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Existence of Semi-Periodic Solutions of Steady Navier-Stokes Equations in a Half Space with an Exponential Decay at Infinity.

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ABSTRACT - We prove the existence of a solution of steady Navier-Stokes equations in a half-space which is periodic in the directions parallel to the boundary and which decays exponentially fast with respect to the distance to the boundary. In fact, the decay holds as soon as the energy over a period is finite.

1. Introduction.

Our objective is to prove the existence of a solution (u, p) of the steady Navier-Stokes equations

$$-\nu\Delta u + u\cdot\nabla u + \nabla p = 0\,, \qquad \nabla\cdot u = 0\,, \qquad u|_{x_3=0} = \gamma \tag{1}$$

in the half space $\mathbb{R}^3_+$ which is periodic with respect to x_1 and x_2, and which decay exponentially fast as well as all its derivatives as $x_3 \to \infty$.

This study has been motivated by the analysis of the asymptotic behaviour of hydrodynamic drag of a plate covered by periodic asperities, when their size goes to 0. We refer to [3], [4] in the case of Stokes flow.

In a first part, we will prove the existence of a solution whose

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energy is finite by a contradiction method due to J. Leray [10], see also C. Amick [2], W. Borchers and K. Pileckas [5].

In a second part, we will prove the exponential decay of every such solution by a technique similar to the one in K. A. Ames, L. E. Payne [1], G. P. Galdi [6], C. O. Horgan, L. T. Wheeler [8], O. A. Ladyshenskaya, V. A. Solonnikov [9] for Navier-Stokes equations in a semi-infinite channel with Dirichlet condition on the lateral boundary. There is an additional difficulty here, which is that we can no longer use the Poincaré inequality to bound the integral $\int_{\Sigma_t} |u|^2$ over any cross section Σ_t of the channel, since u does no longer cancel on the boundary when the Dirichlet condition is replaced by periodicity. We will overcome this difficulty by establishing (Lemma 5) a convenient estimate of the mean value of u over Σ_t, which allows us to use the Poincaré-Wirtinger inequality instead of the Poincaré one.

2. Functional spaces and notations.

In all the paper long, for vector valued functions $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$, $w = (w_1, w_2, w_3)$, we denote $u \cdot v = \sum_i u_i v_i$, $\nabla u \cdot \nabla v = \sum_{ij} \partial_i u_j \partial_i v_j$ and $u \cdot \nabla v$ and $v \cdot \nabla u \cdot w$ the vector and scalar valued functions respectively defined by

$$(u \cdot \nabla v)_j = \sum_i u_i \partial_i v_j, \qquad u \cdot \nabla v \cdot w = \sum_{ij} u_i \partial_i v_j w_j .$$

We denote $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$, $x' = (x_1, x_2)$, $S = (0, l_1) \times (0, l_2)$ and, for $t \geqslant 0$,

$$\Theta_t = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 > t\},$$

$$\Sigma_t = \{x \in \mathbb{R}^3 : x' \in S,\ x_3 = t\},$$

$\Theta = \Theta_0$, $\Sigma = \Sigma_0$, and $\Lambda_t = \Theta \setminus \overline{\Theta}_t$, that is

$$\Lambda_t = \{x \in \mathbb{R}^3 : x' \in S,\ 0 < x_3 < t\}.$$

For any subset Ω of $\mathbb{R}^3$, we denote $\mathcal{D}(\Omega)$ the space of $\mathcal{C}^\infty$ functions with compact support in Ω and $\mathcal{D}'(\Omega)$ the space of distributions on Ω.

For $1 \leqslant r \leqslant \infty$, we denote $W^{1,r}(\Omega) = \{v \in L^r(\Omega)\colon \nabla v \in (L^r(\Omega))^3\}$,

$$W^{-1,r}(\Omega) = \left\{v \in \mathcal{D}'(\Omega) : v = v_0 + \sum_{i=1}^{3} \partial_i v_i,\ v_j \in L^r(\Omega),\ j = 0, \dots, 3\right\},$$

$H^1(\Omega) = W^{1,2}(\Omega)$ and $H^{-1}(\Omega) = W^{-1,2}(\Omega)$.

We define the space of functions which are periodic with respect to x_1 and x_2 with periods l_1 and l_2 by

$$L^r_{\mathrm{per}}(\Theta) = \{f \in L^r_{\mathrm{loc}}(\mathbb{R}^3_+) : f \in L^r(\Theta),$$
$$f(x_1, x_2, x_3) = f(x_1 + l_1, x_2, x_3) = f(x_1, x_2 + l_2, x_3)\}$$

$$W^{1,r}_{\mathrm{per}}(\Theta) = \{f \in W^{1,r}_{\mathrm{loc}}(\mathbb{R}^3_+) : f \in W^{1,r}(\Theta),$$
$$f(x_1, x_2, x_3) = f(x_1 + l_1, x_2, x_3) = f(x_1, x_2 + l_2, x_3)\}$$

equipped respectively with the norm of $L^r(\Theta)$ and $W^{1,r}(\Theta)$.

We define similar spaces $L^r_{\mathrm{per}}(\Lambda_t)$ and $W^{1,r}_{\mathrm{per}}(\Lambda_t)$ by replacing in the above definitions $\mathbb{R}^3_+$ by $\{x \in \mathbb{R}^3\colon 0 < x_3 < t\}$, and we define $W^{1-1/r,r}_{\mathrm{per}}(\Sigma)$ as the trace spaces of $W^{1,r}_{\mathrm{per}}(\Theta)$.

Finally, we denote

$$V_t = \{v \in (H^1_{\mathrm{per}}(\Lambda_t))^3\colon \nabla \cdot v = 0,\ v|_{\Sigma \cup \Sigma_t} = 0\}.$$

3. Main results.

All this paper long, we assume that the prescribed velocity γ on the boundary Σ is periodic and that its flux across Σ cancels. More precisely

$$\gamma \in (H^{1/2}_{\mathrm{per}}(\Sigma))^3, \qquad \int_\Sigma \gamma_3 \, ds = 0. \tag{2}$$

Our first result gives the existence of a periodic solution in the whole space $\mathbb{R}^3_+$ with a finite enstrophy over the «period» Θ.

THEOREM 1. *There exists a solution of* (1) *satisfying*

$$u \in (H^1_{\mathrm{per,\,loc}}(\Theta))^3, \qquad p \in L^2_{\mathrm{per,\,loc}}(\Theta), \qquad \int_\Theta |\nabla u|^2 < \infty. \quad \blacksquare \tag{3}$$

Our second main result gives the exponential decay of such a solution and of all its derivatives.

THEOREM 2. *Let* (u, p) *satisfy* (1), (2) *and* (3). *Then, there exist* $u_\infty \in \mathbb{R}^3$ *and* $p_\infty \in \mathbb{R}$ *such that*

$$|\partial^\alpha (u(x) - u_\infty)| + |\partial^\alpha (p(x) - p_\infty)| \leqslant c \exp(-\sigma x_3) \tag{4}$$

for all x *such that* $x_3 \geqslant 1$ *and for all* $|\alpha| \geqslant 0$, *where* c *and* σ *are positive numbers depending only on* l, γ, ν, α *and on any arbitrary bound* E *of* $\int_\Theta |\nabla u|^2$. ■

This means that, given E, (4) is satisfied for all (u, p) such that $\int_\Theta |\nabla u|^2 \leqslant E$.

It follows from (4) that u, p and all their derivatives are uniformly continuous in Θ_t for any $t > 0$. The uniform continuity up to the boundary Σ, that is the case $t = 0$, requires, in addition, γ to be smooth.

Moreover, by integration (4) gives, for all $m \geqslant 0$, $1 \leqslant q \leqslant \infty$ and $t > 0$,

$$u - u_\infty \in (W^{m,q}(\Theta_t))^3, \qquad p - p_\infty \in W^{m,q}(\Theta_t),$$

with a norm bounded for all $t \geqslant 1$ by $c' \exp(-\sigma t)$ where $c' = (\sigma q)^{-1/q} c$.

