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Quasibases of p-Groups.

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ABSTRACT - The concept of a quasibasis is reduced to that of an inductive quasibasis and abelian p-groups are explicitly described by the corresponding diagonal relation arrays. For a basic subgroup B we determine $\mathrm{Pext}(\mathbb{Z}(p^\infty), B)$ in terms of diagonal relation arrays. Independent diagonal relation arrays are shown to correspond uniquely to reduced, separable groups.

1. Introduction.

This paper [1] deals with abelian p-groups which are considered as extensions of a basic subgroup by a divisible group. We describe them by generators and relations as given by the concept of a quasibasis [3, 33.5] and the methods established in [6]. We define the concept of an inductive quasibasis and obtain results similar to those of Boyer and Mader [2] and Griffith [4] on the embedding of p-groups in their completions. We then give explicit examples by generators and relations. We introduce diagonal relation arrays and show that smallness is equivalent to splitting, cf. [6]. For a basic subgroup B we determine $\mathrm{Pext}(\mathbb{Z}(p^\infty), B)$ in terms of diagonal relation arrays. Finally we show that independent diagonal relation arrays correspond uniquely to reduced, separable groups. Moreover, we

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determine whether a group is a direct sum of cyclics and show that it depends on the cardinality of G/B relative to $2^{|B|}$.

2. Preliminaries.

Let $\mathbb{N}$ denote the set of the natural numbers, without 0, and $\mathbb{Z}_p$ the localization of the ring of integers at the prime p. Let us quickly recall the main definitions and conventions. We adapt to $\mathbb{Z}_p$-modules the description by generators and relations used for of mixed modules in [6]. In particular, we consider p-groups as $\mathbb{Z}_p$-modules. Let G be a p-group with a basic subgroup $B = B_\lambda = \bigoplus_{j \in \mathbb{N}} \bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u$ of isomorphism type $\lambda = (\lambda_j \,|\, j \in \mathbb{N})$ where $\lambda_j = |I_j|$ and order $o(x_j^u) = p^j$ for $i \in \mathbb{N}$, $u \in I_j$. Later we write $B = \bigoplus_{j \in \mathbb{N}} B_j$, where $B_j = \bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u$ is homocyclic of exponent p^j and rank λ_j. Suppose the quotient G/B is a divisible group of rank d and let I be a set of cardinality d. The subset

$$Q = \{x_j^u, a_i^k \,|\, i, j \in \mathbb{N}, u \in I_j, k \in I\} \subset G$$

is called a *quasibasis of* G, if

(i) $\{x_j^u \,|\, j \in \mathbb{N}, u \in I_j\}$ is a basis of the basic subgroup B with $o(x_j^u) = p^j$ for all $j \in \mathbb{N}$, $u \in I_j$;

(ii) $G/B = \bigoplus_{k \in I} A^k$, where $A^k = \langle a_i^k + B \,|\, i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$ and $pa_{i+1}^k + B = a_i^k + B$, for all $i \in \mathbb{N}$, $k \in I$;

(iii) $o(a_i^k) = p^i$ for all $i \in \mathbb{N}$, $k \in I$.

It is easy to see that

$$G = \langle x_j^u, a_i^k \,|\, i, j \in \mathbb{N}, u \in I_j, k \in I\rangle.$$

By [3, 33.5] every p-group G has a quasibasis and a quasibasis intrinsically defines a series of equations with coefficients in $\mathbb{Z}_p$ that describe the relations among the generators, namely

$$pa_{i+1}^k = a_i^k - \sum_{j \in \mathbb{N}} \sum_{u \in I_j} \alpha_{i,j}^{k,u} x_j^u, \qquad i \in \mathbb{N}, \; k \in I. \tag{1}$$

The choice of a_i^k uniquely determines $\alpha_{i,j}^{k,u}$ modulo $p^j\mathbb{Z}_p$. Modeling upon these relations we introduce the concept of an abstract array $\alpha = (\alpha_{i,j}^{k,u})$ with $i, j \in \mathbb{N}$, $u \in I_j$, $k \in I$ having entries in $\mathbb{Z}_p$ and which is *row finite in* j *and* u, i.e. for a fixed pair (k, i), $\alpha_{i,j}^{k,u} \in p^j\mathbb{Z}_p$ for almost all pairs (j, u).

Such arrays α are called *relation arrays*. If $\lambda = (|I_j| \, | j \in \mathbb{N})$ is the isomorphism type of the basic subgroup B and $d = |I|$ then we call the relation array α of *format* (λ, d).

When (1) holds for a quasibasis Q of a p-group G we say that the relation array $(\alpha_{i,j}^{k,u})$ *corresponds to the quasibasis* Q. A relation array is called *realizable* if there is a p-group with a quasibasis Q and corresponding relation array $(\alpha_{i,j}^{k,u})$. In such cases we say that a relation array is *realized* by a p-group.

REMARK. If $(\alpha_{i,j}^{k,u})$ is the relation array corresponding to a quasibasis of a p-group then $\alpha_{i,j}^{k,u} \in p^{j-i}\mathbb{Z}_p$ for all $j \geq i$, since by the definition of a quasibasis $pa_{i+1}^k - a_i^k = -\sum_{j \in \mathbb{N}} \sum_{u \in I_j} \alpha_{i,j}^{k,u} x_j^u$ and $o(a_i^k) = p^i$. This is a necessary property for a relation array to be realizable. This property is also sufficient, but the proof is of computational type and we omit it.

Let $\{g_i \, | i\}$ be a generating system of a group G and let P be a system of relations of the elements g_i. If all relations between the elements in $\{g_i \, | i\}$ follow from P then P is called a *system of defining relations*. This is equivalent to the fact that the quotient of the free group H generated by $\{g_i \, | i\}$ and the subgroup generated by P is isomorphic to G, i.e. $G \cong H/\langle P \rangle$. Since a quasibasis induces relations it is natural to ask if these relations determine the p-group completely, i.e., do they present the group. This is shown in the following theorem.

THEOREM 1. *Let* $\{x_j^u, a_i^k \, | i, j \in \mathbb{N}, u \in I_j, k \in I\}$ *be a quasibasis of some p-group with corresponding relation array* $(\alpha_{i,j}^{k,u})$. *Then*

$$P = \left\{ p^j x_j^u, \, p^i a_i^k, \, pa_{i+1}^k - a_i^k + \sum_{j \in \mathbb{N}} \sum_{u \in I_j} \alpha_{i,j}^{k,u} x_j^u \, | i, j \in \mathbb{N}, \, u \in I_j, \, k \in I \right\}$$

is a system of defining relations.

