prof. Andreas Baudisch.

2. Algebraic characterizations.

Let $G = C(p)^{(\aleph_0)} \oplus H$ where p is a prime and H is a finite group whose order q is prime to p. We are looking for a group theoretic characterization of $\mathcal{K}(G)$. Notice that H is definable in G as the subgroup of the elements whose period divides q. A (comparatively easy) algebraic definition of $\mathcal{K}(G)$ is obviously provided by the fact that the first order theory of G is totally categorical: up to isomorphism, its models are just those of the form $C(p)^{(\alpha)} \oplus H$ where α is an infinite cardinal. Accordingly, «$\equiv G$» means «$\simeq C(p)^{(\alpha)} \oplus H$» for some infinite α. But now we wish to propose another algebraic characterization, which involves $S'S^p$ as a sort of intermediate group, and which will be useful later.

THEOREM 2.1. Let $G = C(p)^{(\aleph_0)} \oplus H$ where p is a prime, H is a finite group and the order q of H is prime to p. Then a group S is in $\mathcal{K}(G)$ if and only if S satisfies the following assumptions:

(i) $S'S^p$ has infinite index in S;

(ii) $S'S^p$ has an abelian subgroup N such that N is normal in S, $S'S^p/N \simeq H$ and every element $s \in S$ whose period modulo N divides p commutes with both commutators and p-th powers modulo N.

PROOF. First assume $S \in \mathcal{K}(G)$. Then there is a normal abelian subgroup A of S such that $S/A \simeq C(p)^{(\alpha)} \oplus H$ for some infinite cardinal α. Choose $b_0, \dots, b_r$ in $S - A$ such that $b_0 A, \dots, b_r A$ generate the copy of H in S/A with respect to some fixed presentation. Put

$$B = \langle A, b_0, \dots, b_r \rangle .$$

Since A is normal in S, every element in B can be expressed as ab where $a \in A$ and b is a word on $b_0, \dots, b_r$. We claim that B is normal in S. In fact, let $s \in S$, $a \in A$, b be a word on $b_0, \dots, b_r$; notice that $b^q \in A$. Owing to the structure of S, there exist $a' \in A$, a word b' on $b_0, \dots, b_r$ and $x \in S$ such that xA is in the copy of $C(p)^{(\alpha)}$ in S/A (hence $x^p \in A$) and

$$a^s b^s = (ab)^s = a' b' x .$$

As $b^q \in A$ and A is normal, $(ab)^q \in A$, hence $((ab)^s)^q = ((ab)^q)^s \in A^s = A$; consequently $(b'x)^q \in A$. Notice that b' and x commute with each other modulo A and $b'^q \in A$. Accordingly $x^q \in A$. But we know $x^p \in A$, hence, as p and q are coprime, we can deduce $x \in A$, and $(ab)^s \in B$. Clearly A is a normal subgroup of B and the quotient group B/A is isomorphic to H. Moreover

$$S/B \simeq \frac{S/A}{B/A} \simeq \frac{C(p)^{(\alpha)} \oplus H}{H} \simeq C(p)^{(\alpha)} .$$

Consequently, for every $s \in S$,

$$s^p B = (sB)^p = B .$$

So S^p is a normal subgroup of B. Furthermore, as S/B is abelian, S' is included in B, and hence $S'S^p$ is a (normal) subgroup of B. In particular the index of $S'S^p$ in S is infinite, namely (i) holds. Now consider $A \cap$ $\cap S'S^p \leqslant S'S^p$. We know

$$\frac{S'S^p}{A \cap S'S^p} \simeq \frac{\langle S'S^p, A \rangle}{A}$$

where the latter quotient is a subgroup of $B/A \simeq H$. Suppose

$$\frac{\langle S'S^p, A\rangle}{A} \neq \frac{B}{A}.$$

Then $|B:\langle S'S^p, A\rangle| > 1$; but $|B:\langle S'S^p, A\rangle|$ divides $|B:A| = |H|$, and this is prime to p. On the other side, every element in B has a period dividing p modulo $\langle S'S^p, A\rangle \geq S'S^p$. So we get a contradiction. Consequently

$$\frac{\langle S'S^p, A\rangle}{A} = \frac{B}{A} \simeq H$$

and

$$\frac{S'S^p}{A \cap S'S^p} \simeq H.$$

Put $N = S'S^p \cap A$. N is abelian because N is a subgroup of A; N is normal because both $S'S^p$ and A are. We have just seen that $S'S^p/N \simeq H$. Finally let $s \in S$ satisfy $s^p \in N$; hence $s^p \in A$ and the period of sA in S/A divides p. Accordingly sA commutes with any element in B/A, in particular with any element tA in $S'S^p/A$: for all $t \in S'S^p$, $[s, t] \in A$. But $[s, t] \in S'S^p$, hence $[s, t] \in N$. So (ii) holds.

