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# Varieties without Boundary in Rigid Analytic Geometry.

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## Introduction.

In [L], Lütkebohmert suggests a definition for rigid analytic varieties without boundary and conjectures a duality theorem. In this article we put forward a new definition of varities without boundary and the more general definition of morphisms without boundary. This definition appears weaker than Lütkebohmert's, but has the advantage of being of local nature. Nevertheless, in the case of a discrete valuation, for separated paracompact rigid analytic varieties, the two definitions coincide. In the first paragraph, we introduce the notion of morphism without boundary and show that it behaves quite well with respect to the classic operations of geometry. In the next paragraph, we give some topological properties of quasi-paracompact admissible formal schemes. Paragraph three deals with generalizing the theorem of Raynaud which shows that the category of analytic rigid quasi-separated and paracompact varieties is equivalent to the cartegory of the generically paracompact (quasi-paracompact) formal schemes localized with respect to admissible formal blowing-ups. The fourth paragraph shows that under this equivalence, the (locally quasi-compact) morphisms without boundary correspond to (locally quasi-compact) morphisms between formal admissible schemes which are universally closed and separated (for technical reasons, we only consider the case of a discrete valuation). Then we deduce that our notion of morphism without boundary coincides with Lütkebohmert's in the separated and paracompact case, and that a Zariski open subset of a paracompact variety without boundary is without boundary.

In a forthcoming paper, we prove Serre's duality theorem for para-

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compact rigid varieties without boundary which are countable at infinity.

Remark that proper varieties, Stein varieties and algebraic varieties are separated, paracompact and without boundary. This fact illustrates the interest of the notion of variety without boundary.

In this paper, an analytic variety shall mean a rigid analytic variety and all open subsets are assumed to be *admissible*.

By $K$ we denote a non trivial complete non-Archemedean valued field with absolute value $| \ |$. We write $R := \{\lambda \in K / |\lambda| \leq 1\}$ its ring of integers and we fix $\pi \in R$, with $0 < |\pi| < 1$.

## 1. Rigid morphisms without boundary.

DEFINITION 1.1. Let $X = \mathrm{Spm}(A)$ and $Y = \mathrm{Spm}(B)$ be two affinoid varieties. An affinoid open subset $U$ of $X$ is called *relatively compact* in $X$ over $Y$ if there exists an affinoid generating system $F = \{f_1, \ldots, f_r\}$ of $A$ over $B$ such that $U$ is contained in $\{x \in X / |f_i(x)| < 1, i = 1, \ldots, r\}$. We then write $U \subset\subset_Y X$. When $B = K$, we simply say that $U$ is *relatively compact* in $X$ and we write $U \subset\subset X$.

REMARKS 1.2. (i) According to the Principal Maximum Modulus ([BGR], 6.2.1/4), we have $U \subset\subset_Y X$ if and only if there exists an affinoid system $\{f_1, \ldots, f_r\}$ of $A$ over $B$ and a real number $0 < \varepsilon < 1$, such that

$$\varepsilon \in \sqrt{|k^*|} \quad \text{and} \quad U \subset X(\varepsilon^{-1} f_1, \ldots, \varepsilon^{-1} f_r) =$$
$$= \{x \in X / |f_1(x)| < \varepsilon, \ldots, |f_r(x)| < \varepsilon\}.$$

(ii) Let $U = \mathrm{Spm}(A_U)$ be an affinoid open subset of $X$. We write $\| \ \|_U$ the spectral semi-norm on the affinoid $K$-algebra $A_U$. We have $U \subset\subset_Y X$ if and only if there exists an affinoid generating system $\{f_1, \ldots, f_r\}$ of $A$ over $B$ such that, if we denote by $\varphi_U : A \to A_U$ the canonical morphism from $A$ to $A_U$, then for every $1 \leq i \leq r$, $\|\varphi_U(f_i)\|_U < 1$.

Let $A$ be an affinoid algebra. If we denote by $\| \ \|_{sp}$ the spectral semi-norm on $A$, then $A^0 = \{f \in A / \|f\|_{sp} \leq 1\}$ is a ring and $A^{00} = \{f \in A / \|f\|_{sp} < 1\}$ is an ideal in $A^0$. The residual ring $A^0 / A^{00}$ is written $\tilde{A}$. Any morphism $\varphi : A \to B$ of affinoid algebras induces a morphism $\tilde{\varphi} : \tilde{A} \to \tilde{B}$; we say that $\tilde{\varphi}$ is obtained by reduction.