In addition, $u - u_\infty \in (H^1(\Theta))^3 \cap (L^6(\Theta))^3$ and $p - p_\infty \in H^{-1}(\Theta)$ since the enstrophy is finite (for fluids, the energy is $\int |u|^2$ and $\int |\nabla u|^2$ is named the enstrophy).

Finally, let us remark that (u_∞, p_∞) is obviously unique for a given (u, p). It satisfies, *cf.* (32),

$$|u_\infty| \leqslant \left|\int_\Sigma \gamma\right| + c(l, \nu) \int_\Theta |\nabla u|^2.$$

4. Proof of Theorem 1.

We will use the following two lemmas, on the existence of a solenoidal extension of γ and on Euler equation, which are proved at the end of this section.

LEMMA 3. *There exists* $h \in (H^1_{\text{per}}(\Theta))^3$ *such that*

$$h = \gamma \text{ on } \Sigma, \qquad h = 0 \text{ outside } \Lambda_1, \qquad \nabla \cdot h = 0. \quad ■$$

LEMMA 4. (i) *Let* $b \in V_m$ *satisfy, for all* $v \in V_m$,

$$\int_{\Lambda_m} b \cdot \nabla b \cdot v = 0 .$$

Then there exists $q \in W^{1,3/2}_{\text{per}}(\Lambda_m)$ *such that*

$$b \cdot \nabla b + \nabla q = 0 . \tag{5}$$

Moreover q *is constant on* Σ *and on* Σ_m.

(ii) *Let* $b \in (H^1_{\text{per, loc}}(\Theta))^3$ *satisfy* $\nabla \cdot b = 0$, $b|_\Sigma = 0$ *and, for all* m *and for all* $v \in V_m$,

$$\int_{\Lambda_m} b \cdot \nabla b \cdot v = 0 .$$

Then there exists $q \in W^{1,3/2}_{\text{per, loc}}(\Theta)$ *such that*

$$b \cdot \nabla b + \nabla q = 0 .$$

Moreover q *is constant on* Σ. ■

FIRST STEP: WEAK SOLUTION. First of all, let us remark, setting $w = u - h$ where h is defined by Lemma 3, that the equation (1) is equivalent to

$$\begin{gathered} -\nu \Delta w + w \cdot \nabla w + h \cdot \nabla w + w \cdot \nabla h + \nabla p = \nu \Delta h - h \cdot \nabla h , \\ \nabla \cdot w = 0 , \qquad w|_{x_3 = 0} = 0 . \end{gathered} \tag{6}$$

We will find a weak solution w of these equations, which means that it satisfies (6),

$$w \in (H^1_{\text{per, loc}}(\Theta))^3, \qquad \int_\Theta |\nabla w|^2 < \infty , \tag{7}$$

and, denoting $\tilde{}$ the extension by 0 for $x_3 \geq m$: for all $v \in V_m$, for all m,

$$\int_\Theta \nu \nabla w \cdot \nabla \tilde{v} + w \cdot \nabla w \cdot \tilde{v} + h \cdot \nabla w \cdot \tilde{v} + w \cdot \nabla h \cdot \tilde{v} = -\int_\Theta \nu \nabla h \cdot \nabla \tilde{v} + h \cdot \nabla h \cdot \tilde{v} . \tag{8}$$

At first, given an integer number $m \geq 1$, we will prove the existence of an approached solution $w_m \in V_m$ defined on the bounded set Λ_m such that, for all $v \in V_m$,

$$\text{(9)} \qquad \int_{\Lambda_m} \nu \nabla w_m \cdot \nabla v + w_m \cdot \nabla w_m \cdot v + h \cdot \nabla w_m \cdot v + w_m \cdot \nabla h \cdot v = = - \int_{\Lambda_m} \nu \nabla h \cdot \nabla v + h \cdot \nabla h \cdot v \, .$$

Then, we will see that $\int_{\Lambda_m} |\nabla w_m|^2$ remains bounded. Finally, we will get a solution w of (6)-(8) by passing to the limit on the solutions w_m.

Existence of a weak solution w_m of (9). To prove it, we argue as in J. Leray [10] (see also C. Amick [2], W. Borchers and K. Pileckas [5]). The space V_m being equipped with the inner product

$$(u, v) = \int_{\Lambda_m} \nabla u \cdot \nabla v \, ,$$

a compact operator N_m from V_m into itself and a function $F_m \in V_m$ are defined by

$$(N_m w_m, v) = - \int_{\Lambda_m} w_m \cdot \nabla w_m \cdot v + h \cdot \nabla w_m \cdot v + w_m \cdot \nabla h \cdot v \, ,$$

$$(F_m, v) = - \int_{\Lambda_m} \nu \nabla h \cdot \nabla v + h \cdot \nabla h \cdot v \, .$$

Then, by the Riesz representation theorem, the variational equation (9) is equivalent to the following equation in V_m:

$$\text{(10)} \qquad \nu w_m - N_m w_m = F_m .$$

To prove the existence of a solution w_m of (10), we use the Leray-Schauder principle, that is we have to prove that the set of all possible solutions of the equation

$$\text{(11)} \qquad w^\lambda - \lambda N_m w^\lambda = \nu^{-1} F_m, \qquad \lambda \in [0, \nu^{-1}]$$

is bounded in V_m. In order to conclude by contradiction, let us suppose that there exists a sequence $(\lambda_s)_{s \geq 1}$ in $[0, \nu^{-1}]$ converging to $\lambda_0 \in$ $\in [0, \nu^{-1}]$ and $w^s = w^{\lambda_s}$ such that

$$\|w^s\|_{V_m} \to \infty \qquad \text{as } s \to \infty \, .$$

Denoting $b^s = \|w^s\|_{V_m}^{-1} w^s$ we get, since $\|b^s\|_{V_m} = 1$ for all s, the existence of a subsequence, still denoted (b^s), converging weakly in V_m to some b. Let us check that b satisfies the two contradictory equations

$$1 - \lambda_0 \int_{\Lambda_m} b \cdot \nabla b \cdot h = 0 \,, \tag{12}$$

and

$$\int_{\Lambda_m} b \cdot \nabla b \cdot h = 0 \,. \tag{13}$$

The equation (11) for $\lambda = \lambda_s$ gives, for all $v \in V_m$,

$$\int_{\Lambda_m} \nabla w^s \cdot \nabla v + \lambda_s \int_{\Lambda_m} w^s \cdot \nabla w^s \cdot v + h \cdot \nabla w^s \cdot v + w^s \cdot \nabla h \cdot v = \tag{14}$$

$$= -\frac{1}{\nu} \int_{\Lambda_m} \nu \nabla h \cdot \nabla v + h \cdot \nabla h \cdot v \,.$$

Moreover, for all v, w and z in V_m, since $\nabla \cdot w = 0$ and $\nabla \cdot z = 0$,

$$\int_{\Lambda_m} w \cdot \nabla v \cdot z = -\int_{\Lambda_m} w \cdot \nabla z \cdot v \,, \tag{15}$$

and then

$$\int_{\Lambda_m} w \cdot \nabla v \cdot v = 0 \,. \tag{16}$$

Thus, choosing $v = \|w^s\|_{V_m}^{-1} b^s$, we get

$$1 - \lambda_s \int_{\Lambda_m} b^s \cdot \nabla b^s \cdot h = -\frac{1}{\nu \|w^s\|_{V_m}} \int_{\Lambda_m} \nu \nabla h \cdot \nabla b^s + h \cdot \nabla \cdot h \cdot b^s.$$

Therefore, letting $s \to \infty$ and using $b^s \to b$ in $(L^6(\Lambda_m))^3$, we obtain (12).