Conversely, suppose that S satisfies both (i) and (ii). Then N is a normal subgroup of S and the quotient S/N is an extension of $S'S^p/N \simeq H$ by $S/S'S^p$. $S/S'S^p$ is abelian (because $S' \leq S'S^p$), infinite (owing to (i)), of exponent p (because $S^p \leq S'S^p$); hence $S/S'S^p$ is an infinite elementary abelian p-group. As the order q of H is prime to p, we can apply a Schur-Zassenhaus argument to deduce that S/N is a semidirect product of $S'S^p/N \simeq H$ and $S/S'S^p \simeq C(p)^{(\alpha)}$ for some infinite α. Accordingly put

$$\frac{S}{N} = \frac{S'S^p}{N} \rtimes \frac{K}{N}$$

for a suitable K. In order to show that actually S/N is a direct product of $S'S^p/N$ and K/N (and hence to finish the proof), we have to prove that K/N acts identically on $S'S^p/N$. But any element $s \in K$ has a period dividing p modulo N and so, by (ii), commutes modulo N with all the elements of $S'S^p$. This is just what we need to show. ∎

When H (hence G) is abelian, we can obtain the following, more direct characterization.

COROLLARY 2.2. Let G be as in Theorem 2.1, H be abelian. Then a group S is in $\mathcal{K}(G)$ if and only if S satisfies

(i) $S'S^p$ has infinite index in S;

(ii)$'$ there is an abelian subgroup N of $S'S^p$ such that N contains S' and $S'S^p/N \simeq H$.

PROOF. It suffices to show that, for H (and G) abelian, (ii)$'$ is equivalent to (ii) in Theorem 2.1. First assume (ii). For $S \in \mathcal{K}(G)$, $S/A \equiv G$ is abelian; hence S' is a subgroup of A, and consequently of $N = A \cap S'S^p$. So (ii)$'$ holds. Conversely, assume (ii)$'$. As $S' \leq N$, N is normal in S; the same reason proves S/N abelian; consequently the final condition in (ii) is trivially satisfied. ■

Let $\mathcal{K}'(G)$, $\mathcal{K}''(G)$ respectively denote the classes of groups satisfying (i), (ii) (or (ii)$'$). Clearly, if both $\mathcal{K}'(G)$ and $\mathcal{K}''(G)$ are elementary, then $\mathcal{K}(G)$ is. The following sections will be devoted to discussing (separatedly or jointly) first order axiomatizability for $\mathcal{K}'(G)$ and $\mathcal{K}''(G)$. But, before concluding this section, let us underline that, for H abelian and $S \in \mathcal{K}(G)$, $S' \leq A$, hence S' is abelian. In other words, any group $S \in \mathcal{K}(G)$ is solvable of class 2.

3. Definability and stability.

The reference to S' and S^p for $G = C(p)^{(\aleph_0)} \oplus H$ in Theorem 2.1 and the necessity of translating (i), (ii) or (ii)$'$ in a first order way suggest to explore if these subgroups -S' and S^p- are definable in our setting.

It is known that a preliminary assumption ensuring some more definability is stability. By the way, recall that, for G finite, any group $S \in \mathcal{K}(G)$ is stable, because most model theory of S reduces to A, viewed as a $\mathbf{Z}[G]$-module with respect to the action of $S/A \simeq G$ on A, and every module is stable. So let us deal briefly with the connection with modules when G is infinite. Let $S \in \mathcal{K}(G)$, A be a normal abelian subgroup satisfying $S/A \equiv G$. Then S/A acts on A, and consequently A still inherits a structure of module over the group ring $\mathbf{Z}[S/A]$; but this ring depends on S, and may be uncountable. Of course, by replacing (S, A) with a countable elementary substructure, we can assume S countable; notice that

this does not affect, for instance, stability; moreover we have seen in § 1 that, in order to establish the first order axiomatizability of $\mathcal{K}(G)$, in some sense it is sufficient to examine the countable groups in $\mathcal{K}(G)$; finally, when G is of the form $C(p)^{(\aleph_0)} \oplus H$ for p prime, H finite and $(|H|, p) = 1$, then G is $\aleph_0$-categorical and hence, for every countable $S \in \mathcal{K}(G)$, S/A is fixed up to isomorphism. Nevertheless, A, as a $\mathbb{Z}[G]$-module, is always stable, and we will see soon that S may be not. So, in any case, the connection fails.
Now let us treat stability. We have said that, for G finite, every group in $\mathcal{K}(G)$ is stable. But nothing similar is preserved for G infinite, even in the simplest case $G = C(p)^{(\aleph_0)}$ with p prime, as the following proposition shows.

PROPOSITION 3.1. Let $G = C(p)^{(\aleph_0)}$ with p prime. Then there are groups $S \in \mathcal{K}(G)$ in all stability classes. Moreover, for p prime, the problem of characterizing a stability class is equivalent to characterizing the groups $S \in \mathcal{K}(G)$ belonging to the class.