PROOF. Let G be a p-group with basic subgroup B with the given quasibasis. Let

$$\sum_{i,k} \beta_i^k a_i^k + \sum_{j,u} \beta_j^u x_j^u = 0$$

be some relation in G. Modifying with elements in P we may assume that $0 \leq \beta_i^k < p$ and $0 \leq \beta_j^u < p^j$ for all $i, j \in \mathbb{N}$, $u \in I_j$, $k \in I$. Now we consider this relation modulo B. Since $G/B = \bigoplus_{k \in I} \langle a_i^k + B \, | i \in \mathbb{N} \rangle$, all $\beta_i^k = 0$. Furthermore, since the elements x_j^u form a basis of B, we also obtain $\beta_j^u = 0$

for all $j \in \mathbb{N}$, $u \in I_j$. Hence the given relation was generated by the elements in P, as desired. ■

In view of Theorem 1 a quasibasis of a p-group with corresponding relation array is a *presentation* of this p-group. We consider two examples one using the standard-B group $\mathcal{B} = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_p a_i$, $o(a_i) = p^i$, and the other the generalized Prüfer group $H_{\omega+n}$ of length $\omega + n$.

EXAMPLE ([3, 35.1]). The standard-B group $\mathcal{B}$ has quasibases with corresponding relation arrays $\alpha = (\alpha_{i,j})$ and $\beta = (\beta_{i,j})$ of the form

$$(2) \qquad \alpha = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{bmatrix}.$$

To see this we proceed as in the proof of [3, 35.1] and define $x_i = a_i - pa_{i+1}$ for all $i \in \mathbb{N}$. The subgroup $B = \bigoplus_{i \in \mathbb{N}} \mathbb{Z}_p x_i$ is basic in $\mathcal{B}$ with quotient $\mathbb{Z}(p^\infty)$ and $\{x_i, a_i \mid i \in \mathbb{N}\}$ is a quasibasis. The induced relations are $pa_{i+1} = a_i - x_i$, hence the corresponding relation array is α.

To realize β as a relation array let

$$x_i = \begin{cases} a_i - p^2 a_{i+2} & \text{if } i \text{ is odd}, \\ a_i & \text{if } i \text{ is even}, \end{cases} \qquad b_i = \begin{cases} a_i & \text{if } i \text{ is odd}, \\ pa_{i+1} & \text{if } i \text{ is even}. \end{cases}$$

It is straightforward to verify that $\{x_i \mid i \in \mathbb{N}\}$ is a p-independent set. This proves that the group $B = \sum_{i \in \mathbb{N}} \mathbb{Z}_p x_i$ is a direct sum of cyclics and pure in $\mathcal{B}$. Moreover, $\mathcal{B}/B$ is divisible, namely

$$pb_{i+1} = \begin{cases} p^2 a_{i+2} = a_i - x_i = b_i - x_i & \text{if } i \text{ is odd}, \\ pa_{i+1} = b_i & \text{if } i \text{ is even}. \end{cases}$$

Thus $\mathcal{B}/B \cong \mathbb{Z}(p^\infty)$, B is a basic subgroup, $\{x_i, b_i \mid i \in \mathbb{N}\}$ is a quasibasis of $\mathcal{B}$ and β is the corresponding relation array.

REMARK. The argument that $\mathcal{B}$ has the relation array β can be modified to show that $\mathcal{B}$ allows all 0 and 1 sequences on the diagonal as long as there are infinitely many entries equal to 1. We will prove this fact later without any calculation, cf. the note after Corollary 16.

EXAMPLE 3. ([3, Section 35, Example]) The generalized Prüfer group $H_{\omega+n}$ of length $\omega+n$, $n\in\mathbb{N}$, has a quasibasis with corresponding relation array $\alpha=(\alpha_{i,j})$ of the form

$$\alpha=\begin{bmatrix} p^n & & & \\ & p^n & & \\ & & p^n & \\ & & & \ddots \end{bmatrix}. \tag{3}$$

This can be seen as follows. In the example in [3, Section 35] $H_{\omega+n}$ is generated by the set $\{b_0, b_1, b_2, \dots\}$ with the relations $p^n b_0=0$, $p^i b_i = b_0$ for all $i\in\mathbb{N}$. Clearly $o(b_i)=p^{n+i}$ for all $i\geqslant 0$. Define $x_i=b_i-pb_{i+1}$ and $a_i=p^n b_i$ for all $i\in\mathbb{N}$. It is routine to show that the set $\{x_i \mid i\in\mathbb{N}\}$ is p-independent. Moreover, $B=\bigoplus_{i\in\mathbb{N}}\mathbb{Z}_p x_i$ is a basic subgroup and $\{x_i, a_i \mid i\in\mathbb{N}\}$ is a quasibasis since $pa_{i+1}=p^{n+1}b_{i+1}=p^n(b_i-x_i) = a_i-p^n x_i$, i.e. $H_{\omega+n}/B\cong\mathbb{Z}(p^\infty)$. Thus we have the corresponding relation array α as indicated.

3. Inductive Quasibases.

We begin by recalling the Baer-Boyer decomposition, [3, 32.4], which simplifies the relations (1).

Let B be a basic subgroup of the p-group G with a homogeneous decomposition $B=\bigoplus_{n\in\mathbb{N}}B_n$. For $m\in\mathbb{N}$ let the subgroup G_m be defined by $G_m=(\bigoplus_{n>m}B_n)+p^m G$. Then

$$G=\left(\bigoplus_{n=1}^{m}B_n\right)\oplus G_m \quad \textit{and} \quad G_m=B_{m+1}\oplus G_{m+1}.$$

A quasibasis $\{x_j^u, a_i^k \mid i,j\in\mathbb{N}, u\in I_j, k\in I\}$ is called an *inductive quasibasis* if the corresponding relations are $pa_{i+1}^k=a_i^k-b_i^k$, $i\in\mathbb{N}$, $k\in I$, with the condition that $b_i^k\in B_i=\bigoplus_{u\in I_i}\mathbb{Z}_p x_i^u$.

THEOREM 4. *If B is a basic subgroup of a p-group with corresponding quasibasis*

$$\{x_j^u, a_i'^k \mid i,j\in\mathbb{N}, u\in I_j, k\in I\},$$

then there are $a_1^k = a_1'^k$ *and* $a_i^k \in a_i'^k + B$ *for all* $i > 1$ *and* $k \in I$*, such that* $\{x_j^u, a_i^k \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ *is an inductive quasibasis.*

In particular, every p-group has an inductive quasibasis.

