Rendiconti *del* Seminario Matematico *della* Università di Padova

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Rendiconti del Seminario Matematico della Università di Padova, tome 102 (1999), p. 67-75

<http://www.numdam.org/item?id=RSMUP_1999__102__67_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 102 (1999)

Rigid Meromorphic Foliations on Complex Surfaces.

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Introduction.

We are interested in the problem of existence and density of foliations without algebraic leaves. Here we give a construction (see Theorem 1.1) of singular meromorphic foliations without algebraic leaves on every smooth projective surface. In sections 2 and 3 we consider the related problem of «rigidity» or «persistency» of a singular meromorphic foliation on a compact complex surface X. We study the case of a foliation coming from a fibration, i.e. from a morphism $X \to B$ with B smooth curve. In section 2 we study the case of a surface with Kodaira dimension $-\infty$, $X \neq \boldsymbol{P}^2$ and give (see Theorem 2.1) another proof of the theorem proved in [15]. In section 3 we consider the case of an elliptic fibration.

The author want to thank the referees for the remarks to the previous versions of this paper. The author was partially supported by MURST and GNSAGA of CNR (Italy).

1. Foliations without algebraic leaves.

Recall that a meromorphic foliation by curves on a smooth complex manifold M is given by a non zero morphism $\boldsymbol{i}: L \to TM$ with L line bundle on M. Of course, if $\dim(M) = 2$ this is a codimension 1 meromorphic foliation with singularities on M. We will call «foliation» any meromorphic foliation with singularities. The singular set $\operatorname{Sing}(F)_{\text{red}}$ (or just $\operatorname{Sing}(F)$) of the foliation F is the set of points of M where $\boldsymbol{i}$ drops rank. The foliation is called *saturated* if $\boldsymbol{i}$ drops rank at most in codimension 2.

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If M is a surface and F is saturated we have an exact sequence

(1) $$0 \to L \to TM \to I_Z \otimes K_M^{-1} \otimes L^{-1} \to 0$$

with $\dim(Z) = 0$, $Z := \mathrm{Sing}(F)$ with its scheme structure (see e.g. [8]); if $\dim(M) > 2$ or one is interested in foliations on singular surfaces, the best background material on saturated subsheaves is probably contained in the first section of [12].

We may move the foliation either varying L or fixing L and choosing a nearby non proportional section of $H^0(M, TM \otimes L^{-1})$. Note that in every small deformation of the foliation F the algebraic, numerical and topological equivalence class of the line bundle L remain constant. Hence if $F \subset M$ is a curve, we have $\deg(L_t|F) = \deg(L|F)$ for all t.

For the theory of deformations of singular foliations, see [7] or [6] or [16] or [17] or [2]. For the particular case of deformations of foliations by curves, see [8] and [9]. Hence we will say that a singular meromorphic foliation is *rigid* if every flat deformation of it parametrized by a reduced space is trivial.

Let M be a complex projective surface. In this section we give a construction of families of singular meromorphic foliations on M with large dimension in which the set of foliations without algebraic leaves is dense. We will prove the following result.

THEOREM 1.1. *Let M be a smooth complex projective surface. Fix a very ample line bundle R on M. Set $x := h^0(M, R)$. For every integer $r \geq 4$ the moduli space of saturated singular meromorphic foliations associated to a non zero* map $R^{\otimes(-r)} \to TM$ *contains a Zariski open subset of a projective space of dimension* $\max\{3(x-3), r^2+6r+8\}$ *in which the set of foliations without any algebraic leaf is dense in the euclidean topology and a Zariski open non-empty subset of a projective space of dimension* $3(x-3)$ *formed by foliations without any algebraic leaf.*

PROOF. Recall that the Grassmannian $G(3, x)$ of 3-dimensional subspaces of $\mathbf{C}^x$ has dimension $3(x-3)$. Consider M embedded by R in the projective space $|R| \cong \mathbf{P}^{x-1}$ and let $p\colon M \to \mathbf{P}^2$ be a general projection. Hence $R \cong p^*(O_{\mathbf{P}}(1))$. Call $A \subset M$ (resp. $D \subset \mathbf{P}^2$) the ramification locus (resp. the discriminant divisor). Fix a singular meromorphic saturated foliation by curves G on $\mathbf{P}^2$ of degree $r \geq 4$. By a form of the Bertini theorem (see e.g. [14]) there is $g \in \mathrm{Aut}(\mathbf{P}^2)$ such that $g(D)$ is transversal to