On the other hand, dividing (14) by $(\|w^s\|_{V_m})^2$ and passing to the limit with respect to s, we obtain

$$\int_{\Lambda_m} b \cdot \nabla b \cdot v = 0 \qquad \forall v \in V_m.$$

By Lemma 4, we deduce that there exists $q \in W^{1,3/2}_{\mathrm{per}}(\Lambda_m)$ such that (b, q)

is solution of the Euler equation (5). Therefore, multiplying (5) by h and integrating over Λ_m, we get

$$\int_{\Lambda_m} b\cdot\nabla b\cdot h = -\int_{\Lambda_m} \nabla q\cdot h = -\int_{\partial\Lambda_m} qh\cdot n\, ds\,. \tag{17}$$

Since q is constant on Σ and on Σ_m and $\int_{\Sigma\cup\Sigma_m} h\cdot n\, ds = 0$, we have

$$\int_{\Sigma\cup\Sigma_m} qh\cdot n\, ds = 0\,.$$

Reminding that for all $q\in W^{1,3/2}_{\text{per}}(\Lambda_m)$ and $h\in(H^1_{\text{per}}(\Lambda_m))^3$, we have

$$\int_{\partial_{lat}\Lambda_m} qh\cdot n\, ds = 0\,,$$

where $\partial_{\text{lat}}\Lambda_m = \partial\Lambda_m\backslash(\Sigma\cup\Sigma_m)$ is the lateral boundary. Thus the right-hand side of (17) vanishes, which proves (13).

Thus, since (12) and (13) are contradictory, the set of all possible solutions w^λ of (11) is bounded in V_m and we obtain, by Leray-Schauder principle, the existence of a solution $w_m\in V_m$ of (9).

Uniform bound for $\int_{\Lambda_m} |\nabla w_m|^2$. We proceed, as previously, by a contradiction argument. Let us suppose that there exists m such that

$$\|w_m\|_{V_m}\to\infty \qquad \text{as } m\to\infty\,.$$

Defining $b^m = \|w_m\|^{-1}_{V_m} w_m$, we have $\|b^m\|_{V_m} = 1$ for all $m\geq 1$. Therefore, denoting $\tilde{b}^m$ the extension of b^m by 0 for $x_3\geq m$, we deduce the existence of a subsequence denoted again $\tilde{b}^m$ such that

$$\nabla\tilde{b}^m\to\nabla b \text{ in } (L^2(\Theta))^9 \text{ weak}\,,$$

$$\tilde{b}^m\to b \text{ in } (L^r_{\text{loc}}(\Theta))^3, \qquad \forall r<6\,.$$

Let us check that b satisfies the two contradictory equations

$$\nu - \int_{\Theta} b\cdot\nabla b\cdot h = 0\,, \tag{18}$$

and

$$\int_{\Theta} b \cdot \nabla b \cdot h = 0 . \tag{19}$$

Since w_m satisfies, for all $v \in V_m$,

$$\int_{\Lambda_m} \nu \nabla w_m \cdot \nabla v + w_m \cdot \nabla w_m \cdot v + h \cdot \nabla w_m \cdot v + w_m \cdot \nabla h \cdot v = \tag{20}$$

$$= - \int_{\Lambda_m} \nu \nabla h \cdot \nabla v + h \cdot \nabla h \cdot v$$

choosing $v = \|w^m\|_{V_m}^{-1} b^m$ we obtain, using (16) and (15),

$$\nu - \int_{\Lambda_m} b^m \cdot \nabla b^m \cdot h = - \frac{1}{\|w^m\|_{V_m}} \int_{\Lambda_m} \nu \nabla h \cdot \nabla b^m + h \cdot \nabla h \cdot b^m ,$$

and therefore,

$$\nu - \int_{\Theta} \tilde{b}^m \cdot \nabla \tilde{b}^m \cdot h = - \frac{1}{\|w^m\|_{V_m}} \int_{\Theta} \nu \nabla h \cdot \nabla \tilde{b}^m + h \cdot \nabla h \cdot \tilde{b}^m .$$

Passing to the limit in this equation, since h has a compact support, we obtain (18).

Now, dividing (20) by $(\|w_m\|_{V_m})^2$, extending it on Θ and passing to the limit with respect to m, we obtain

$$\int_{\Theta} b \cdot \nabla b \cdot v = 0 \qquad \text{for all } v \in V_m, \text{ for all } m \geq 1 .$$

Using Lemma 4 and arguing as previously, we deduce (19). Thus, $\int_{\Lambda_m} |\nabla w_m|^2$ is uniformly bounded with respect to m.

Existence of a solution w of (6)-(8). Let $\tilde{w}_m$ be the extension of w_m by 0 for $x_3 \geq m$. Its gradient remains bounded in $(L^2(\Theta))^9$, then there exist $w \in (H^1_{\text{loc}}(\Theta))^3$ and a subsequence still denoted $(\tilde{w}_m)$ such that

$$\nabla \tilde{w}_m \to \nabla w \text{ in } (L^2(\Theta))^9 \text{ weak} ,$$

$$\tilde{w}_m \to w \text{ in } (L^r_{\text{loc}}(\Theta))^3, \qquad \forall r < 6 ,$$

as $m \to \infty$. The equation $\nabla \cdot \tilde{w}_m = 0$ gives $\nabla \cdot w = 0$, and the periodicity of

$\widetilde{w}_m$ gives the one of w, and thus $w \in (H^1_{\text{per, loc}}(\Theta))^3$. The equation (9) is satisfied for any $v \in V_{m_0}$ provided that $m_0 \leqslant m$, then we get (8) by passing to the limit, m_0 remaining fixed.

SECOND STEP: STRONG SOLUTION. *Existence of* p. Setting $u = w + h$, and choosing $v \in (\mathcal{D}(\Theta))^3$ such that $\nabla \cdot v = 0$, the equation (8) yields

$$\langle -\nu \Delta u + u \cdot \nabla u,\ v \rangle_{(\mathcal{D}'(\Theta))^3 \times (\mathcal{D}(\Theta))^3} = 0\ .$$

Then, by de Rham Theorem, see for instance [12], there exists $p \in \mathcal{D}'(\Theta)$ such that

$$-\nu \Delta u + u \cdot \nabla u + \nabla p = 0\ .$$

Since $u \in (H^1_{\text{per, loc}}(\Theta))^3$, this equation gives $\nabla p \in (H^{-1}(\Lambda_t))^3$ for every finite t, which implies (see for example [13] or [14]) that $p \in L^2(\Lambda_t)$ since Λ_t is a Lipschitz bounded domain.

Extension of p. By definition of H^1_{per}, the function u is defined in the whole set $\mathbb{R}^3_+$, but p is only defined in the set Θ. Its gradient has a periodic extension due to equation (1), but this does not ensure the existence of a periodic extension of p itself. By (8), u satisfies, for all $v \in V_m$ and for all $m \geqslant 1$,

$$\int_\Theta \nu \nabla u \cdot \nabla v + u \cdot \nabla u \cdot v = 0\ . \tag{21}$$

The functions u and v being periodic, $F = \nu \nabla u \cdot \nabla v + (u \cdot \nabla)\, u \cdot v$ is periodic. Its integral over the period Θ being null by (21), so does its integral over any translation of Θ, and in particular over $\Theta' = \{x \in \mathbb{R}^3:\ -l_1/2 < x_1 < l_1/2,\ 0 < x_2 < l_2,\ x_3 > 0\}$.