PROOF. Let $\{x_j^u, a_i'^k \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ be a quasibasis of the p-group G. We continue in three steps. First, we define a new quasibasis by changing the elements $a_i'^k$ to $c_i^k \in G_{i-1}$. By the Baer-Boyer decomposition we have for each pair (i, k):

$$a_i'^k = b_{i,1}^k + b_{i,2}^k + \ldots + b_{i,i-1}^k + g_{i,i-1}^k \quad \text{where} \quad b_{i,n}^k \in B_n \quad \text{and} \quad g_{i,i-1}^k \in G_{i-1}.$$

Let $c_1^k = a_1'^k$ and $c_i^k = g_{i,i-1}^k$ if $i > 1$; then the set $\{x_j^u, c_i^k \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ is also a quasibasis of G.

Second, we show certain properties of the elements $b_{i,n}^k \in B_n$ occurring in the relations

$$c_i^k - pc_{i+1}^k = b_i^k = \sum_{n \in \mathbb{N}} b_{i,n}^k, \qquad i \in \mathbb{N},\ k \in I, \tag{4}$$

namely

(i) $b_{i,n}^k = 0$ for $n < i$,

(ii) $o(b_{i,n}^k) \leq p^i$ for all $n \geq i$; in particular, p divides $b_{i,n}^k$ for $n > i$.

Since $b_i^k \in B$ there is an $m \in \mathbb{N}$ such that $b_{i,n}^k = 0$ for all $n > m$. With $c_{i+1}^k \in G_i$, $c_i^k \in G_{i-1}$ and $G_i \subset G_{i-1}$ we get $b_i^k = c_i^k - pc_{i+1}^k \in G_{i-1}$. Using the decomposition

$$G = \Big(\bigoplus_{n=1}^{i-1} B_n\Big) \oplus \Big(\bigoplus_{n=i}^{m} B_n\Big) \oplus G_{m-1},$$

and the fact that $\Big(\bigoplus_{n=i}^{m} B_n\Big) \oplus G_{m-1} = G_{i-1}$, we obtain (i).

Write $c_i^k - pc_{i+1}^k = b_i^k = b_{i,1}^k + b_{i,2}^k + \ldots + b_{i,m}^k$ for some m. Since $o(c_i^k) = o(pc_{i+1}^k) = p^i$, we get $o(b_{i,n}^k) \leq p^i$ for all n, because $B = \bigoplus_{n \in \mathbb{N}} B_n$. In particular, we immediately have that p divides $b_{i,n}^k$ for $n > i$, since $b_{i,n}^k \in B_n$.

The third step will be to define an inductive quasibasis $\{x_j^u, a_i^k \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ starting with the special quasibasis $\{x_j^u, c_i^k \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ already obtained. For this let $a_i^k = c_i^k + b_{i-1}'^k$, where $b_i'^k \in \bigoplus_{j \geq i+1} B_j \subset G_i$ and we induct on i. If $i = 1$, let $b_0'^k = 0$ and then $a_1^k = c_1^k = a_1'^k$. The induction hypothesis is that the set $\{x_j^u, a_{n_1}^k, c_{n_2}^k \mid j \in \mathbb{N}, u \in I_j, k \in I, 1 \leq n_1 \leq i, n_2 \geq i+1\}$ is a quasibasis of G, where $pa_{j+1}^k = a_j^k - b_j^k$

with $b_j^k \in B_j$ for all $j < i$ and a_l^k, $c_l^k \in G_{l-1}$ for all l, respectively. We may assume that the elements $b_{i,n}^k$ have the Properties (i) and (ii). This results in the following equation

$$pc_{i+1}^k = a_i^k - b_i^k = a_i^k - \left(b_{i,i}^k + \sum_{j \geq i+1} b_{i,j}^k\right),$$

where the sum $\sum_{j \geq i+1} b_{i,j}^k$ is divisible by p, i.e. there is some $b_i'^k \in \bigoplus_{j \geq i+1} B_j \subset \subset G_i$ such that $pb_i'^k = \sum_{j \geq i+1} b_{i,j}^k$. Let $a_{i+1}^k = c_{i+1}^k + b_i'^k$. Then $a_{i+1}^k \in G_i$ and

$$pa_{i+1}^k = a_i^k - b_{i,i}^k. \tag{5}$$

Hence $o(a_{i+1}) \leq p^{i+1}$ since $o(a_i^k) = p^i$ and $o(b_{i,i}^k) \leq p^i$. But $o(a_{i+1}^k + B) = o(c_{i+1}^k + B) = p^{i+1}$ and, by (5), $p^i a_{i+1}^k = p^{i-1} a_i^k - p^{i-1} b_{i,i}^k \notin B$ since $p^{i-1} a_i^k \notin B$. Thus $o(a_{i+1}^k) = p^{i+1}$, and this concludes the proof. ■

A relation array $\alpha = (\alpha_{i,j}^{k,u})$ is called *diagonal* if $\alpha_{i,j}^{k,u} = 0$ for $i \neq j$. We denote a diagonal relation array as $\alpha = (\alpha_i^{k,u})$.

Corollary 5. *Every p-group has a diagonal relation array. In particular, a relation array corresponding to an inductive quasibasis is diagonal.*

Proof. By Theorem 4 every p-group has an inductive quasibasis and a relation array corresponding to a inductive quasibasis is always diagonal. ■

4. A presentation of p-groups.

First we need some notation.

Notation. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a sequence of cardinal numbers and let d be a cardinal number. Recall that

$$B_\lambda = B = \bigoplus_{j \in \mathbb{N}} \bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u,$$

where $o(x_j^u) = p^j$ for all $u \in I_j$, $|I_j| = \lambda_j$. Furthermore, let $pc_{i+1}^k = c_i^k$,

$pc_1^k = 0$ and $\langle c_i^k \mid i \in \mathbb{N} \rangle \cong \mathbb{Z}(p^\infty)$ for all $k \in I$, where $|I| = d$. Let the p-groups

$$G_{\lambda, d} = \bigoplus_{k \in I} \langle c_i^k \mid i \in \mathbb{N} \rangle \oplus \Big(\bigoplus_{j \in \mathbb{N}} \Big(\bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u \Big)\Big) \cong \Big(\bigoplus_d \mathbb{Z}(p^\infty)\Big) \oplus B_\lambda ,$$

and

$$H_{\lambda, d} = \bigoplus_{k \in I} \langle c_i^k \mid i \in \mathbb{N} \rangle \oplus \boldsymbol{t} \Big(\prod_{j \in \mathbb{N}} \Big(\bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u \Big)\Big) \cong \Big(\bigoplus_d \mathbb{Z}(p^\infty)\Big) \oplus \overline{B}_\lambda ,$$

where $\overline{B}_\lambda = \boldsymbol{t}\Big(\prod_{j \in \mathbb{N}} \Big(\bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u \Big)\Big)$ is torsion-complete, cf. [3, Section 68]. The elements $h \in H_{\lambda, d}$ are infinite sequences of the form $h = (h_0, h_1, h_2, \ldots)$ where

$$h_0 \in \bigoplus_{k \in I} \langle c_i^k \mid i \in \mathbb{N} \rangle \quad \text{and} \quad h_j \in \bigoplus_{u \in I_j} \mathbb{Z}_p x_j^u \ \text{ for all } j \in \mathbb{N} .$$