G outside finitely many points. Taking $g^*(G)$ instead of G we may assume that the discriminant D is transversal to G outside finitely many points. Let ω be the meromorphic 1-form inducing G and let E be the foliation induced by $p^*(\omega)$. In general E may be non saturated. Let F be the saturation of E. Note that for every algebraic leaf T of F on $M \backslash A$, the closure of $p(T)$ is an algebraic leaf of G. Hence every algebraic leaf of F is either contained in the counterimage of an algebraic leaf of G or it is contained in A. Since the discriminant divisor D is transversal to G outside finitely many points, there is no algebraic leaf of F contained in A and $E = F$ is saturated. By a theorem of Jouanolou ([13], Ch. 4, Th. 1.1) every euclidean neighborhood of G contains foliations without algebraic leaves. Fixing the projection p, we obtain a Zariski open subset U of a projective space of dimension $r^2 + 6r + 8$ in which the set of foliations without any algebraic leaf is dense in the euclidean topology. Viceversa, fixing any such foliation G of degree r on $\boldsymbol{P}^2$ and varying the projections we find a Zariski open dense subset of $G(3, x)$ parametryzing one to one foliations without algebraic leaves. Indeed, to check that the parametrization is one to one it is sufficient to look at the singularities of the pull-backs of G arising as counterimages of the singularities of G. By a theorem of Gomez-Mont and Kempf ([10]) every degree r non-degenerate (i.e. such that all its singularities have multiplicity one) foliation on $\boldsymbol{P}^2$ is uniquely determined by the set of its singularities. By [13], part 2) of Th. 2.3 at p. 87, a Zariski open non empty subset of U parametrizes non-degenerate foliations. ■

Usually under the assumptions of Theorem 1.1 the integer x is small with respect to r and hence $3(x-3)$ is much smaller than $r^2 + 6r + 8$. A family of exceptional cases is given taking $R \cong M^{\otimes m}$ with $M \in \mathrm{Pic}(X)$, M very ample, and m very large. This family of examples is interesting only for the last assertion of Theorem 1.1, because usually $h^0(X, M)$ is much smaller than mr.

2. Rigid and ruled fibrations.

In this section we will study the meromorphic foliations with singularities on a smooth projective surface X with Kodaira dimension $-\infty$, $X \neq \boldsymbol{P}^2$. This is the class of all surfaces with a morphism $u: X \to B$, B smooth curve with general fiber isomorphic to $\boldsymbol{P}^1$ (a ruling of X). We will say that such a surface is ruled; we will say that X is geometrically ruled if all the fibers of u are smooth (hence isomorphic to $\boldsymbol{P}^1$). Some authors call birationally ruled our general set up and call ruled surfaces only the geometrically ruled surfaces. Any such X is obtained from a geometrically ruled surface, Y, with a finite number of blowing ups; the surface Y and the morphism $\pi: X \to Y$ is uniquely determined if B has genus $g > 0$. The case of a geometrically ruled surface was considered in [8]. As a consequence of our analysis we will prove Theorem 2.1 below, i.e. we will give another proof of the theorem proved in [14]. The local analysis of what happens to a holomorphic foliation making a blowing up (strict transform of the foliation) was made in [9], § 6. We will fix the following notations. Let g be the genus of B and $t \geqslant 0$ the number of blowing ups whose composition gives π. We will identify divisors and line bundles and often use the additive notation for both. Let $v: Y \to B$ be the ruling of Y. As a base for the Neron Severi group $NS(Y)$ of divisors of Y (i.e. divisors modulo numerical equivalence) we will give the classes h and f with $h^2 = 0$, $h \cdot f = 1$, $f^2 = 0$, f class of a fiber of v, h class of a section, up to multiples of f (i.e. there may not be any effective curve with numerical class h and, even if there is one, it may consist of an irreducible section plus a few fibers). We will denote by $-e$ the minimal self-intersection of a section of v; by a theorem of Nagata we have $e \geqslant -g$. Call H (resp. F) the total transform of h (resp. f) on X; hence $H^2 = 0$, $H \cdot F = 1$ and $F^2 = = 0$; F will denote also a general fiber of u (hence a general fiber of v). As a base of the Neron Severi group $NS(X)$ of X we will take H, F and the following divisors E_i, $1 \leqslant i \leqslant t$, with $H \cdot E_i = F \cdot E_i = 0$, $K_X \cdot E_i = E_i^2 = -1$ for all t. Decompose π into t blowing ups and call $\pi(i)$: $X(i) \to Y$, $0 \leqslant i \leqslant t$, the composition of the first i of these blowing ups; assume to have defined E_j, $j \leqslant i$, for some $i < t$ as a class on $X(i)$; as classes E_a on $X(i+1)$ takje the total transform of the classes E_a on $X(i)$ for $a \leqslant i$ and the class of the exceptional divisor of the blowing up $X(i+1) \to (X(i)$ as class of E_{i+1}. Note that every E_j is effective, but may be reducible.