Choosing now $v \in (\mathcal{D}(\Theta'))^3$ such that $\nabla \cdot v = 0$, the equation $\int_{\Theta'} F = 0$ gives the existence of $p' \in \mathcal{D}'(\Theta')$ such that $-\nu \Delta u + (u \cdot \nabla)\, u + \nabla p' = 0$ in Θ'. In $\Theta \cap \Theta'$, the gradient of $p - p'$ cancels, therefore $p - p'$ is constant. By addition of this constant to p', we obtain an extension of p which satisfies the equation in $\Theta \cup \Theta'$. By repeated extensions, we obtain an extension $p \in L^2_{\text{loc}}(\mathbb{R}^3_+)$ which satisfies equation (1) in the whole set $\mathbb{R}^3_+$.

Periodicity of p. We denote p_{l_1} the translated function of p by l_1 in the direction x_1. Since u is periodic, (21) gives now $\nabla(p_{l_1} - p) = 0$, hence $p_{l_1} - p$ is constant, say c_1. Moreover, the regularity results for Navier-Stokes equation (see [7] or [15] for instance) show that u and p are smooth. Let us multiply the equation (1) by a function v such that

$$v \in (H^1_{\text{per}}(\Theta))^3, \qquad \nabla \cdot v = 0, \quad \text{support } v \subset \{x : r < x_3 < t\},$$

where $0 < r < t < \infty$. Integrating by parts over Θ, we get

$$\int_{\Theta} \nu \nabla u \cdot \nabla v + u \cdot \nabla u \cdot v + \int_{\partial\Theta} \left(- \frac{\partial u}{\partial n} \cdot v + pv \cdot n \right) ds = 0$$

where n is the outer normal. The integral over Θ cancels due to the variational equation (21). In the integral over $\partial\Theta$, the integral over the lower boundary Σ vanishes since v cancels, and the integral of $\partial u/\partial n \cdot v$ over two opposite lateral boundaries cancels since $\partial u/\partial n \cdot v$ is antiperiodic. Finally, the equation reduces to

$$\int_{\partial_{\text{lat}}\Theta} pv \cdot n \, ds = 0$$

where $\partial_{\text{lat}}\Theta = \partial\Theta \backslash \Sigma$. Now, we choose $v = (\psi(x_3), 0, 0)$ where $\psi \in \mathcal{D}(]0, \infty[)$. Then, this reduces to

$$\int_0^{l_2} \int_0^{\infty} (p_{l_1} - p)(0, x_2, x_3)\, \psi(x_3)\, dx_3\, dx_2 = 0 \,.$$

This proves that the constant value of $p_{l_1} - p$ is 0 and therefore that p is periodic with respect to x_1. Similarly, it is periodic with respect to x_2, and the proof of Theorem 1 is complete.

PROOF OF LEMMA 3. Let $\eta \in (H^1_{\text{per}}(\Lambda_1))^3$ be such that $\eta = \gamma$ on Σ, $\eta = 0$ on Σ_1. Since $\int_{\Lambda_1} \nabla \cdot \eta = 0$ and Λ_1 is a Lipschitz domain, there exists (see for instance [6] Theorem 3.1 p. 127) $v \in (H^1_0(\Lambda_1))^3$ such that

$$\nabla \cdot v = -\nabla \cdot \eta \text{ in } \Lambda_1 .$$

Let us denote v_{per} the periodic extension of v over the horizontal directions. Since $\nabla \cdot v_{\text{per}} = (\nabla \cdot v)_{\text{per}}$, the desired properties are satisfied by choosing

$$h = \begin{cases} \eta + v_{\text{per}} & \text{in } \Lambda_1, \\ 0 & \text{in } \Theta \backslash \overline{\Lambda}_1 . \end{cases} \qquad \blacksquare$$

PROOF OF LEMMA 4. *Part* (i). Let $b \in V_m$ satisfy

$$\int_{\Lambda_m} b \cdot \nabla b \cdot v = 0 \qquad \text{for all } v \in V_m.$$

By de Rham Theorem (see for example [12]), there exists $q \in \mathcal{D}'(\Lambda_m)$ such that

$$b \cdot \nabla b + \nabla q = 0\,. \tag{22}$$

This equation gives $\nabla q \in (L^{3/2}(\Lambda_m))^3$ which implies (see for example [13]), since Λ_m is a Lipschitz bounded domain, that $q \in W^{1,3/2}(\Lambda_m)$. Using the same arguments than in the proof of Theorem 1, we deduce the existence of a periodic extension $q \in W^{1,3/2}_{\text{per}}(\Lambda_m)$ which satisfies (22) in the whole set $\{x \in \mathbb{R}^3 \colon 0 < x_3 < m\}$.

Let us now prove that q is constant on Σ and on Σ_m (the constants can be differents). This property has been observed by J. Leray [10]. Here, we adapt the proof given by Ch. J. Amick [2] in the two dimensional case. We have, for all $0 < \varepsilon < m$,

$$\int_{\Lambda_\varepsilon} \frac{|b \cdot \nabla b|}{|x_3|} \leqslant \|\nabla b\|_{(L^2(\Lambda_\varepsilon))^9} \left\| \frac{b}{x_3} \right\|_{(L^2(\Lambda_\varepsilon))^3}.$$

Since $b = 0$ on Σ, we have

$$\int_{\Lambda_\varepsilon} \left| \frac{b}{x_3} \right|^2 \leqslant \int_\Sigma dx' \int_0^\varepsilon \frac{1}{|x_3|^2} \left(\int_0^{x_3} |\partial_3 b(x', \eta)|\, d\eta \right)^2 dx_3$$

then, using Hardy's inequality (see for instance Lemma 5.1 p. 94 in [11]), we get

$$\int_{\Lambda_\varepsilon} \left| \frac{b}{x_3} \right|^2 \leqslant 4 \int_{\Lambda_\varepsilon} |\partial_3 b|^2.$$

Therefore, since $x_3 < \varepsilon$, we obtain

$$\frac{1}{\varepsilon} \int_{\Lambda_\varepsilon} |b \cdot \nabla b| \leqslant 2(\|\nabla b\|_{(L^2(\Lambda_\varepsilon))^3})^2. \tag{23}$$

By the Poincaré-Wirtinger inequality the mean value $\overline{q}(t)$ over Σ_t satisfies, for almost all t,

$$\int_{\Sigma_t} |q(x', t) - \overline{q}(t)|\, dx' \leqslant c \int_{\Sigma_t} |\nabla_{x'} q(x', t)|\, dx'$$

and therefore

$$\frac{1}{\varepsilon} \int_0^{\varepsilon} dt \int_{\Sigma_t} |q - \overline{q}(t)|\, dx' \leqslant \frac{c}{\varepsilon} \int_{\Lambda_\varepsilon} |\nabla q|\, dx .$$

By (22) and (23), the right hand side goes to 0 as $\varepsilon \to 0$ and therefore the trace satisfies $\int_{\Sigma} |q - \overline{q}(0)| = 0$ which proves that q is constant on Σ. Similarly, q is constant on Σ_m since b vanishes on it.

Part (ii). Now, b is defined in the whole set $\mathbb{R}^3_+$ and it vanishes on Σ then, using the same method, we get a periodic function q in $\mathbb{R}^3_+$ which is constant on Σ. ■

5. Proof of Theorem 2.

We will prove (Lemmas 5 and 6) that $\overline{u}(t)$ and therefore $u(t)$ go to a limit $\overline{u}_\infty$. Then we will prove (Lemma 7) an identity which implies (Lemma 8) the exponential decay of $\int_{\Theta_t} |\nabla u|^2$. Finally, the pointwise decay will be deduced from regularity properties for Stokes equations given at Lemma 9.