In particular, B_λ is embedded in $H_{\lambda, d}$ as a basic subgroup, cf. [3, Section 33] where we identify x_j^u with $(0, \ldots, 0, \underbrace{x_j^u}_{j^{th}}, 0, \ldots)$ and c_i^k with $(c_i^k, 0, \ldots)$. Let

$$D = \bigoplus_{k \in I} \langle c_i^k \mid i \in \mathbb{N} \rangle = \{ h = (h_0, h_1, h_2, \ldots) \in H_{\lambda, d} \mid 0 = h_1 = h_2 = \ldots \} \cong \bigoplus_d \mathbb{Z}(p^\infty) .$$

Then D is the maximal divisible subgroup of $H_{\lambda, d}$, which is also the first Ulm subgroup of $H_{\lambda, d}$ and its set of elements of infinite height, cf. [3, Section 37]. Note that $B_\lambda = G_{\lambda, 0} \subset H_{\lambda, 0} = \overline{B}_\lambda$, $G_{\lambda, d} \subset H_{\lambda, d}$ and that B_λ is a basic subgroup of $\overline{B}_\lambda$.

For a diagonal relation array $\alpha = (\alpha_i^{k, u})$ of format (λ, d) we define a subgroup $G(\alpha)$ of $H_{\lambda, d}$ as

$$G(\alpha) = \langle x_j^u , a_i^k(\alpha) \mid i, j \in \mathbb{N}, u \in I_j, k \in I \rangle ,$$

where

$$a_i^k(\alpha) = (a_{i,0}^k(\alpha), a_{i,1}^k(\alpha), a_{i,2}^k(\alpha), \ldots)$$

and

$$a_{i,j}^k(\alpha) = \begin{cases} c_i^k & \text{if } j = 0 , \\ 0 & \text{if } 0 < j < i , \\ p^{j-i} \sum_{u \in I_j} \alpha_j^{k, u} x_j^u & \text{if } j \geq i . \end{cases}$$

Observe that by the row finiteness of a relation array the sum above is finite, i.e. the elements $a^k_{i,j}(\alpha)$ and $a^k_i(\alpha)$ are well defined.

PROPOSITION 6. *Let $\alpha = (\alpha^{k,u}_i)$ be a diagonal relation array of format (λ, d). Then B_λ is a basic subgroup of $G(\alpha)$ and $\{x^u_j, a^k_i(\alpha) \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ is an inductive quasibasis of $G(\alpha)$ with α as its corresponding relation array.*

In particular, all diagonal relation arrays are realizable.

PROOF. First we show that B_λ is a basic subgroup of $G(\alpha)$. Since B_λ is already a basic subgroup of $H_{\lambda,d}$ it suffices to show that $G(\alpha)/B_\lambda$ is divisible. This is implied by the identity

$$pa^k_{i+1}(\alpha) = a^k_i(\alpha) - \sum_{u \in I_i} \alpha^{k,u}_i x^u_i \qquad \text{for all } i \in \mathbb{N},\ k \in I. \tag{6}$$

Furthermore, $G(\alpha)/B_\lambda = \bigoplus_{k \in I} \langle a^k_i(\alpha) + B_\lambda \mid i \in \mathbb{N} \rangle \cong \bigoplus_d \mathbb{Z}(p^\infty)$. Thus the set $\{x^u_j, a^k_i(\alpha) \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ has the Properties (i) and (ii) of a quasibasis. It remains to show Property (iii) of a quasibasis, i.e. that $o(a^k_i(\alpha)) = p^i$. This follows from the fact that all components of the sequence $a^k_i(\alpha) = (a^k_{i,0}(\alpha), a^k_{i,1}(\alpha), a^k_{i,2}(\alpha), \ldots)$ have order less than or equal to p^i and the first entry $a^k_{i,0}(\alpha) = c^k_i$ has precisely order p^i. By (6) the relation array α corresponds to this quasibasis and since α is diagonal this quasibasis is inductive. ■

The following corollary allows us to say that a p-group G is *presented by* $G(\alpha)$ or *presented by* α.

COROLLARY 7. *Each p-group G with basic subgroup B of isomorphism type λ and quotient $G/B \cong \bigoplus_d \mathbb{Z}(p^\infty)$ can be embedded in $H_{\lambda,d}$.*

More precisely, if $\{x^u_j, a^k_i \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ is an inductive quasibasis of G with corresponding diagonal relation array α, then the mapping

$$x^u_j \mapsto x^u_j, \qquad a^k_i \mapsto a^k_i(\alpha)$$

extends to an isomorphism of G with $G(\alpha)$ in $H_{\lambda,d}$, identifying B with B_λ.

PROOF. By Theorem 4, a p-group G with basic subgroup B of isomorphism type λ, quotient $G/B \cong \bigoplus_d \mathbb{Z}(p^\infty)$ has an inductive quasibasis. By

Corollary 5, the corresponding relation array α is diagonal and of format (λ, d).

More precisely, the indicated mapping is an isomorphism, since the inductive quasibasis $\{x_j^u, a_i^k \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ of G is mapped to the inductive quasibasis $\{x_j^u, a_i^k(\alpha) \mid i, j \in \mathbb{N}, u \in I_j, k \in I\}$ of $G(\alpha)$ and the corresponding diagonal relation arrays are for both quasibases equal to α. By Theorem 1, the relation arrays induce a system of defining relations, thus G and $G(\alpha)$ must be isomorphic. ■

Corollary 7 is in the spirit of the Boyer and Mader result in [2] and provides another proof of the well known embedding of a p-group G in $D(p^\omega G) \oplus \overline{B}$, cf. Griffith [4, Theorem 25], where $D(p^\omega G)$ is the injective hull of $p^\omega G$ and B is a basic subgroup of G. Furthermore, this embedding is a pure embedding.