PROOF. It is easy to provide ω-stable, superstable non-ω-stable, stable unsuperstable groups in $\mathcal{K}(G)$: just take an ω-stable, superstable non-ω-stable, stable unsuperstable abelian group A and form $S = A \oplus G$. With respect to unstable examples in $\mathcal{K}(G)$ for $p = 2$, look at $S_3^{\aleph_0}$; it is known that this group is not stable; but it admits a normal abelian subgroup $A_3^{\aleph_0}$ whose quotient group is an (abelian) elementary infinite 2-group. In order to handle the odd case and to prove the second statement of our proposition, let us recall some facts from [Me]. In that paper, fixed a prime $p > 2$, it is described an effective procedure providing, for every (infinite) structure M in a finite language, a nil-2 goup $S(M)$ of exponent p such that M is first order definable in $S(M)$ and M, $S(M)$ are in the same stability class. In more detail, given M, firstly one defines an (infinite) graph $\Gamma(M)$ satisfying some suitable assumptions; then one replaces every node γ in $\Gamma(M)$ with a copy S_γ of $C(p)$, and one forms the free nil-2 product $\mathbf{2}_{\gamma \in \Gamma(M)} S_\gamma$. $S(M)$ is just the quotient of $\mathbf{2}_{\gamma \in \Gamma(M)} S_\gamma$ with respect to the normal subgroup generated by the commutators $[a_\gamma, b_\delta]$ where γ and δ are adjacent nodes in the graph, $a_\gamma \in S_\gamma$ and $b_\delta \in S_\delta$. So $S = S(M)$ is in $\mathcal{K}(G)$ because S is nil-2 (hence S' is abelian) and S/S' is an infinite elementary abelian p-group. ■

Now let us discuss the definability of S' and S^p for $S \in \mathcal{K}(G)$. It is known that, as a consequence of Zil'ber Indecomposability Theorem, the derived subgroup S' is definable when S is ω-stable of finite Morley rank. But this definability result already fails for ω-stable groups of infinite Morley rank, in particular for a free infinite group S in the class of nil-2 groups of prime exponent $p > 2$ [B], and we have seen that this counterexample is in $\mathcal{K}(G)$ for $G = C(p)^{(\aleph_0)}$. In any case, let us quote some very partial results concerning the definability of S'; these facts will be useful later.

LEMMA 3.2 [T]. Let $S \in \mathcal{K}(G)$ where G is a finite abelian group of order n and let p be a positive integer. Then every element in $S' S^p$ can be expressed as the product of (at most) $2n^2 + n^{2pn}$ commutators and 2 p-th powers.

PROOF. Recall that every element c in $S' S^p$ can be written as a product of commutators and p-th powers; we can arrange these factors and to obtain that commutators precede p-th powers. The problem is to bound uniformly the number of commutators and p-th powers occurring in these decompositions. For instance, one can factorize c using at most one p-th power following suitably many commutators, but this does not bind - a priori- the number of involved commutators. However let us fix such a decomposition. Let A be a normal abelian subgroup of S such that S/A is isomorphic to G. Since G is abelian, S' is included in A, and hence S' is abelian. Let $x_1, \dots, x_n$ be a set of representatives for the cosets of A in S, then every element in S decomposes uniquely as ax_j with $a \in A$ and $1 \leqslant \leqslant j \leqslant n$. Now consider the commutators in the decomposition of c. Using some basic identities and the fact that A is abelian and includes S', one sees that, for a, b in A and $1 \leqslant i, j \leqslant n$,

$$[ax_i, bx_j] = [a, x_i]^{x_j}[x_j, x_i][x_j, b]^{x_i}$$

and

$$[a, x_i][b, x_i] = [ab, x_i].$$

Hence every element in S' can be expressed as a product of

$$[a, x_i]^{x_j}, [x_j, b]^{x_i}$$

with $a \in A$ and $1 \leqslant i, j \leqslant n$, and

$$[x_j, x_i]^h$$

where $1 \leqslant i, j \leqslant n$ and h ranges over the integers; consequently every element in S' is a product of at most $2n^2$ commutators of the former kind, at most n^{2pm} commutators of the latter kind (with $0 \leqslant h < pn$) and pn-th powers. Actually, as S' is abelian, a unique pn-th power occurs. Clearly any pn-th power is also a p-th power. Hence the given element c is the product of $2n^2 + n^{2pm}$ commutators and 2 p-th powers. ■

(Notice that, in [T], this adaptation of the final Lemma still ensures the definability of $S'S^p$ when $S'S^p$ is abelian and has finite index, and hence still guarantees the Main Theorem, quoted as (b) in § 1).

LEMMA 3.3. Let $S \in \mathcal{K}(G)$ where $G = C(p)^{(\aleph_0)} \oplus H$, p is a prime and H is a finite group whose order q is prime to p. Let N be a normal abelian subgroup of S such that $N \leqslant S'S^p$ and $S'S^p/N \simeq H$ (see Theorem 2.1). Then every element in S' is a product of (at most) q commutators modulo N.

(For the proof, just use the fact that $S'S^p/N \simeq H$ is finite of order q).

With respect to S^r, there are some definability results in [O1] and [O2], mainly concerning polyciclic-by-finite groups, hence, in particular, finitely generated nilpotent-by-finite groups (see [O1], Proposition 2.1, and [O2], Proposition 1). But notice that no group $S \in \mathcal{K}(G)$ is finitely generated when G is of the form $C(p)^{(\aleph_0)} \oplus H$ for some prime p.

3. The simple nonabelian case.

Throughout this section, assume $G = C(p)^{(\aleph_0)} \oplus H$ where p is a prime and H is a simple nonabelian finite group of order prime to p. Our aim is to study the elementarity of $\mathcal{K}(G)$ under this hypothesis. We will express (ii) in a first order way, and then we will discuss (i). First let us underline the following fact.

Fact 4.1. Let F be a group, K_0, K_1 be normal subgroups of F such that K_0 is abelian and both F/K_0 and F/K_1 are isomorphic to H. Then $K_0 = K_1$.