The case of singular foliations on Y was studied in detail in [8]. Call G the foliation of fibration type induced by the ruling of X and let L'' be the associated saturated line subsheaf of TX. Note that the ruling (and hence the foliation) is unique except in the cases $g = 0$, $e = 0$, $t = 0$ or 1, in which there are exactly two rulings. For every smooth fiber $F \cong \boldsymbol{P}^1$ we have $(\deg(L''|F) = 2$. We claim that the numerical equivalence class of L'' is $2H - \sum_{1 \leqslant i \leqslant t} E_i$. To check the claim, use [8], Lemma 1.4, for the case $y = X$, the local analysis of the behaviour of tangent bundles on surfaces by blowing ups made in [9], § 4, and the fact that $c_1(TX) - L''$ is numerically equivalent to $(2-2g)\,F$. Consider a small deformation $\{G_t\}_{t \in \Delta}$, Δ the unit disc of $\boldsymbol{C}$, of G with X fixed. Note that in any small deformation of a foliation by curves the numerical equivalence class of the saturated line subsheaf of TX remains constant. Let L_t'' be the line bundle corresponding to L'' for the foliation G_t. Since $c_1(TX) - L''$ is numerically equivalent to the pull-back of a line bundle on the curve B (the base of the ruling) for a general fiber, F, of the ruling u we have $\deg(L_t'' \mid F) = 0$. Since $F \cong \boldsymbol{P}^1$, $L_t'' \mid F$ is trivial. Hence F is a leaf of G_t. Thus G is persistent, giving another proof of the following theorem proved in [14].

THEOREM 2.1. *On every smooth projective surface with Kodaira dimension* $-\infty$ *except the projective plane there is a singular meromorphic foliation (the foliation induced by a ruling) which is rigid.*

There is an inclusion between (saturated) singular foliations by curves in Y and X ([9], § 6); with the terminology of [9], § 6, the foliation on X corresponding to a foliation A on Y is called the strict transform of A. As in [9], Def. 2.5, we will give the following definition of foliation on X of Riccati type.

DEFINITION 2.2. A saturated foliation on X induced by an inclusion $L \to TX$ is called a *Riccati foliation* if there is $M \in \operatorname{Pic}(B)$ with $c_1(L) = c_1(u^*(M))$.

Fix a Riccati foliation F on X. Since we have $H^2(X, \boldsymbol{O}_X) = 0$, the exponential sequence