If the standard-B group $\mathcal{B}$ is presented by the diagonal relation array α given in (2) then $\lambda = (1, 1, \ldots)$, $d = 1$, and the embedding $\mathcal{B}(\alpha)$ in $H_{\lambda, d}$ is given by

$$a_i(\alpha) = (c_i, 0, \ldots, 0, \underbrace{x_i}_{i^{th}}, px_{i+1}, p^2 x_{i+2}, p^3 x_{i+3}, \ldots).$$

If the generalized Prüfer group $H_{\omega+1}$ is presented by the diagonal relation array α as in (3) then again $\lambda = (1, 1, \ldots)$, $d = 1$, and the embedding $H_{\omega+1}(\alpha)$ in $H_{\lambda, d}$ is given by

$$a_i(\alpha) = (c_i, 0, \ldots, 0, \underbrace{px_i}_{i^{th}}, p^2 x_{i+1}, p^3 x_{i+2}, p^4 x_{i+3}, \ldots).$$

We now prove a useful technical lemma.

LEMMA 8. *For each element $z \in G(\alpha)$ there is a natural number l such that*

$$z = \sum_k \mu^k a_l^k(\alpha) + b_l, \qquad \mu^k \in \mathbb{Z}_p \qquad \text{and} \qquad b_l \in \bigoplus_{i=1}^{l-1} B_i. \tag{7}$$

Furthermore, for each element $z \in D \cap G(\alpha) \setminus \{0\}$ there is a natural number l such that for all integers $m \geq 0$

$$z = p^m \Big(\sum_k \mu^k a_{l+m}^k(\alpha) \Big), \qquad \mu^k \in \mathbb{Z}_p. \tag{8}$$

In particular, if $D \cap G(\alpha) \neq 0$ then the set

$$\left\{\pi(a_1^k(\alpha)) + B \in \overline{B}_\lambda / B \,|\, k \in I\right\}$$

is dependent, where $\pi\colon H_{\lambda, d} \to \overline{B}_\lambda$ is the projection with kernel D.

PROOF. A general element $z \in G(\alpha)$ has the form $z = \sum_{i,k} \mu_i^k a_i^k(\alpha) + b \in G(\alpha)$, where $\mu_i^k \in \mathbb{Z}_p$, $b \in B$. By the relation (6) in $G(\alpha)$ there is a natural number l such that

$$z = \sum_k \mu^k a_l^k(\alpha) + b_l,$$

where $\mu^k \in \mathbb{Z}_p$, $b_l \in B$. Moreover, since the relation array is diagonal we may assume that $b_l \in \bigoplus_{i=1}^{l-1} B_i$.

Furthermore, again by (6) we have for all $m \in \mathbb{N}$

$$z = p^m \left(\sum_k \mu^k a_{l+m}^k(\alpha)\right) + b_{l+m},$$

where $b_{l+m} \in \bigoplus_{i=1}^{l+m-1} B_i$. Now let $y = (y_0, 0, \ldots) = z \in D \cap G(\alpha)$. Using the form of the tuples $a_i^k(\alpha)$ and the fact that $b_{l+m} = 0$ for all m we obtain the desired formula.

If $z \in D \cap G(\alpha) \setminus \{0\}$ then we apply the projection π to (8) with $m = 0$, i.e.

$$0 = \pi(z) = \sum_k \mu^k \pi(a_l^k(\alpha)) .$$

Thus the elements $\pi(a_l^k(\alpha))$ are dependent since not all μ^k are 0 since $z \neq 0$. This implies by (6) that the elements $\pi(a_1^k(\alpha))$ are dependent modulo B, as desired. ■

We close this section by describing the first Ulm subgroup and the zeroth Ulm factor of $G(\alpha)$.

LEMMA 9. *The first Ulm subgroup of $G(\alpha)$ is $p^\omega G(\alpha) = D \cap G(\alpha)$ and the zeroth Ulm factor is $G(\alpha)/(D \cap G(\alpha)) \cong \pi(G(\alpha))$. In particular, $G(\alpha)$ is reduced if and only if $D \cap G(\alpha)$ is reduced.*

PROOF. It is clear that elements of infinite height in $G(\alpha)$ have infinite height in $H_{\lambda, d}$, hence they are in $D \cap G(\alpha)$. Conversely, by (8) in Lemma 8 all elements in $D \cap G(\alpha)$ have infinite height.

The divisible part of a group is contained in its first Ulm subgroup.

Thus a group is reduced if and only if its first Ulm subgroup is reduced. ■

5. Small diagonal relation arrays.

We follow the usual convention to define a sequence $(\alpha_i)_{i\in\mathbb{N}}$ in $\mathbb{Z}_p$ to be a *p-adic zero sequence* if there is for all $m\in\mathbb{N}$ some $N\in\mathbb{N}$ such that $\alpha_i\in p^m\mathbb{Z}_p$ for all $i\geq N$. Let I_i be sets for all $i\in\mathbb{N}$ and let $\alpha_i^u\in\mathbb{Z}_p$ for all $i\in\mathbb{N}$, $u\in I_i$. We extend this to say that the sequence $((\alpha_i^u\mid u\in I_i))_{i\in\mathbb{N}}$ of tuples of group elements is called a *p-adic zero sequence* if there is for all $m\in\mathbb{N}$ some $N\in\mathbb{N}$ such that $\alpha_i^u\in p^m\mathbb{Z}_p$ for all $i\geq N$, $u\in I_i$. A relation array $(\alpha_{i,j}^{k,u})$, not necessarily diagonal, is called *small*, cf. [6], if $\left(\left(\sum_{i=0}^{j-1}\alpha_{s+i,j}^{k,u}p^i\right)_{s,j}^{k,u}\right)$ is also a relation array. The last array is in general not a relation array since the row finiteness in j and u is not guaranteed. If the relation array is diagonal, i.e. $\alpha=(\alpha_i^{k,u})$, then smallness amounts to the simple fact that the sequence $((\alpha_i^{k,u}\mid u\in I_i))_{i\in\mathbb{N}}$ is a p-adic zero sequence for all $k\in I$. We call a diagonal relation array *almost equal to* 0 if for each $k\in I$, $\alpha_i^{k,u}=0$ for almost all pairs (i,u). In particular, if a diagonal relation array is almost equal to 0, then it is small.

We note that the prototype of a small relation array is given by $\alpha = (\alpha_i)$ of the form

$$\alpha=\begin{bmatrix}1 & & & & \\ & p & & & \\ & & p^2 & & \\ & & & p^3 & \\ & & & & \ddots\end{bmatrix}. \tag{9}$$

If a p-group G is presented by this diagonal relation array α then $\lambda = (1,1,\dots)$, $d=1$, and $G(\alpha)$ is given by

$$a_i(\alpha)=(c_i,0,\dots,0,\underbrace{x_i}_{i^{th}},p^2x_{i+1},\dots,\underbrace{p^{2l}x_{i+l}}_{(i+l)^{th}},\dots,\underbrace{0}_{(2i)^{th}},0,\dots)\in D\oplus B,$$

where $p^{2l}x_{i+l}=0$ for $l\geq i$.