Otherwise $K_0 \cdot K_1$ properly includes both K_0 and K_1, so, as H is simple, $K_0 \cdot K_1 = F$. Accordingly

$$\frac{K_0}{K_0 \cap K_1} \simeq \frac{K_0 \cdot K_1}{K_1} = \frac{F}{K_1} \simeq H$$

where K_0 is abelian, and H is not -a contradiction-.

THEOREM 4.2. Let $G = C(p)^{(\aleph_0)} \oplus H$ where p is a prime, H is a finite group of order q prime to p, and H is simple and not abelian. Then $\mathcal{K}''(G)$ is elementary.

PROOF. Assume $S \in \mathcal{K}''(G)$. Then there is a normal abelian subgroup N of S such that $N \leqslant S'S^p$ and $S'S^p/N \simeq H$; furthermore, for all $s \in S$, if $s^p \in N$, then, for every choice of a and b in S, s commutes with both a^p and $[a, b]$ modulo N. Notice that «H nonabelian» implies «$S'S^p$ nonabelian», and that every element in $S'S^p$ can be expressed in the form $s^p c$ for some $s \in S$ and $c \in S'$. Choose $b_0, \ldots, b_r$ in a set of representatives of cosets of N in S such that

$$b_0 N, \ldots, b_r N$$

generate $S'S^p/N \simeq H$ with respect to a fixed presentation of H. Lemma 3.3 applies to our setting, and so every element in S' is a product of (at most) q commutators modulo N. Hence we can put, for every $j \leqslant r$,

$$b_j = s_j^p c_j$$

where $s_j \in S$ and c_j is a product of q commutators. Actually, as S/N is not abelian, N is properly included in $\langle N, S' \rangle$, and so in $S'S^p$; so the simplicity of H forces $\langle N, S' \rangle = S'S^p \geqslant S^p$; consequently every element in S' is directly a product of q commutators modulo N, and we can suppose that b_j is the product of q commutators for all $j \leqslant r$. Now we distinguish two cases.

Case 1: *for all $a \in N$ and $j \leqslant r$, a centralizes b_j.* This means that the centralizer of a in $S'S^p$ $C_{S'S^p}(a)$ equals $S'S^p$ for all $a \in N$, hence $C_{S'S^p}(N) = S'S^p$. Consequently N is a (normal) subgroup of the center $Z(S'S^p)$ of $S'S^p$. On the other side, $S'S^p$ is not abelian, so $Z(S'S^p) \neq$ $\neq S'S^p$, and the simplicity of H implies $Z(S'S^p) = N$. Recall that $C_S(S'S^p)$ is $\emptyset$-definable and includes $Z(S'S^p)$. Now consider the first order sentences in the language L of groups stating what follows:

(α_1) every product of $q+1$ commutators is expressible as a product of q commutators modulo $C_S(S'S^p)$ (hence modulo $Z(S'S^p)$);

(α_2) there are $b_0, \ldots, b_r$ in S such that, for every $j \leqslant r$, b_j is a product of q commutators, $b_j \notin C_S(S'S^p)$, b_j centralizes any product of q commutators and a p-th power modulo $C_S(S'S^p)$ (hence modulo $Z(S'S^p)$), and finally $b_0, \ldots, b_r$ just satisfy modulo $C_S(S'S^p)$ (hence $Z(S'S^p)$) the relations in the given presentation of H;

(α_3) for every $s \in S$ such that s^p is in $C_S(S'S^p)$ (hence in $Z(S'S^p)$), s centralizes the commutators and the p-th powers modulo $C_S(S'S^p)$ (so môdulo $Z(S'S^p)$).

It is clear that α_1, α_2 and α_3 are first order sentences of L. Let α denotes their conjunction. Also, under our assumptions, (ii) implies α. Conversely, let α hold in S, and put $N = Z(S'S^p)$. Then N is a normal abelian subgroup of S, $N \leqslant S'S^p$; α_2 (and α_1) imply that $S'S^p/N$ is a homomorphic image of H; α_2 again ensures $S'S^p \neq N$; so the simplicity of H forces $S'S^p/N \simeq H$. These facts and α_3 yield (ii).

Case 2: there are $a \in N$ and $j \leqslant r$ such that $[a, b_j] \neq 1$. Then

$$S'S^p > C_{S'S^p}(a) \geqslant C_{S'S^p}(N) \geqslant N .$$

Notice that $C_{S'S^p}(N)$ is a normal subgroup of $S'S^p$ (because N is) and, using again the simplicity of H, one can conclude $C_{S'S^p}(N) = N$, or also

$$N = C_S(N) \cap S'S^p$$

where $C_S(N)$ is normal, too. As the index of N in $S'S^p$ is q, one can find a natural number $t < q$ and $a_0, \ldots, a_{t-1}$ in N such that

$$C_S(a_0, \ldots, a_{t-1}) \cap S'S^p = C_S(N) \cap S'S^p = N .$$

In fact, take $a_0 \in N$ and look at $C_S(a_0) \cap S'S^p$. If $C_S(a_1) \supseteq C_S(a_0) \cap S'S^p$ for all $a_1 \in N$, then $C_S(N) \supseteq C_S(a_0) \cap S'S^p$, and we are done. Otherwise, pick $a_1 \in N$ satisfying $C_S(a_1) \not\supseteq C_S(a_0) \cap S'S^p$ and form $C_S(a_0, a_1) \cap \cap S'S^p$. Repeat this procedure, and notice that the machinery must stop in at most q steps, producing $t \leqslant q$ and $a_0, \ldots, a_{t-1} \in N$ as required. Of course we can assume $t = q$. Recall that $b_0, \ldots, b_r \in S'S^p - N$, so, for every $j \leqslant r$, there is some $i < q$ for which $a_i b_j \neq b_j a_i$. In conclusion, there are $a_0, \ldots, a_{q-1}$ and $b_0, \ldots, b_r$ in S such that the following conditions hold:

(β_1) for all $j \leqslant r$, there is $i < q$ satisfying $[a_i, b_j] \neq [b_j, a_i]$;

(β_2) for all $j \leqslant r$, b_j is a product of q commutators;

(β_3) for every a and b in S, both a^p and $[a, b]$ can be expressed in the form

$$sw(b_0, \ldots, b_r)$$

where $w(b_0, \ldots, b_r)$ is in a fixed set of words on $b_0, \ldots, b_r$ (depending only on H) and s belongs to $C_S(a_0, \ldots, a_{q-1})$ (hence to $C_S(a_0, \ldots, a_{q-1}) \cap S'S^p$);

(β_4) $b_0, \ldots, b_r$ just satisfy modulo $C_S(a_0, \ldots, a_{q-1})$ (hence modulo $C_S(a_0, \ldots, a_{q-1}) \cap S'S^p$) the relations in the given presentation of H;

(β_5) $C_S(a_0, \ldots, a_{q-1})$ is normal in S;

(β_6) if $s, a, b \in S$ and $s^p \in C_S(a_0, \ldots, a_{q-1})$ (so $s^p \in C_S(a_0, \ldots, a_{q-1}) \cap S'S^p$), then s commutes with both a^p and $[a, b]$ modulo $C_S(a_0, \ldots, a_{q-1})$ (hence modulo $C_S(a_0, \ldots, a_{q-1}) \cap S'S^p$).

Notice that $\beta_1, \ldots, \beta_6$ can be expressed as first order formulas in L. Let

$$\beta_0 \colon \beta_0(v_0, \ldots, v_{q-1}, w_0, \ldots, w_r)$$

be their conjunction. So

$$\beta : \exists v_0 \ldots \exists v_{q-1} \exists w_0 \ldots \exists w_r \beta_0(v_0, \ldots, v_{q-1}, w_0, \ldots, w_r)$$

just translates the previous condition. Now choose

$$a_0', \ldots, a_{q-1}', b_0', \ldots, b_r' \in S$$

satisfying β_0. Hence $K_1 = C_S(a_0', \ldots, a_{q-1}') \cap S'S^p$ is a subgroup of $S'S^p$ normal (by β_5) and proper (owing to β_1 and β_2); moreover $S'S^p/K_1$ is a homomorphic image of H (by β_3 and β_4), and so is just isomorphic to H because H is simple. Then we can apply Fact 4.1 to

$$F = S'S^p, \ K_0 = C_S(a_0, \ldots, a_{q-1}) \cap S'S^p$$

and

$$K_1 = C_S(a_0', \ldots, a_{q-1}') \cap S'S^p;$$

recalling that K_0 is abelian, we can conclude

(γ) for every choice of $(a_0', \ldots, a_{q-1}', b_0', \ldots, b_r')$ in $\beta_0(S^{q+r+1})$, $C_S(a_0', \ldots, a_{q-1}') \cap S'S^p$ is abelian.

Notice that γ may need infinitely many L-sentences to be translated in a first order way, because $S'S^p$ is not necessarily definable. However this (possibly infinite) translation can be done.

In conclusion, when $S \in \mathcal{K}(G)$ satisfies Case 2, then

$$S \models \beta \wedge \gamma .$$

Let us check that the converse is also true. So assume that there are $a_0, \ldots, a_{q-1}$ and $b_0, \ldots, b_r$ in S satisfying β_0 and that, for all $(a_0', \ldots, a_{q-1}', b_0', \ldots, b_r')$ in $\beta_0(S^{q+r+1})$, $C_S(a_0', \ldots, a_{q-1}') \cap S'S^p$ is abelian. First of all, β_2 ensures $b_0, \ldots, b_r \in S'S^p$. Now put

$$N = C_S(a_0, \ldots, a_{q-1}) \cap S'S^p.$$

By β_1, $b_0, \ldots, b_r \notin N$. By β_3 $S'S^p = \langle N, b_0, \ldots, b_r \rangle$: for, every commutator or p-th power in S decomposes as a product

$$sw(b_0, \ldots, b_r)$$

where $w(b_0, \ldots, b_r)$ is a word on $b_0, \ldots, b_r$ and s belongs to $C_S(a_0, \ldots, a_{q-1})$, so that $s \in N$. By β_5, N is normal in S, hence in $S'S^p$. By β_4, $S'S^p/N$ is a homomorphic image of H; but $S'S^p / = N$ -as stated in β_1 and β_2-, so $S'S^p/N$ is just isomorphic to N because H is simple; β_6 ensures the last condition in (ii). Finally, N is abelian owing to γ.