In all the sequel, (u, p), being a solution of Navier-Stokes equations, is smooth and therefore any of its derivatives is pointwise defined and integrable over every bounded domain or surface. For $t_2 \geqslant t_1$, we denote $\Lambda_{t_1}^{t_2} = \Theta_{t_1} \setminus \overline{\Theta}_{t_2}$ that is

$$\Lambda_{t_1}^{t_2} = \{x \in \mathbb{R}^3 : x' \in S,\ t_1 < x_3 < t_2\} .$$

We will use the mean values over Λ_t^{t+1} and over the cross section Σ_t which are defined by

$$\overset{\ominus}{v}(t) = \frac{1}{l_1 l_2} \int_{\Lambda_t^{t+1}} v(x)\, dx , \qquad \overline{v}(t) = \frac{1}{l_1 l_2} \int_{\Sigma_t} v(x', t)\, dx' .$$

The Poincaré-Wirtinger inequality gives, for any $r \leqslant 6$,

$$\|v - \hat{\bar{v}}(t)\|_{L^r(\Lambda_t^{t+1})} \leqslant k \|\nabla v\|_{(L^2(\Lambda_t^{t+1}))^3}, \tag{24}$$

$$\|v - \bar{v}\|_{L^r(\Sigma_t)} \leqslant k \|\nabla_{x'} v\|_{(L^2(\Sigma_t))^2}. \tag{25}$$

At first, let us that $\bar{u}$ goes to a limit $\bar{u}_\infty$.

LEMMA 5. *Let* (u, p) *satisfy* (1), (2) *and* (3). *Then, for all* t,

$$\bar{u}_3(t) = 0 \tag{26}$$

and there exists a unique $\bar{u}_\infty \in \mathbb{R}^3$ *such that, for all* $t > 0$,

$$|\bar{u}(t) - \bar{u}_\infty| \leqslant c(l, \nu) \int_{\Theta_t} |\nabla u|^2. \quad \blacksquare \tag{27}$$

PROOF. *Proof of* (26). Since u is periodic with respect to x_1 and x_2, for all positive t we have $\int_{\Sigma_t} \partial_1 u \, dx' = \int_{\Sigma_t} \partial_2 u \, dx' = 0$. Therefore, $\nabla \cdot u = 0$ implies

$$0 = \int_{\Sigma_t} \nabla \cdot u(x', t) \, dx' = \int_{\Sigma_t} \partial_3 u_3(x', t) \, dx'$$

which may be written as $\dfrac{d}{dt} \int_{\Sigma_t} u_3 \, dx' = 0$. Then,

$$\int_{\Sigma_t} u_3(x', t) \, dx' = \int_{\Sigma_0} u_3(x', 0) \, dx' = \int_{\Sigma} \gamma_3(x') \, dx' = 0,$$

which proves (26).

Proof of (27). Since $\nabla \cdot u = 0$, the nonlinear term in (1) may be written as $u \cdot \nabla u = \partial_1(u_1 u) + \partial_2(u_2 u) + \partial_3(u_3 u)$. Then, integrating the first and second components of equation (1) and using the periodicity of u and p with respect to x_1 and x_2, we obtain, for $i = 1$ or 2,

$$0 = \int_{\Sigma_t} (-\nu \Delta u_i + \Sigma_j \partial_j (u_j u_i) + \partial_i p)(x', t) \, dx' =$$

$$= \int_{\Sigma_t} (-\nu \partial_3^2 u_i + \partial_3(u_3 u_i))(x', t) \, dx'.$$

Thus, there exist real numbers c_1 and c_2 such that, for all $t > 0$,

$$\int_{\Sigma_t} (-\nu\partial_3 u_i + u_3 u_i)(x', t)\, dx' = c_i. \tag{28}$$

Thanks to (26), $\int_{\Sigma_t} u_3 u_i = \int_{\Sigma_t} (u_3 - \overline{u}_3)(u_i - \overline{u}_i)$ thus, by (25),

$$\left| \int_{\Sigma_t} u_3 u_i \right| \leqslant k^2 \int_{\Sigma_t} |\nabla u|^2 \tag{29}$$

and therefore

$$|c_i| \leqslant \int_{\Sigma_t} \nu |\partial_3 u_i| + k^2 |\nabla u|^2.$$

Integrating from 0 to t and using $\int_{\Lambda_t} |\partial_3 u_i| \leqslant \sqrt{l_1 l_2 t} \left(\int_{\Lambda_t} |\partial_3 u_i|^2 \right)^{1/2}$, we get, for all $t > 0$,

$$|c_i|\, t \leqslant \int_{\Lambda_t} \nu |\partial_3 u_i| + k^2 |\nabla u|^2 \leqslant \nu\sqrt{l_1 l_2 E t} + k^2 E$$

which implies $c_i = 0$. Now, according to (28), we may write

$$\frac{d\overline{u}_i}{dt}(t) = \frac{1}{l_1 l_2} \int_{\Sigma_t} \partial_3 u_i(x', t)\, dx' = \frac{1}{\nu l_1 l_2} \int_{\Sigma_t} u_3 u_i\, dx'.$$

Using (29), we obtain, for all $t > 0$,

$$\int_t^\infty \left| \frac{d\overline{u}_i}{d\tau}(\tau) \right| d\tau \leqslant \frac{k^2}{\nu l_1 l_2} \int_{\Theta_t} |\nabla u|^2 \leqslant \frac{k^2 E}{\nu l_1 l_2}.$$

Therefore $\overline{u}_i$ possesses a limit $\overline{u}_{i,\infty}$, as $t \to \infty$, which satisfies

$$|\overline{u}_i(t) - \overline{u}_{i,\infty}| \leqslant \frac{k^2}{\nu l_1 l_2} \int_{\Theta_t} |\nabla u|^2.$$

Since $\overline{u}_3 \equiv 0$, this give (27). ■

LEMMA 6. *Let* (u, p) *satisfy* (1), (2) *and* (3). *Then, for all* $t > 0$,

$$\|u - \overline{u}_\infty\|_{(L^6(\Lambda_t^{t+1}))^3} \leq c(l, \gamma, \nu, E)\Big(\int_{\Theta_t} |\nabla u|^2\Big)^{1/2}, \tag{30}$$

$$\|u\|_{(L^6(\Lambda_t^{t+1}))^3} \leq c(l, \gamma, \nu, E). \quad \blacksquare \tag{31}$$

PROOF. The inequality (27) implies

$$|\hat{\overline{u}}(t) - \overline{u}_\infty| = \left|\int_t^{t+1} \overline{u}(\tau) - \overline{u}_\infty \, d\tau\right| \leq c(l, \nu)\int_{\Theta_t} |\nabla u|^2.$$

Therefore, by the Poincaré-Wirtinger inequality (24),

$$\begin{aligned}\|u - \overline{u}_\infty\|_{(L^6(\Lambda_t^{t+1}))^3} &\leq \|u - \hat{\overline{u}}_\infty\|_{(L^6(\Lambda_t^{t+1}))^3} + (l_1 l_2)^{1/6}\,|\hat{\overline{u}} - \overline{u}_\infty| \\ &\leq (k + (l_1 l_2)^{1/6} c(l, \nu)\sqrt{E})\Big(\int_{\Theta_t} |\nabla u|^2\Big)^{1/2}\end{aligned}$$

which proves (30). On the other hand, (27) for $t = 0$ gives

$$|\overline{u}_\infty| \leq |\overline{\gamma}| + c(l, \nu)\int_\Theta |\nabla u|^2 \tag{32}$$

from which we obtain (31). $\blacksquare$

From now, we will denote c various positive numbers depending at most on l, ν, γ and E. Let us prove the following identity.