LEMMA 10. *Let α and β be diagonal relation arrays. If the difference $\alpha-\beta$ is small then $G(\alpha)=G(\beta)$.*

PROOF. By Proposition 6, $B \subset G(\alpha)$, $G(\beta)$. If $\alpha - \beta$ is small then

$$a_i^k(\alpha) - a_i^k(\beta) = (a_{i,j}^k(\alpha) - a_{i,j}^k(\beta) \mid j \in \mathbb{N}) =$$

$$= \left(p^{j-i} \sum_{u \in I_j} (\alpha_j^{k,u} - \beta_j^{k,u})\, x_j^u \mid j \in \mathbb{N}\right) \in B$$

and $a_{i,0}^k(\alpha) = a_{i,0}^k(\beta)$, for all $i \in \mathbb{N}$, $k \in I$. Thus

$$G(\alpha) = \langle x_j^u,\ a_i^k(\alpha) \mid i, j \in \mathbb{N},\ u \in I_j,\ k \in I \rangle =$$

$$= \langle x_j^u,\ a_i^k(\beta) \mid i, j \in \mathbb{N},\ u \in I_j,\ k \in I \rangle = G(\beta). \qquad \blacksquare$$

PROPOSITION 11. *A basic subgroup B of a p-group G with corresponding diagonal relation array α is a direct summand of G if and only if α is small.*

In particular, $G \cong G_{\lambda, d}$ if the isomorphism type of B is λ, G/B is of rank d and α is small.

PROOF. Since a group G is isomorphic to its presentation $G(\alpha)$ we may consider $G(\alpha)$ instead. If α is small then the generating elements $a_i^k(\alpha)$ of $G(\alpha)$ are in $D \oplus B$, hence B is a direct summand of $G(\alpha)$. Otherwise, if α is not small, then these elements $a_i^k(\alpha)$ are not in $D \oplus B$, i.e. B is not a direct summand. $\blacksquare$

6. The module Pext.

The module $\mathrm{Pext}(\mathbb{Z}(p^\infty), B_\lambda)$ is the first Ulm subgroup of $\mathrm{Ext}(\mathbb{Z}(p^\infty), B_\lambda)$, cf. [3, 53.3], and since in general Ext(C, A) is a bimodule over the endomorphism rings of A and C, cf. [3, Section 52], $\mathrm{Pext}(\mathbb{Z}(p^\infty), B_\lambda)$ is a p-adic module. Let $\widehat{\mathbb{Z}}_p$ denote the ring of p-adic integers. In this section we fix the isomorphism type $\lambda = (\lambda_i \mid i \in \mathbb{N})$ where $\lambda_i = |I_i|$. Let A_ω be the torsion-free p-adic module

$$A_\omega = \prod_{i \in \mathbb{N}} \left(\bigoplus_{u \in I_i} \widehat{\mathbb{Z}}_p y_i^u \right)$$

which is the product of the free p-adic modules $\bigoplus_{u \in I_i} \widehat{\mathbb{Z}}_p y_i^u \cong \bigoplus_{|I_i|} \widehat{\mathbb{Z}}_p$. A diagonal relation array $\alpha = (\alpha_i^u)$ of format $(\lambda, 1)$ can be considered an element of A_ω by identifying

$$\alpha = \left(\sum_{u \in I_i} \alpha_i^u y_i^u \mid i \in \mathbb{N} \right).$$

This is the usual way to consider the set of matrices over some ring R and of the same format as an R-module with respect to addition and multiplication by scalars. Let A_{small} denote the subset of all small diagonal relation arrays.

LEMMA 12. *A_{small} is a pure submodule of A_ω such that the quotient module*

$$\mathfrak{A}_\omega = A_\omega / A_{\text{small}}$$

is a reduced torsion-free $\widehat{\mathbb{Z}}_p$-module with $p^\omega \mathfrak{A}_\omega = 0$.

PROOF. Let A_0 be the submodule of A_ω consisting of all diagonal relation arrays $\alpha = (\alpha_i^u)$ of format $(\lambda, 1)$ where almost all α_i^u are 0. The submodules A_0 and A_{small} are obviously pure and A_0 is contained in A_{small}. Moreover, A_{small}/A_0 is a divisible submodule of A_ω/A_0. It is even the maximal divisible submodule since a diagonal relation array $\alpha = (\alpha_i^u)$ of format $(\lambda, 1)$ is small if and only if $((\alpha_i^u \mid u \in I_i))_{i \in \mathbb{N}}$ is a p-adic zero sequence, cf. Section 5; hence $\alpha + A_0$ is an element of the divisible part of A_ω/A_0 if and only if α is small. Thus $\mathfrak{A}_\omega = A_\omega/A_{\text{small}} \cong (A_\omega/A_0)/(A_{\text{small}}/A_0)$ is a reduced torsion-free $\widehat{\mathbb{Z}}_p$-module, i.e. $p^\omega \mathfrak{A}_\omega = 0$.

Next we will show that $\operatorname{Pext}(\mathbb{Z}(p^\infty), B_\lambda) \cong \mathfrak{A}_\omega$. Let

$$0 \to B_\lambda \to G \to \mathbb{Z}(p^\infty) \to 0$$

be pure exact, i.e. a short exact sequence in $\operatorname{Pext}(\mathbb{Z}(p^\infty), B_\lambda)$. We follow the notation in [3, Section 49]. Define $g\colon \mathbb{Z}(p^\infty) = \langle c_i \mid i \in \mathbb{N} \rangle \to G$ such that $g(u)$ is a representative of the coset of u, and in particular $g(c_i) = a_i$, where we may choose the element a_i such that $o(a_i) = p^i$ since the exact sequence was pure, cf. [3, Section 33]. Then $\{x_i^u, a_i \mid i \in \mathbb{N}, u \in I_i\}$ is a quasibasis of G with corresponding relation array α of format $(\lambda, 1)$, which is not necessarily diagonal. We can associate g with α, i.e. $\alpha = \alpha(g)$. Moreover, the function g also determines a factor set $f = f(g)$ by