Hence $\mathcal{K}''(G)$ is the class of the groups satisfying either α or β and the sentences in γ; so $\mathcal{K}''(G)$ is elementary. ■

Now let us discuss the elementarity of $\mathcal{K}'(G)$ for G as before. In order to express (i) in a first order way, we cannot use Lemma 3.2 and the related approach already followed in [T] for $\mathcal{K}(C(p)^{(\aleph_0)})$; for, neither $S'S^p$ nor S/N are abelian. By the way, notice that no group $S \in \mathcal{K}(G)$ is nilpotent (because S has a subgroup $S'S^p$ projecting onto H). On the other side, if $|S : S'S^p|$ is finite, then $|S : S'S^p|$ is a power p^m of p and the quotient group $S/S'S^p$ is of the form $C(p)^m$; furthermore, if $S \in \mathcal{K}''(G)$, then S/N is $C(p)^m \oplus H$, and hence S is abelian-by-$(C(p)^m \oplus H)$, more generally S is abelian-by-finite because N is abelian. So we can state at least this positive result.

COROLLARY 4.3. Let G be as above. Then the class of groups $S \in \mathcal{K}(G)$ which are not abelian-by-finite (as well as the class of groups which are not abelian-by-$C(p)^m \oplus H$ for any nonnegative integer m) is elementary.

PROOF. Just recall that the class of groups which are not abelian-by-finite is elementary, as well as the class of the groups which are not abelian-by-$(C(p)^m \oplus H)$, and use the previous remarks. ■

5. The squarefree order abelian case.

In the previous section, we discussed the elementarity of $\mathcal{K}(G)$ when $G = C(p)^{(\aleph_0)} \oplus H$ where H is finite simple nonabelian and the order of H is prime to p. Now let us deal with

$$G = C(p)^{(\aleph_0)} \oplus H$$

where H is simple abelian, hence $H = C(q)$ for some prime q. Of course, we still assume $q \neq p$. By the way, notice that, for H abelian, G is abelian, too; and recall that, in this case, a group $S \in \mathcal{K}(G)$ is solvable of class 2.

With respect to the elementarity problem, we divide the groups $S \in \in \mathcal{K}(G)$ in two subclasses, according to whether $S' S^p$ is abelian or not. Notice that the condition «$S' S^p$ abelian» can be easily expressed by a first order sentence in the language L of groups. Moreover, for $S \in \in \mathcal{K}(G)$,

$$S' S^p \text{ is abelian if and only if } S \in C(p)^{(\aleph_0)}.$$

In fact, let $S \in \mathcal{K}(G)$, so S satisfies the conditions (i) and (ii)′ in Corollary 2.2. If S is not in $\mathcal{K}(C(p)^{(\aleph_0)})$, then $S' S^p$ cannot be commutative because $S' S^p$ has infinite index in S owing to (i) and the quotient group is an elementary abelian p-group. Conversely, let $S \in \mathcal{K}(C(p)^{(\aleph_0)})$, and let A be a normal abelian subgroup such that S/A is an infinite elementary abelian p-group. Then both S' and S^p are subgroups of A, $S' S^p \leqslant A$ and $S' S^p$ is abelian.

First let us treat the groups $S \in \mathcal{K}(G)$ for which $S' S^p$ is not abelian. A simplified version of the procedure in Theorem 4.2 and Lemma 3.2 yields

THEOREM 5.1. Let $G = C(p)^{(\aleph_0)} \oplus C(q)$ where $p \neq q$ are prime. Then the class of the groups $S \in \mathcal{K}(G)$ such that $S' S^p$ is not abelian is elementary.

PROOF. In order to write (ii) (or also (ii)′) in a first order way, we have simply to adapt Theorem 4.2 and use our hypothesis that $S' S^p$ is not abelian. But now the obvious remark that G is abelian improves the situation with respect to (i). In fact, what we have to prove is that (i) can be written by suitable first order sentences in L. Of course we can assume that (ii)′ holds. Assume that S does not satisfy (i). Consequently $S' S^p$ has a finite index n in S. Then the subgroup N in (ii)′ has a finite index qn in S. So we can apply Lemma 3.2; in fact, S has a normal abelian subgroup N such that S/N is abelian and has a finite order qn. Accordingly every element of $S'S^p$ is the product of a uniformly bounded number of commutators and p-th powers. This lets us define in a first order way $S'S^p$ inside S. Hence, if the index of $S'S^p$ in S is n, then we can write this property by a first order sentence δ_n in L. It follows that $\{\neg\delta_n : n \in \omega, n > 0\}$ expresses (i) in L. ∎

So we can limit our analysis to the groups $S \in \mathcal{K}(G)$ satisfying «$S'S^p$ abelian». In this case, we can slightly enlarge our setting and assume

$$G = C(p)^{(\aleph_0)} \oplus C(q)$$

where q is squarefree (and prime to p). Even under this assumption, Corollary 2.2 ensures that there is a normal abelian subgroup N of S such that $S'S^p \geqslant N$ and $S'S^p/N \simeq C(q)$ (consequently, $S/N \simeq C(p)^{(\alpha)} \oplus \oplus C(q)$ for some infinite cardinal α). Put

$$C = C_S(S'S^p)$$

(the centralizer of $S'S^p$ in S); C includes $S'S^p$ (because $S'S^p$ is abelian), is normal (because $S'S^p$ is) and $\emptyset$-definable (even if $S'S^p$ is not). However C may be nonabelian. So consider

$$Z = Z(C)$$

(the center of C); Z still includes $S'S^p$ and is normal in S and $\emptyset$-definable; furthermore Z is abelian. In particular, as $Z \geqslant S'S^p$, S/Z is a (possibly finite) elementary abelian p-group because S/Z is a homomorphic image of $S/S'S^p$. Z/N is a subgroup of S/N including $S'S^p/N \simeq C(q)$, and so is of the form

$$C(p)^{(\beta)} \oplus C(q)$$

for some (possibly finite) cardinal β.