LEMMA 7. *Let* (u, p) *satisfy* (1), (2) *and* (3). *Then, for all* $t > 0$,

$$\nu\int_{\Theta_t} |\nabla u|^2 = -\frac{\nu}{2}\int_{\Sigma_t} \partial_3(|u - \overline{u}_\infty|^2) + \frac{1}{2}\int_{\Sigma_t} u_3\,|u - \overline{u}_\infty|^2 + \int_{\Sigma_t} pu_3. \quad \blacksquare \tag{33}$$

PROOF. We remark that (u, p) is a solution of

$$-\nu\Delta(u - \overline{u}_\infty) + u\cdot\nabla(u - \overline{u}_\infty) + \nabla p = 0\,. \tag{34}$$

Multiplying by $u - \overline{u}_\infty$, integrating over Σ_t and using the identities

$$-\nu\Delta v\cdot v = \nu|\nabla v|^2 - \frac{\nu}{2}\Delta(|v|^2), \qquad w\cdot\nabla v\cdot v = \frac{1}{2}\nabla\cdot(w|v|^2),$$

for all smooth vectors fields v and w such that $\nabla\cdot w=0$, we get

$$\nu\int_{\Sigma_t}|\nabla u|^2-\frac{\nu}{2}\int_{\Sigma_t}\Delta(|u-\overline{u}_\infty|^2)+\frac{1}{2}\int_{\Sigma_t}\nabla\cdot(u|u-\overline{u}_\infty|^2)+\int_{\Sigma_t}\nabla\cdot(p(u-\overline{u}_\infty))=0\,.$$

Since (u, p) is periodic in the directions x_1 and x_2 and $(\overline{u}_\infty)_3=0$, it follows that

$$\nu\int_{\Sigma_t}|\nabla u|^2=\frac{\nu}{2}\int_{\Sigma_t}\partial_3^2(|u-\overline{u}_\infty|^2)-\frac{1}{2}\int_{\Sigma_t}\partial_3(u_3|u-\overline{u}_\infty|^2)-\int_{\Sigma_t}\partial_3(pu_3)\,.$$

Then, integrating from τ to τ', we get, for $0\leqslant\tau<\tau'$,

$$\nu\int_{\Lambda_\tau^{\tau'}}|\nabla u|^2=F(\tau')-F(\tau)\,,\tag{35}$$

where

$$F(t)=\int_{\Sigma_t}\left(\frac{\nu}{2}\partial_3(|u-\overline{u}_\infty|^2)-\frac{1}{2}u_3|u-\overline{u}_\infty|^2-pu_3\right).$$

Due to the hypothesis (3), it follows from (35) that $F(t)$ has a limit F_∞ as $t\to\infty$ which satisfies, for all $\tau>0$,

$$F_\infty=\nu\int_{\Theta_\tau}|\nabla u|^2+F(\tau)\,.\tag{36}$$

To get the lemma, it remains to prove that $F_\infty=0$. For this, let us integrate the both sides of (36) from t to $t+1$. We obtain, for $t\geqslant 0$,

$$F_\infty=\nu\int_t^{t+1}\int_{\Theta_\tau}|\nabla u|^2+\int_{\Lambda_t^{t+1}}\frac{\nu}{2}\partial_3(|u-\overline{u}_\infty|^2)-\frac{1}{2}u_3|u-\overline{u}_\infty|^2-pu_3\,.\tag{37}$$

Let us prove now that each term of the right-hand side of (37) goes to 0 as $t \to \infty$. At first, by assumption (3), $\int_{\Theta_t} |\nabla u|^2 \to 0$ thus

$$\int_t^{t+1} \int_{\Theta_t} |\nabla u|^2 \to 0 . \tag{38}$$

For the second term, using (30), we bound

$$\frac{1}{2} \left| \int_{\Lambda_t^{t+1}} \partial_3 (|u - \bar{u}_\infty|^2) \right| \leqslant \|u - \bar{u}_\infty\|_{(L^2(\Lambda_t^{t+1}))^3} \|\partial_3 u\|_{(L^2(\Lambda_t^{t+1}))^3} \leqslant \tag{39}$$

$$\leqslant c \|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9}.$$

For the third term, we remark that $\overset{\ominus}{\bar{u}}_3 = 0$ since $\bar{u}_3 = 0$, thus, using the Poincaré-Wirtinger inequality (24) and (30), we bound

$$\left| \int_{\Lambda_t^{t+1}} u_3 \, |u - \bar{u}_\infty|^2 \right| \leqslant \|u_3 - \overset{\ominus}{\bar{u}}_3\|_{L^2(\Lambda_t^{t+1})} (\|u - \bar{u}(t)\|_{(L^4(\Lambda_t^{t+1}))^3})^2 \tag{40}$$

$$\leqslant c \|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9}.$$

To estimate the last term in (37), we will use the Poincaré-Wirtinger inequality

$$\|p - \overset{\ominus}{\bar{p}}(t)\|_{L^2(\Lambda_t^{t+1})} \leqslant k' \|\nabla p\|_{(H^{-1}(\Lambda_t^{t+1}))^3}.$$

Using again $\overset{\ominus}{\bar{u}}_3 \equiv 0$ and (24), we bound

$$\left| \int_{\Lambda_t^{t+1}} p u_3 \right| = \left| \int_{\Lambda_t^{t+1}} (p - \overset{\ominus}{\bar{p}}(t))(u_3 - \overset{\ominus}{\bar{u}}_3(t)) \right| \tag{41}$$

$$\leqslant k k' \|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9} \|\nabla p\|_{(H^{-1}(\Lambda_t^{t+1}))^3}.$$

Let us now give an estimate for $\|\nabla p\|_{(H^{-1}(\Lambda_t^{t+1}))^3}$. The space $H^{-1}(\Lambda_t^{t+1})$ being equipped with the norm

(42) $\quad \|\Psi\|_{H^{-1}(\Lambda_t^{t+1})} =$

$$= \inf \left\{ \left(\int_{\Lambda_t^{t+1}} \sum_{i=0}^{3} |\Psi_i|^2 \right)^{1/2} : \Psi = \Psi_0 + \sum_{j=1}^{3} \partial_j \Psi_j, \ \Psi_j \in L^2(\Lambda_t^{t+1}), \ j = 0, \ldots, 3 \right\},$$

from the equation $\nabla p = \nu \nabla \cdot \nabla u - u \cdot \nabla u$, we deduce that

$$\|\nabla p\|_{(H^{-1}(\Lambda_t^{t+1}))^3} \leq \nu \|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9} + \|u \cdot \nabla u\|_{(H^{-1}(\Lambda_t^{t+1}))^3}. \tag{42}$$

By the Sobolev and Hölder inequalities, we bound

$$\|u \cdot \nabla u\|_{(H^{-1}(\Lambda_t^{t+1}))^3} \leq c \|u \cdot \nabla u\|_{(L^{3/2}(\Lambda_t^{t+1}))^3} \leq c \|u\|_{(L^6(\Lambda_t^{t+1}))^3} \|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9}.$$

Thus, using (41), (42) and (31), we get

$$\left| \int_{\Lambda_t^{t+1}} p u_3 \right| \leq kk'(\nu + c)(\|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9})^2. \tag{43}$$

From (3), $\|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9} \to 0$ as $t \to 0$ therefore (38), (39), (40) and (43) imply that each term in the right-hand side of (37) goes to 0 as $t \to \infty$. Consequently $F_\infty = 0$ and (33) holds. ■

Before going further, let us improve (40) for a later use, as follows:

$$\left| \int_{\Lambda_t^{t+1}} u_3 \, |u - \overline{u}_\infty|^2 \right| \leq c(l, \gamma, \nu, E)(\|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9})^2. \tag{44}$$