$$f(u, v) = g(u) + g(v) - g(u+v).$$

Suppose $g'\colon \mathbb{Z}(p^\infty) \to G$ is another function inducing the same factor set, $f(g) = f(g')$ with $g'(c_i) = b_i$, $o(b_i) = p^i$. Then $\{x_i^u, b_i \mid i \in \mathbb{N}, u \in I_i\}$ is another quasibasis with corresponding relation array β. By the definition of the functions g, g' we have $a_i - b_i \in B_\lambda$. This implies that $\alpha - \beta$ is small by [7, Proposition 7]. By Theorem 4 there is a diagonal relation array β such that $\beta - \alpha(g)$ is small and $\{x_i^u, b_i \mid i \in \mathbb{N}, u \in I_i\}$ is a corresponding inductive quasibasis of G. It is straightforward to show that the condition $o(b_i) = p^i$ forces the exact sequence

$$0 \to B_\lambda = \bigoplus_{i \in \mathbb{N},\, u \in I_i} \widehat{\mathbb{Z}}_p x_i^u \to \langle x_i^u, b_i \mid i \in \mathbb{N}, u \in I_i \rangle_{\widehat{\mathbb{Z}}_p} \to \langle b_i + B_\lambda \mid i \in \mathbb{N} \rangle_{\widehat{\mathbb{Z}}_p} \to 0$$

to be pure exact. Let $\mathrm{Fact}_p(\mathbb{Z}(p^\infty), B_\lambda)$ denote the factor sets corresponding to pure exact sequences. Then a mapping is defined

$$\mu\colon \mathrm{Fact}_p(\mathbb{Z}(p^\infty), B_\lambda) \to \mathfrak{A}_\omega, \quad \text{given by } \mu(f) = \alpha + A_{\mathrm{small}}.$$

Let $\mathrm{Trans}_p(\mathbb{Z}(p^\infty), B_\lambda) = \mathrm{Trans}(\mathbb{Z}(p^\infty), B_\lambda) \cap \mathrm{Fact}_p(\mathbb{Z}(p^\infty), B_\lambda)$, cf. [3, Section 49]. Next we show that this mapping μ induces an isomorphism between $\mathrm{Pext}(\mathbb{Z}(p^\infty), B_\lambda)$ and $\mathfrak{A}_\omega$.

THEOREM 13. *The mapping μ given above is an epimorphism with kernel* $\mathrm{Trans}_p(\mathbb{Z}(p^\infty), B_\lambda)$. *Thus*

$$\mathrm{Pext}(\mathbb{Z}(p^\infty), B_\lambda) \cong \mathfrak{A}_\omega.$$

PROOF. Let α be a diagonal relation array of format $(\lambda, 1)$. Then α is realized by a p-group G with corresponding quasibasis $\{x_i^u, a_i \mid i \in \mathbb{N}, u \in I_i\}$, cf. Corollary 5. Define $g\colon \mathbb{Z}(p^\infty) = \langle c_i \mid i \in \mathbb{N} \rangle \to G$ by $g(c_i) = a_i$ and extend the definition of g arbitrarily such that g maps elements to their cosets. We thus obtain that the image of the factor set $f(g)$ under μ is $\alpha + A_{\mathrm{small}}$, i.e.$\mu(f) = \alpha + A_{\mathrm{small}}$. Hence μ is epic.

The kernel of μ is the set $\mathrm{Trans}_p(\mathbb{Z}(p^\infty), B_\lambda)$ of transformation sets, since the transformation sets are the factor sets of the splitting extension. By Proposition 11 these correspond to the diagonal relation arrays which are small. Thus

$$\mu(\mathrm{Trans}_p(\mathbb{Z}(p^\infty), B_\lambda)) = A_{\mathrm{small}}.$$

Next we show that μ is a homomorphism. For this we write down precisely how the factor set f can be determined by α. Let $u, v \in \langle d_i \mid i \in \mathbb{N} \rangle \cong \widehat{\mathbb{Z}}_p$ and presented in the form $u = r(u)\, d_{i(u)}$, where $i(u) \in \mathbb{N} \cup \{0\}$

and $r(u) \in \widehat{\mathbb{Z}}_p \backslash p\widehat{\mathbb{Z}}_p$. Thus

$$f(u, v) = g(u) + g(v) - g(u + v) = r(u)\, a_{i(u)} + r(v)\, a_{i(v)} - r(u + v)\, a_{i(u+v)}.$$

It is straightforward to determine $i(u + v)$ and $r(u + v)$. We know that $f(u, v) \in B_\lambda$, hence the sum representation of f on the right-hand side of the equation above must be of the form $r(p^l a_{l+h} - a_h)$ and by (1) we obtain

$$r(p^l a_{l+h} - a_h) = r\left(\sum_{i=h}^{l+h-1} \sum_{u \in I_i} \alpha_i^u p^{i-h} x_i^u\right),$$

where $r \in \widehat{\mathbb{Z}}_p$. In particular, r and the indices l, h depend only on u, v. Now let $f = f(g)$, $f' = f'(g')$ be two factor sets with $\mu(f) = \alpha + A_{\text{small}}$ and $\mu(f') = \beta + A_{\text{small}}$. The sum of f and f' is given by

$$f(u, v) + f'(u, v) = (g(u) + g(v) - g(u+v)) + (g'(u) + g'(v) - g'(u+v)).$$

Writing this in terms of α and β and observing that the parameters r and the indices h, l depend only on u, v, we get

$$f(u, v) + f'(u, v) = r\left(\sum_{i=h}^{l+h-1} \sum_{u \in I_i} \alpha_i^u p^{i-h} x_i^u + \sum_{i=h}^{l+h-1} \sum_{u \in I_i} \beta_i^u p^{i-h} x_i^u\right)$$
$$= r\left(\sum_{i=h}^{l+h-1} \sum_{u \in I_i} (\alpha_i^u + \beta_i^u)\, p^{i-h} x_i^u\right).$$

This shows that the mapping μ is additive. The proof of $\mu(rf) = r\mu(f)$ for $r \in \widehat{\mathbb{Z}}_p$ is analogous. Hence μ is a homomorphism and induces an isomorphism between $\text{Pext}(\mathbb{Z}(p^\infty), B_\lambda)$ and $\mathfrak{A}_\omega$.

7. Independent diagonal relation arrays.

A diagonal relation array $\alpha = (\alpha_i^{k,u})$ is called *independent* if the tuple $(\overline{\alpha}^k + p\mathfrak{A}_\omega \mid k \in I)$ forms an independent set of $\mathfrak{A}_\omega / p\mathfrak{A}_\omega$.

Recall that in Lemma 8 we proved that under certain conditions cosets of the projection map $\pi\colon H_{\lambda,d} \to \overline{B}_\lambda$ are dependent. In the next lemma we discuss when these cosets are independent.