$$0 \to \boldsymbol{Z} \to \boldsymbol{O}_X \to \boldsymbol{O}_X^* \to 0$$

shows that numerical equivalence, algebraic equivalence and topological equivalence of line bundles on X coincide. Furthermore, $\operatorname{Pic}^0(X) \cong u^*(\operatorname{Pic}^0(B))$. Hence in the definition 2.2 of Riccati foliation we may assume $L \cong u^*(M)$. Since $u = v \circ \pi$, there is $L' \in \operatorname{Pic}(Y)$ with $\pi^*(L') \cong L$. Let $U \subset Y$ be the Zariski open subset of Y with card $(Y \setminus U)$ finite and such that $\pi | \pi^{-1}(U)$: $\pi^{-1}(U) \to U$ is an isomorphism. The restriction to $\pi^{-1}(U)$ of the Riccati foliation F induced a singular holomorphic foliation G on U. However, a priori this singular holomorphic foliation does not extend to a singular meromorphic foliation on all X. Since it is sufficient to check the integrability condition for a foliation defined by a meromorphic 1-form on a Zariski open dense subset, to obtain the extension of G to Y (as singular foliation) it is sufficient to show the existence of $L'' \in \mathrm{Pic}(Y)$ and of a map $\boldsymbol{i}$: $L'' \to TY$ such that $\boldsymbol{i}\,|\,U$ induces G. Since $\mathrm{codim}\,(Y \backslash U) = 2$, for every $M \in \mathrm{Pic}(Y)$ we have $h^0(Y, TY \otimes M) = h^0(U, (TY \otimes M)\,|\,U)$. Hence it is sufficient to find $L'' \in \mathrm{Pic}(Y)$ and $\boldsymbol{r}$: $L''\,|\,U \to TU$ inducing G. We claim that we may take L' as such line bundle L''. Indeed by the definition of U and L' the morphism $\pi\,|\,\pi^{-1}(U)$ induces an isomorphism of $\pi^{-1}(U)$ onto U and $\pi^*(L''\,|\,U) = L$, $\pi^*(TU) = T\pi^{-1}(U)$. Since $u = v \circ \pi$ we have $L' \in v^*(\mathrm{Pic}(B))$. Thus every Riccati foliation is the strict transform of a Riccati foliation on the geometrically ruled surface Y. Such foliations on Y are studied in [8], § 2.

3. Rigid and elliptic fibrations.

In this section we consider the meromorphic singular foliation induced by an elliptic fibration π: $X \to B$ with B smooth curve of genus $g \geqslant 0$. Hence the general fiber of π is a smooth elliptic curve. The main difference with respect to the case of a ruling considered in section 2 is that now the general fiber of the fibration is not simply connected. Let $T_{X/B}$ be the relative tangent sheaf of π (see e.g. [18], pages 408-409). We assume that the following exact sequence

$$0 \to L \to TX \to N \otimes \boldsymbol{I}_Z \to 0 \tag{2}$$

with $L \cong T_{X/B}$ and $N^{-1} \cong K_X \otimes L$ defines the foliation F induced by π.

Note that the fibration π of X is rigid as fibration and that the irreducible component of the Hilbert scheme $\mathrm{Hilb}\,(X)$ of X containing a smooth fiber F of the fibration is given by the fibers of the fibration π. Hence the foliation induced by π is rigid if the following two conditions are satisfied:

($a1$) Every nearby foliation is induced by the same line bundle L.

($a2$) We have $h^0(X, TX \otimes L^{-1}) = 1$.

REMARK 3.1. Assume that the elliptic fibration is relatively minimal. Then $L = T_{X/B} \cong \pi^*(A) \otimes \boldsymbol{O}(\Sigma(1 - m_i)\, F_i)$ where the sum Σ is over all multiple fibers F_i's, F_i has multiplicity m_i and $A \in \mathrm{Pic}\,(B)$, $\deg\,(A) = -\chi(\boldsymbol{O}_X)$, $N = \pi^*(A')$ with $\deg\,(A') = 2(1 - g(B))$ ([3], p. 162).

Now consider the restriction of (2) to a smooth fiber F of the fibration π. By the adjunction formula we obtain the following exact sequence

$$0 \to \boldsymbol{O}_F \to TX\,|\,F \to \boldsymbol{O}_F \to 0\,. \tag{3}$$

Two cases are possible: either (3) splits or not. We will call the fibration of *indecomposable type* if the exact sequence (3) does not split. Assume that (3) does not split. By Atiyah's classification of vector bundles on an elliptic curve ([1]), this is equivalent to the fact that $TX|F$ is isomorphic to tyhe unique indecomposable rank 2 vector bundle of F with trivial determinant. This implies $h^0(F,(TX|F)\otimes D)=0$ for every $D\in \mathrm{Pic}^0(F)$ with $D\neq \boldsymbol{O}_F$ and $h^0(F, TX|F)=1$. Thus for every nearby foliation F_t induced, say, by $s_t\colon L_t\to TX$, $s_t|F=s|F$ is uniquely determined, i.e. the tangent direction of the foliation F_t at any point of F is the same as the one for F, i.e. F is a leaf of F_t. Thus $F_t=F$ and F is rigid. Hence we have proved the following result.