Indeed

$$|u - \overline{u}_\infty|^2 = |u - \hat{\overline{u}}(t)|^2 + 2(u - \hat{\overline{u}}(t)) \cdot (\hat{\overline{u}}(t) - \overline{u}_\infty) + |\hat{\overline{u}}(t) - \overline{u}_\infty|^2,$$

therefore

$$\int_{\Lambda_t^{t+1}} u_3 \, |u - \overline{u}_\infty|^2 = \int_{\Lambda_t^{t+1}} u_3 \, |u - \hat{\overline{u}}(t)|^2 + 2 \int_{\Lambda_t^{t+1}} u_3 (u - \hat{\overline{u}}(t)) \cdot (\hat{\overline{u}}(t) - \overline{u}_\infty)$$

since $|\hat{\overline{u}}(t) - \overline{u}_\infty|^2$ is constant on Λ_t^{t+1} and $\int_{\Lambda_t^{t+1}} u_3 = \hat{\overline{u}}(t) = 0$. Thus,

$$\left| \int_{\Lambda_t^{t+1}} u_3 \, |u - \overline{u}_\infty|^2 \right| \leq \|u_3 - \hat{\overline{u}}_3\|_{L^2(\Lambda_t^{t+1})} \|u - \hat{\overline{u}}(t)\|_{(L^4(\Lambda_t^{t+1}))^3} \cdot$$

$$\cdot (\|u - \hat{\overline{u}}(t)\|_{(L^4(\Lambda_t^{t+1}))^3} + 2\|\hat{\overline{u}}(t) - \overline{u}_\infty\|_{(L^4(\Lambda_t^{t+1}))^3}) .$$

Integrating (27) we bound $|\hat{\overline{u}}(t) - \overline{u}_\infty| \leq c$ thus, by (32), $|\hat{\overline{u}}(t)| \leq c$. Then, using (31), we get $\|u - \hat{\overline{u}}(t)\|_{(L^4(\Lambda_t^{t+1}))^3} + 2\|\hat{\overline{u}}(t) - \overline{u}_\infty\|_{(L^4(\Lambda_t^{t+1}))^3} \leq c$. Therefore, (44) follows by the Poincaré-Wirtinger inequality (24) for $r =$ $= 2$ and 4.

LEMMA 8. *Let* (u, p) *satisfy* (1), (2) *and* (3). *Then, for all* $t > 0$,

$$\int_{\Theta_t} |\nabla u|^2 \leqslant c \exp(-\sigma_1 t), \tag{45}$$

where c *and* σ_1 *are positive numbers depending only on* l, γ, ν *and* E. ■

PROOF. Integrating the both sides of the identity (33) we get, for $t \geqslant 0$ and $m \in \mathbb{N}$,

$$\nu \int_t^{t+m} \int_{\Theta_\tau} |\nabla u|^2 = -\frac{\nu}{2} \int_{\Sigma_{t+m}} |u - \overline{u}_\infty|^2 + \frac{\nu}{2} \int_{\Sigma_t} |u - \overline{u}_\infty|^2 + \tag{46}$$
$$+ \frac{1}{2} \int_{\Lambda_t^{t+m}} u_3 |u - \overline{u}_\infty|^2 + \int_{\Lambda_t^{t+m}} p u_3 .$$

Using the Poincaré-Wirtinger inequality on Σ_t (25) and (27), we obtain

$$\frac{1}{2} \int_{\Sigma_t} |u - \overline{u}_\infty|^2 \leqslant k^2 \int_{\Sigma_t} |\nabla u|^2 + c\left(\int_{\Theta_t} |\nabla u|^2 \right)^2 .$$

Using respectively (43) and (44) on Λ_{t+n}^{t+n+1} and summing over n from 0 to $m-1$, we get

$$\frac{1}{2} \left| \int_{\Lambda_t^{t+m}} u_3 |u - \overline{u}_\infty|^2 \right| \leqslant c \int_{\Lambda_t^{t+m}} |\nabla u|^2 ,$$

$$\left| \int_{\Lambda_t^{t+m}} p u_3 \right| \leqslant c \int_{\Lambda_t^{t+m}} |\nabla u|^2 .$$

Then (46) gives, as $m \to \infty$,

$$\int_t^{\infty} \int_{\Theta_\tau} |\nabla u|^2 \leqslant c\left(\int_{\Sigma_t} |\nabla u|^2 + \int_{\Theta_t} |\nabla u|^2 \right).$$

Setting

$$y(\tau) = \int_{\Theta_\tau} |\nabla u|^2$$

this gives, since $y'(t) = -\int_{\Sigma_t} |\nabla u|^2$, for all $t > 0$:

$$cy'(t) + \int_t^\infty y(\tau)\, d\tau \leqslant cy(t).$$

This inequality and $y(t) \to 0$ as $t \to \infty$ imply, by a lemma of [8], see [6] Lemma 2.2 p. 315,

$$y(t) \leqslant c' y(0) \exp(-\sigma t)$$

that is (45). ■

Let us now focus on a regularity property of periodic solutions of Stokes equations.

LEMMA 9. *Given* $f \in (\mathcal{C}^\infty_{\text{per}}(\Theta))^3$, *let* $(v, \phi) \in (\mathcal{C}^\infty_{\text{per}}(\Theta))^3 \times \mathcal{C}^\infty_{\text{per}}(\Theta)$ *be a solution of*

$$\nu \Delta v = \nabla\phi + f, \qquad \nabla \cdot v = 0 \qquad \textit{in } \mathbb{R}^3_+ \tag{47}$$

and let $0 < \delta \leqslant 1$ *and* $s \geqslant \delta$. *Then, for all* $m \geqslant 0$ *and* $1 < q < \infty$,

$$\|v\|_{(W^{m+2,q}(\Lambda_s^{s+1}))^3} + \|\nabla\phi\|_{(W^{m,q}(\Lambda_s^{s+1}))^3} \leqslant \tag{48}$$

$$\leqslant c(m, q, \delta, \nu, l)\left(\|f\|_{(W^{m,q}(\Lambda_{s-\delta}^{s+1+\delta}))^3} + \|v\|_{(W^{1,q}(\Lambda_{s-\delta}^{s+1+\delta}))^3}\right).$$

From now, the numbers c may depend on m, in addition to l, ν, γ and E.

PROOF. We will deduce this property from a similar one for Dirichlet boundary condition, see [6], by localization. We first note that it suffices to prove (48) for $s = 1$ since the general result follows by the translation $x_3 \mapsto x_3 - s + 1$.

Given a positive integer k, we denote

$$U_k = \Lambda_{1-1/k}^{2+1/k} = \left\{ x \in \mathbb{R}^3 \colon 0 < x_1 < l_1,\ 0 < x_2 < l_2,\ 1 - \frac{1}{k} < x_3 < 2 + \frac{1}{k} \right\},$$

$$V_k = \left\{ x \in \mathbb{R}^3 \colon -\frac{1}{k} < x_1 < l_1 + \frac{1}{k},\ -\frac{1}{k} < x_2 < l_2 + \frac{1}{k},\ 1 - \frac{1}{k} < x_3 < 2 + \frac{1}{k} \right\}.$$

We consider a localization function $\psi_k \in \mathcal{C}^\infty(\mathbb{R}^3)$ such that $\psi_k \equiv 1$ in V_{k+1}, with $\operatorname{supp} \psi_k \subset V_k$ and we denote $v_k = \psi_k v$, $\phi_k = \psi_k \phi$. Then

$$\nu \Delta v_k = \nabla \phi_k + f_k, \qquad \nabla \cdot v_k = g_k$$

where $f_k = \psi_k f + 2\nu \nabla \psi_k \cdot \nabla v + \nu v \Delta \psi_k - \phi \nabla \psi_k$, and $g_k = \nabla \psi_k \cdot v$.