LEMMA 14. *Let G be a p-group presented by the diagonal relation array α and the group $G(\alpha)$, respectively. Then α is independent if and* *only if*

$$\{\pi(a_1^k(\alpha)) + B \,|\, k \in I\}$$

is independent.

PROOF. The general element of $(\overline{B}_\lambda/B)[p]$ is

$$\left(\sum_{u \in I_i} \mu_i^u p^{i-1} x_i^u \,|\, i \in \mathbb{N}\right) + B\,,$$

where $\mu_i^u \in \mathbb{Z}_p$. Define the mapping $\varphi\colon (\overline{B}_\lambda/B)[p] \to \mathfrak{A}_\omega/p\mathfrak{A}_\omega$ by

$$\varphi\left(\left(\sum_{u \in I_i} \mu_i^u p^{i-1} x_i^u \,|\, i \in \mathbb{N}\right) + B\right) = \left(\sum_{u \in I_i} \mu_i^u y_i^u \,|\, i \in \mathbb{N}\right) + pA_\omega .$$

The elementary abelian p-groups $(\overline{B}_\lambda/B)[p]$ and $\mathfrak{A}_\omega/p\mathfrak{A}_\omega$ are vector spaces over $\mathbb{Z}/p\mathbb{Z}$. It is easy to verify that φ is a vector space isomorphism. Since

$$\pi(a_1^k(\alpha)) = \left(\sum_{u \in I_i} a_i^{k,u} p^{i-1} x_i^u \,|\, i \in \mathbb{N}\right) \in \overline{B}_\lambda/B\,,$$

cf. Section 4, we obtain that

$$\varphi(\pi(a_1^k(\alpha)) + B) = \left(\sum_{u \in I_i} a_i^{k,u} y_i^u \,|\, i \in \mathbb{N}\right) + pA_\omega .$$

Since φ is a vector space isomorphism the lemma is shown. ■

PROPOSITION 15. *For a p-group G presented by a diagonal relation array α the following are equivalent:*

(i) *α is independent;*

(ii) $p^\omega G = 0$;

(iii) *G is reduced and separable;*

(iv) *G is embeddable in $\overline{B}$, where B is a basic subgroup of G.*

PROOF. (i) ⇒ (ii): Let $G(\alpha)$ be the presentation of the diagonal relation array α. By Lemma 9 we have that $p^\omega G(\alpha) = D \cap G(\alpha)$. Hence, if α is independent, Lemma 14 and (8) in Lemma 8 imply that $D \cap G(\alpha) = 0$.

(ii) ⇒ (i): if α is dependent then there are finitely many $\mu^k \in \mathbb{Z}_p$, not all 0, such that $b = \sum_k \mu^k \pi(a_1^k(\alpha)) \in B$. Then $\sum_k \mu^k a_1^k(\alpha) - b \in D$. But this is an element in $D \cap G(\alpha)$, which is not 0, since not all μ^k are 0. However by Lemma 9 this implies that the first Ulm subgroup of $G(\alpha)$, hence of G, is not 0 and we are done.

(ii) $\Leftrightarrow$ (iii): This is shown by [3, 65.1].

(iii) $\Rightarrow$(iv): This is shown by a result of Kulikov [3, 68.2].

(iv) $\Rightarrow$ (ii): This is trivial. ■

In view of Proposition 15 the following Corollary is a reformulation of [3, 68.3].

COROLLARY 16. *A p-group G with independent diagonal relation array of format* (λ, d) *is a direct sum of cyclics if* $d \leq \aleph_0$. *Consequently* $G \cong B_\lambda$.

Moreover, if in addition $|B_\lambda| < d$ *then G is not a direct sum of cyclics.*

PROOF. Since G is presented by α, the format of α tells us that there is basic subgroup $B = B_\lambda$ with quotient G/B of rank d. Since the relation array is independent, G is separable by Proposition 15. Thus if $d \leq \aleph_0$, i.e. G/B is countable, then by a result of Prüfer [3, 68.3] the group G is a direct sum of cyclics hence $G \cong B_\lambda$.

However, if $|B_\lambda| < d$ and if we assume G to be a direct sum of cyclics then $G \cong B_\lambda$ and $|G| = |B_\lambda| < d \leq |G|$ yields a contradiction. ■

Corollary 16 shows in particular that each relation array $\alpha = \operatorname{diag}(\alpha_1, \alpha_2, \ldots)$ with a proper 0-1-sequence $(\alpha_i)_{i \in \mathrm{N}}$, i.e. with infinitely many $\alpha_i = 1$, is a relation array of the standard-B group. This is what we meant by the remark following Example 2. Moreover, with the notation in Section 6, a diagonal relation array of format $(\lambda, 1)$ of ω-type is the presentation of a direct sum of cyclics.

COROLLARY 17. *A p-group G with a diagonal relation array of format* (λ, d) *is not reduced if* $d > 2^{|B_\lambda|}$.

PROOF. There are only $2^{|B_\lambda|}$ different diagonal relation arrays of format $(\lambda, 1)$. Let α be a diagonal relation array of format (λ, d). Then by the pigeon hole principle there are $k, l \in I$ such that $(\alpha_i^k) = (\alpha_i^l)$ for all i. Thus using a presentation $G(\alpha)$ of G, there correspond generating elements $a_i^k(\alpha), a_i^l(\alpha)$ such that $a_i^k(\alpha) - a_i^l(\alpha) = c_i^k - c_i^l$ for all i. Hence the subgroup of $G(\alpha)$ generated by $\{a_i^k(\alpha) - a_i^l(\alpha) \mid i \in \mathbb{N}\}$ is quasicyclic. ■

REMARK. We want to point out what a diagonal relation array may look like when it corresponds to the p-group $\overline{B}_\lambda$. Let α be a diagonal relation array of format (λ, d) where $d = 2^{|B_\lambda|}$. Then $(\overline{B}_\lambda / B_\lambda)[p]$ is a $\mathbb{Z}/p\mathbb{Z}$-vector space of uncountable dimension. For a diagonal relation array α to present $\overline{B}_\lambda = G(\alpha)$ it is necessary for

$$M = \left\{\pi(a_1^k(\alpha)) + B \in \overline{B}_\lambda / B \mid k \in I\right\}$$

to generate the socle $(\overline{B}_\lambda / B)[p]$ of $\overline{B}_\lambda / B$. Since $\overline{B}_\lambda$ is reduced and separable it is also necessary that M is independent, i.e. M is a basis of the socle. It then follows that $G(\alpha) = \overline{B}_\lambda$. As is known in such cases there is no explicit description of a basis for the socle.

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