LEMMA 5.2. Let S be a group such that $S'S^p$ is abelian, r be a positive integer prime to p. Then the following propositions are equivalent:

(1) $S = S^r$,

(2) $S = S'S^r$,

(3) $S'S^p = S'S^{pr}$,

(4) $S'Z^r = Z$.

Proof (1) $\Rightarrow$ (3). For every $a \in S^r$, a can be expressed as $b_0^r \dots b_k^r$ where k is a nonnegative integer and $b_0, \dots, b_k \in S$. So

$$a^p = (b_0^r \dots b_k^r)^p \equiv b_0^{rp} \dots b_k^{rp}$$

modulo S', hence $a^p \in S'S^{pr}$. It follows $S'S^p \leqslant S'S^{pr}$. The converse is clear.

(3) $\Rightarrow$ (2) Let x and y be integers satisfying $1 = px + ry$. For all $a \in S$, $a = a^{px}a^{ry}$; $a^{px} \in S^p \leqslant S'S^{pr}$; hence $a \in S'S^r$.

(2) $\Rightarrow$ (1) Clearly $S = S^pS^r$, so

$$\frac{S}{S^r} = \frac{S^pS^r}{S^r} \simeq \frac{S^p}{S^p \cap S^r}.$$

As S^p is abelian, $S^r \geqslant S'$, hence $S = S'S^r = S^r$.

(1) $\Leftrightarrow$ (4) We know $S = S^pS^r$ and $S^p \leqslant Z$, so $S = ZS^r$. It follows

$$\frac{S}{S^r} = \frac{ZS^r}{S^r} \simeq \frac{Z}{Z \cap S^r}.$$

Accordingly it suffices to show $Z \cap S^r = S'Z^r$. $\geqslant$ is trivial, because $S' \leqslant$ $\leqslant Z$ and we have seen that, for S^p abelian, $S' \leqslant S^r$. Conversely take $a \in$ $\in Z \cap S^r$. Then there are a natural k and $b_0, \dots, b_k \in S$ satisfying

$$a = b_0^r \dots b_k^r \equiv (b_0 \dots b_k)^r$$

modulo S'. Put $b = b_0 \dots b_k$ for simplicity. As $S' \leqslant Z$, $b^r \in Z$. Use again $b = b^{px}b^{ry}$ (and $S^p \leqslant Z$) and deduce $b \in Z$, so $a \in S'Z^r$. ■

Theorem 5.3. Let S be a group, $S'S^p$ be abelian, $G = C(p)^{(\aleph_0)} \oplus$ $\oplus C(q)$ where q is squarefree and prime to p. Then $S \in \mathcal{K}(G)$ if and only if S satisfies the following conditions:

(i) $S'S^p$ has infinite index in S;

(ii)″ for every prime r dividing q, $S \neq S^r$.

(Of course, as q and p are coprime, we can replace $S \neq S^r$ in (ii)″ with one of the equivalent propositions in Lemma 5.2).

Proof. Let $S \in \mathcal{K}(G)$. We know that (i) holds and there exists a normal subgroup N of S such that $N \leqslant S'S^p$ and $S'S^p/N \simeq C(q)$; furthermore $S/N \simeq C(p)^{(\alpha)} \oplus C(q)$ for some infinite cardinal α. Then S projects itself onto $C(q)$. Let r be a prime dividing q; then $C(q) \neq C(q)^r$ and, consequently, $S \neq S^r$; so (ii)″ holds.

Conversely let S be a group satisfying «$S'S^p$ abelian», (i) and (ii)$''$. Hence $S/S'S^p$ is an infinite elementary abelian p-group. Furthermore, for every prime r dividing q, $S'S^p \neq S'S^{pr}$. Choose $a_r \in S'S^p - S'S^{pr}$. Form

$$a = \prod_r a_r \in S'S^p.$$

Notice that, for every prime r dividing q, a_r has period r modulo $S'S^{pr}$, hence a period multiple of r modulo $S'S^{pq} \leq S'S^{pr}$. Then the period of a modulo $S'S^{pq}$ is just q. As $S'S^p/S'S^{pq}$ is an abelian group of exponent dividing q, the cyclic subgroup generated by $aS'S^{pq}$ is a non-zero direct summand of $S'S^p/S'S^{pq}$. Let M be a complement of this summand in $S'S^p/S'S^{pq}$, and let N be the preimage of M in the canonical homomorphism of $S'S^p$ onto $S'S^p/S'S^{pq}$. Then $N \leq S'S^p$ (hence N is abelian) and $N \geq S'S^{pq}$ (hence $S' \leq N$ and N is normal in S); finally

$$\frac{S'S^p}{N} \simeq \frac{S'S^p/S'S^{pq}}{N/S'S^{pq}} \simeq \frac{S'S^p/S'S^{pq}}{M} \simeq C(q).$$

Hence Corollary 2.2 implies $S \in \mathcal{K}(G)$. ■

Theorem 5.3 provides a possible approach to prove the elementarity of $\mathcal{K}(G)$ is our setting. Of course, we have to handle S' and S^r in some definable way. Here is a partial positive result.