Since (v_k, ϕ_k) satisifies the Dirichlet boundary conditions for any smooth domain Ω, such that $\operatorname{supp} \psi_k \subset \Omega \subset V_k$, the Theorem IV.6.1 of [6] gives

$$\|v\|_{(W^{m+2,q}(V_{k+1}))^3} + \|\nabla \phi\|_{(W^{m,q}(V_{k+1}))^3} \leqslant \tag{49}$$

$$\leqslant c(\|f_k\|_{(W^{m,q}(V_k))^3} + \|g_k\|_{(W^{m+1,q}(V_k))^3}).$$

The definitions of f_k and g_k imply

$$\|f_k\|_{(W^{m,q}(V_k))^3} + \|g_k\|_{(W^{m+1,q}(V_k))^3} \leqslant c(\|f\|_{(W^{m,q}(V_k))^3} + \|v\|_{(W^{m+1,q}(V_k))^3} + \|\phi\|_{W^{m,q}(V_k)}).$$

Since the statement of the lemma is unchanged if an arbitrary constant is added to ϕ, we can assume $\int_{V_k} \phi = 0$. Then $\|\phi\|_{W^{m,q}(V_k)} \leqslant c \|\nabla \phi\|_{(W^{m-1,q}(V_k))^3}$ and therefore the equation (47) gives

$$\|\phi\|_{W^{m,q}(V_k)} \leqslant c(\|f\|_{(W^{m,q}(V_k))^3} + \|v\|_{(W^{m+1,q}(V_k))^3}).$$

Since f and v are periodic with respect to x_1 and x_2,

$$\|f\|_{(W^{m,q}(V_k))^3} \leqslant c\|f\|_{(W^{m,q}(U_k))^3}, \qquad \|v\|_{(W^{m+1,q}(V_k))^3} \leqslant c\|v\|_{(W^{m+1,q}(U_k))^3}.$$

Finally, using $U_{k+1} \subset V_{k+1}$, (49) gives

$$\|v\|_{(W^{m+2,q}(U_{k+1}))^3} + \|\nabla \phi\|_{(W^{m+1,q}(U_{k+1}))^3} \leqslant c(\|f\|_{(W^{m,q}(U_k))^3} + \|v\|_{(W^{m+1,q}(U_k))^3}).$$

By a repeated use, this equation gives

$$\|v\|_{(W^{m+2,q}(U_{k+1}))^3} + \|\nabla \phi\|_{(W^{m+1,q}(U_{k+1}))^3} \leqslant c(\|f\|_{(W^{m,q}(U_{k-m}))^3} + \|v\|_{(W^{1,q}(U_{k-m}))^3})$$

which proves (48) for $s = 1$ since $\Lambda_1^2 \subset U_{k+1}$ and, for $k \geqslant m + 1/\delta$, $U_{k-m} \subset \Lambda_{1-\delta}^{2+\delta}$. ■

We are now in a position to prove our second, and last, main result.

PROOF OF THEOREM 2. *Bound on u in $(H^2(\Lambda_t^{t+1}))^3$.* At first, let us check that

$$\|u\|_{(H^2(\Lambda_t^{t+1}))^3} \leqslant c\,. \tag{50}$$

The inequality (48) for $m = 0$, $q = 3/2$ and $\delta = 1/2$ gives

$$\|u\|_{(W^{2,3/2}(\Lambda_t^{t+1}))^3} \leqslant c\|u \cdot \nabla u\|_{(L^{3/2}(\Lambda_{t-1/2}^{t+3/2}))^3} + \|u\|_{(W^{1,3/2}(\Lambda_{t-1/2}^{t+3/2}))^3}. \tag{51}$$

By Hölder inequality, for all $t \geqslant 0$,

$$\|u \cdot \nabla u\|_{(L^{3/2}(\Lambda_t^{t+1}))^3} \leqslant \|u\|_{(L^6(\Lambda_t^{t+1}))^3} \|\nabla u\|_{(L^2(\Lambda_t^{t+1}))^9}$$

which is bounded by (3) and (31), thus the right-hand side of (51) is bounded for all $t \geqslant 1/2$. Therefore, by Sobolev theorem, $\|\nabla u\|_{(L^3(\Lambda_t^{t+1}))^9} \leqslant c$ and thus $\|u \cdot \nabla u\|_{(L^2(\Lambda_t^{t+1}))^3} \leqslant c$. Using now (48) for $m = 0$, $q = 2$ and $\delta = 1/2$, we get (50).

Bound on u in $(H^m(\Lambda_t^{t+1}))^3$ Due to the equation (34), Lemma 9 gives

$$\|u - \overline{u}_\infty\|_{(H^{m+2}(\Lambda_t^{t+1}))^3} \leqslant c(\|u \cdot \nabla(u - \overline{u}_\infty)\|_{(H^m(\Lambda_{t-\delta}^{t+1+\delta}))^3} + \\ + \|u - \overline{u}_\infty\|_{(H^1(\Lambda_{t-\delta}^{t+1+\delta}))^3}). \tag{52}$$

For $m \geqslant 2$, $H^m(\Lambda_{t-\delta}^{t+1+\delta})$ is a multiplicative algebra, thus

$$\|u \cdot \nabla(u - \overline{u}_\infty)\|_{(H^m(\Lambda_{t-\delta}^{t+1+\delta}))^3} \leqslant \|u\|_{(H^m(\Lambda_{t-\delta}^{t+1+\delta}))^3} \|\nabla(u - \overline{u}_\infty)\|_{(H^m(\Lambda_{t-\delta}^{t+1+\delta}))^9}.$$

Since $|\overline{u}_\infty|$ is bounded by (32), we get, for $0 \leqslant \varsigma \leqslant 1$,

$$\|u - \overline{u}_\infty\|_{(H^{m+2}(\Lambda_{t-\varsigma}^{t+1+\varsigma}))^3} \leqslant c(\|u - \overline{u}_\infty\|_{(H^{m+1}(\Lambda_{t-\varsigma-\delta}^{t+1+\varsigma+\delta}))^3})^2 + \\ + \|u - \overline{u}_\infty\|_{(H^1(\Lambda_{t-\varsigma-\delta}^{t+1+\varsigma+\delta}))^3}.$$

By m repeated use of this inequality with $\delta = 1/m$, we deduce from (50) that

$$\|u - \overline{u}_\infty\|_{(H^m(\Lambda_{t-\varsigma}^{t+1+\varsigma}))^3} \leqslant c\,.$$

Exponential decay of $u - \overline{u}_\infty$ Now, the previous inequality yields, for $m \geqslant 2$,

$$\|u - \overline{u}_\infty\|_{(H^{m+2}(\Lambda_{t-\varsigma}^{t+1+\varsigma}))^3} \leqslant c\|u - \overline{u}_\infty\|_{(H^{m+1}(\Lambda_{t-\varsigma-\delta}^{t+1+\varsigma+\delta}))^3}.$$

This inequality also holds for $m = 0$ and 1 since, by Sobolev theorem, $\|u\|_{(W^{2,\infty}(\Lambda_t^{t+1}))^3} \leq c$ which gives a convenient bound for the nonlinear term in the right-hand side of (52). Then, by $m+1$ repeated use with now $\delta = = 1/(m+1)$, we get

$$\|u - \overline{u}_\infty\|_{(H^{m+2}(\Lambda_t^{t+1}))^3} \leq c \|u - \overline{u}_\infty\|_{(H^1(\Lambda_{t-1}^{t+2}))^3}.$$

By (30) the right-hand side is bounded by $c \int_{\Theta_t} |\nabla u|^2$, and by (45) it is bounded by $c \exp(-\sigma_1 t/2)$. Thus, by Sobolev theorem,

$$\|u - \overline{u}_\infty\|_{(W^{m,\infty}(\Lambda_t^{t+1}))^3} \leq c \exp(-\sigma_1 t/2). \tag{53}$$

Exponential decay of $p - \overline{p}_\infty$. By Lemma 9, the pressure term $\|\nabla p\|_{(W^{m,\infty}(\Lambda_t^{t+1}))^3}$ may be added to the left-hand side of (52), and therefore of (53). Thus,

$$\|\nabla p\|_{(W^{m,\infty}(\Lambda_t^{t+1}))^3} \leq c \exp(-\sigma_1 t/2).$$

This inequality for $m = 0$ implies the existence of $p_\infty \in \mathbb{R}$ such that $|p(x) - p_\infty| \leq c \exp(-\sigma_1 t/2)$, which ends the proof of Theorem 2. ■

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