PROPOSITION 3.2. *The foliation induced by an elliptic fibration of indecomposable type is rigid.*

Here is another case in which F is rigid.

PROPOSITION 3.3. *Assume that the elliptic fibration* π *is relatively minimal. Assume* $\chi(\boldsymbol{O}_X)<2g-2$, *i.e.* $c_2<12(2g-2)$. *Then the foliation* F *induced by* π *is rigid.*

PROOF. Let $\{L_t\}_{t\in\Delta}$, Δ the unit disk of $\boldsymbol{C}$, be the family of saturated rank 1 subsheaves of TX associated to a small deformation of the foliation F. Hence L_t is numerically equivalent to L. We have $h^0(X, \mathrm{Hom}(L_t, N))=0$ by the numerical assumptions on A and A' with $L=T_{X/B}\cong\pi^*(A)\otimes\boldsymbol{O}(\Sigma(1-m_i)F_i)$ and $N:=\pi^*(A')$. Thus $L_t\cong L$ for all t. Since $h^0(X, \mathrm{Hom}(L, N))=0$ by the numerical assumptions, we have $h^0(X, TX\otimes L^{-1})=1$. Thus the foliation F is rigid. ∎

Here we will consider the case of hyperelliptic surfaces (also called bielliptic surfaces). For the classification of these surfaces, see [5], p. 36-37, or [3], p. 148 and 189, or [4], pp. 113-114, or [11], pp. 585-590.

THEOREM 3.4. *Let* X *be a surface birational to a hyperelliptic surface. Let* F *be the foliation induced by the elliptic fibration* $\pi\colon X\to B$ *given by the Albanese map and* G *the foliation induced by the unique elliptic pencil* $m\colon X\to\boldsymbol{P}^1$. *Then* F *is rigid. If* X *is minimal, then* G *is rigid.*

PROOF. First assume X minimal. By [3], p. 148 and 168, the fibration π has $g(B)=1$, $\chi(\boldsymbol{P}_X)=0$, $h^0(X, \Omega^1_X)=1$ and $Z=\emptyset$. By [11], p. 585, π is smooth. Hence, taking $A, A' \in \mathrm{Pic}(B)$ with $L \cong \pi^*(A')$ and $N \cong \pi^*(A)$, we have $\deg(A) = \deg(A') = 0$. Since $c_2(X) = 0$ we see that also the fibration m induces an exact sequence (2) with $Z = \emptyset$; the only difference is that now L is not of theform $m^*(A')$, because there is the contribution of the multiple fibers (which are the only singular fibers of the fibration m). Let $\{L_t\}_{t \in \Delta}$ be the family of saturated rank 1 subsheaves of TX associated to a small deformation of the foliation F (or the foliation G). By (2) we have $h^0(X, \mathrm{Hom}(L_t, N)) = 0$ if L_t and N are not isomorphic. Since $\deg(L_t | F) = \deg(N|F) = 0$ for every fiber F of π and for a general fiber F of m, we obtain that L is constant in such small deformation of F (or G). We have $h^0(X, TX \otimes L^{-1}) = 1$ unless $L \cong N$ and $TX \cong L \oplus L$. Thus in order to obtain a contradiction we may assume $TX \cong L \oplus L$. Thus Ω^1_X is the direct sum of two isomorphic line bundles. Hence $h^0(X, \Omega^1_X)$ is even, contradiction. Now we drop the assumption of minimality of X. We use the notations of section 2 for the exceptional divisors. We use that on the minimal model the fibration π is smooth. As in the case of the Riccati foliations on ruled surfaces considered at the end of section 2, now $L = \pi^*(A) \otimes (-\Sigma E_i)$, $N = \pi^*(A')$ with $\deg(A) = \deg(A') = 0$. As in the case of the Riccati foliations we see that every small deformation of F comes from a small deformation of the foliation of fibration type on the minimal model of X. Hence we conclude. ∎

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Manoscritto pervenuto in redazione il 29 luglio 1997.