Corollary 5.4. Let $G = C(p)^{(\aleph_0)} \oplus C(q)$ where q is squarefree and prime to p. Then the class of the nil-2 groups $S \in \mathcal{K}(G)$ for which $S'S^p$ is abelian is elementary.

Recall that S is nil-2 (nilpotent of class 2) if and only if its derived subgroup S' is contained in the center $Z(S)$ of S; this condition can be easily written as a first order sentence in L. Recall also that a group in $\mathcal{K}(G)$ is solvable of class 2.

Proof. Let S be a group such that $S'S^p$ is abelian. As before, put $Z = Z(C_S(S'S^p))$. Let r be a positive integer. Clearly, if $Z = Z^r$, then S satisfies all the equivalent conditions in Lemma 5.2; for, $S' \leq Z$, so $Z^r = S'Z^r$ and $Z = S'Z^r$. We claim that, when S is nil-2, the converse is also true: if $Z = S'Z^r$, then $Z = Z^r$. In fact $S = S'S^r$, so, for every $a \in S$, a can be expressed as

$$a = d^r c$$

where $d \in S$ and $c \in S'$. Let a, $b \in S$ and decompose $a = d^r c$ as before. Then

$$[b, a] = [b, d^r][b, c]^{d^r}$$

(see [R], ex. 5.46 p. 118)

$$= [b, d^r]$$

(because S is nil-2, and hence $c \in Z(S)$)

$$= [b, d]^r$$

(because $[b, d] \in Z(S)$; see [R], 5.42 p. 119). Hence $S' \leqslant S'^r \leqslant Z^r$. Consequently $Z^r = S' Z^r$ and $Z = Z^r$.

In conclusion, for S nil-2 and $S' S^p$ abelian, $S \in \mathcal{K}(G)$ if and only if S satisfies (i) and

$$Z \neq Z^r \text{ for all primes } r \text{ dividing } q\,.$$

We already know how to express (i) in a first order way (see Theorem 5.1). In fact, if (i) fails, then $S' S^p$, hence Z, have finite index in S; in particular let n be the index of $S' S^p$ in S. Notice that both Z and S/Z are abelian; so use Lemma 3.2 to define $S' S^p$ and to express $|S : S' S^p| = n$ by a first order sentence of L. Conclude just as in Theorem 5.1. Now recall that Z is $\emptyset$-definable and abelian, so even Z^r is definable. Hence the statement

$$Z \neq Z^r \text{ for all primes } r \text{ dividing } q\,.$$

can be written as a first order sentence of L. In conclusion, the nil-2 groups $S \in \mathcal{K}(G)$ such that $S' S^p$ is abelian are an elementary class. ■

Finally let us give a short look at the problem of avoiding the nil-2 assumption in Corollary 5.4, and hence showing the elementarity of the whole class $\mathcal{K}(G)$, at least when $G = C(p)^{(\aleph_0)} \oplus C(q)$ with q prime and $q \neq p$ (so $G = C(p)^{(\aleph_0)} \oplus H$ with H simple abelian).

REMARKS 5.5 (1). «S/Z^q abelian» can be easily expressed in a first order way, and «S/Z^q abelian» (namely $S' \leqslant Z^q$) obviously ensures the elementarity of the class of the groups $S \in \mathcal{K}(G)$ satisfying this assumption and «$S' S^p$ abelian» (for, $Z \neq Z^q$ is equivalent to $Z \neq S' Z^q$ when $S' \leqslant Z^q$).

Now assume that S/Z^q is not abelian.

(2) The condition «$|S : Z| = n$» for a given positive integer n is still first order; if S satisfies this assumption, then Lemma 3.2 applies (for, Z is abelian and S/Z is abelian, too, because $S' \leqslant Z$). By adapting the corresponding proof and using $S' \leqslant Z$, one sees that $S' Z^q$ is definable, and so «$Z \neq S' Z^q$» can be translated in a first order sentence.

(3) Also «$|Z : Z^q| = n$», for a fixed $n > 1$, is first order. Let S satisfy this assumption. Notice that n is a power of q, $n = q^h$ with $h > 0$, and Z/Z^q is a $\mathbf{Z}/q\mathbf{Z}$-vectorspace of dimension h. Accordingly one can express «$Z \neq S' Z^q$» in a first order way, by simply stating that there is some element in Z which is not of the form

$$c_0^{q_0} \ldots c_{h-1}^{q_{h-1}} z^q$$

with $z \in Z$, $c_0, \ldots, c_{h-1}$ commutators and $0 \leqslant q_0, \ldots, q_{h-1} < q$. For, if $Z = S' Z^q$, then the commutators generate Z modulo Z^q over $\mathbf{Z}/q\mathbf{Z}$, and one can extract a basis of h commutators of Z modulo Z^q.

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