LEMMA 1.3. *Let $\varphi: B \to A$ and $\psi: A \to C$ be two morphisms of affinoid K-algebras. The following two properties are equivalent:*

(i) *there exists a surjective morphism $\varphi': B\{T_1, \ldots, T_n\} \to A$ which extends $\varphi$ and satisfies $\|\psi(\varphi'(T_i))\|_{sp} < 1$.*

(ii) *The ring $\tilde{\psi}(\tilde{A})$ is integral over $\tilde{\psi}(\varphi(B)^{\sim})$.*

PROOF. See [Ber], 2.5.2 (*a*) and (*d*). ■

PROPOSITION 1.4. *Let $X = \mathrm{Spm}(A)$ be an affinoid variety over another affinoid variety $Y = \mathrm{Spm}(B)$ and $U$ an affinoid open subset of $X$. The following properties are equivalent:*

(i) *$U$ is relatively compact in $X$ over $Y$.*

(ii) *There exists a finite number of affinoid domains $V_i$ such that*

$$U \subset \bigcup_i V_i \quad \text{and} \quad V_i \subset\subset_Y X .$$

(iii) *There exists a finite number of affinoid domains $V_i$ such that*

$$U = \bigcup_i V_i \quad \text{and} \quad V_i \subset\subset_Y X .$$

PROOF. It is clear that (iii) implies (ii). Conversely, assume the existence of a finite number of affinoid domains $V_i$ such that $U \subset \bigcup_{i=1}^{n} V_i$ and $V_i \subset\subset_Y X$. The $V_i \cap U$ are affinoid domains $U_i$ and $U = \bigcup_i U_i$. As each $U_i$ satisfies $U_i \subset\subset_Y X$. It is obvious that (i) implies (iii). Conversely, assume that $U = \mathrm{Spm}(A_U) = \bigcup_{i=1}^{n} V_i$, where the $V_i = \mathrm{Spm}(A_{V_i})$ are affinoid domains such that for any $i$, $V_i \subset\subset_Y X$. Then, there exists a surjective $B$-morphisms $\varphi_i: B\{T_1, \ldots, T_{n_i}\} \to A$ $(i = 1, \ldots, n)$ such that, for any point $x$ in $V_i$, $|\varphi_i(T_j)(x)| < 1$. If we denote by $\varphi_{V_i}$ the restriction morphism $A \to A_{V_i}$ and by $\| \ \|_{V_i}$ the spectral semi-norm in $A_{V_i}$, then $\|\varphi_{V_i}(\varphi_i(T_j))\|_{V_i} < 1$. The operation of reduction applied to the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\varphi_{V_i}} & A_{V_i} \\ {\scriptstyle \varphi_U}\downarrow & & \| \\ A_U & \xrightarrow{\varphi'_{V_i}} & A_{V_i} \end{array}$$

gives the commutative diagram

$$\begin{tikzcd} \widetilde{A} \arrow[r, "\widetilde{\varphi}_{V_i}"] \arrow[d, "\widetilde{\varphi}_U"'] & \widetilde{A}_{V_i} \arrow[d, equal] \\ \widetilde{A}_U \arrow[r, "\widetilde{\varphi}'_{V_i}"] & \widetilde{A}_{V_i} \end{tikzcd}$$

According to the previous lemma, $\widetilde{\varphi}_{V_i}(\widetilde{A})$ is integral over $\varphi_{V_i}(\varphi(B)\widetilde{\ })$, where $\varphi$ denotes the morphism $B \to A$. Let $f \in \widetilde{A}$ and for any $i$, $f_i = \widetilde{\varphi}_{V_i}(f)$. There exists a monic polynomial $P_i$ with coefficients in $\varphi(B)\widetilde{\ }$ such that, $\widetilde{\varphi}_{V_i}(P_i(f_i)) = 0$. The element $\left(\prod_{i=1}^{n} P_i\right)(\widetilde{\varphi}_U(f))$ of $\widetilde{A}_U$ satisfies

$$\widetilde{\varphi}'_{V_i}\left(\prod_{j=1}^{n} P_j\right)(\widetilde{\varphi}_U(f)) =$$

$$= \widetilde{\varphi}_{V_i}\left(\prod_{j=1}^{n} P_j\right)(f_i) = \left(\prod_{j=1,\, j \neq i}^{n} \widetilde{\varphi}_{V_i}(P_j)(f_i)\right) \times \widetilde{\varphi}_{V_i}(P_j)(f_i) = 0\,.$$

Since the morphism $\prod_{i=1}^{n} \widetilde{\varphi}'_{V_i} \colon \widetilde{A}_U \to \prod_{i=1}^{n} \widetilde{A}_{V_i}$ is injective, $\widetilde{\varphi}_U(\widetilde{A})$ is integral over $\widetilde{\varphi}_U(\varphi(B)\widetilde{\ })$. Assertion (i) results from the previous lemma. ■

DEFINITION 1.5. Let $\varphi\colon X \to Y$ be a morphism of analytic varieties. We say that $\varphi$ is *without boundary* if there exists an affinoid covering $\{W_i\}_{i \in I}$ of $Y$ such that for any $i$, there exists two affinoid coverings $\{U_{ij}\}_{j \in J}$ and $\{V_{ij}\}_{j \in J}$ of the open subset $\varphi^{-1}(W_i)$ such that for any $j$, $U_{ij} \subset\subset_{W_i} V_{ij}$. We say that $X$ is *without boundary* if the canonical morphism $X \to \mathrm{Spm}(K)$ is without boundary. Thus we see that $X$ is without boundary if and only if there exists two affinoid coverings $\{U_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ such that for any $i \in I$, $U_i \subset\subset V_i$.

REMARK 1.6. The composition of a closed immersion $\varphi\colon X \to Y$ and a morphism without boundary $\psi\colon Y \to Z$ gives a morphism without boundary. In fact, it is easy to check that if $W$ is an affinoid domain in $Z$ and $\{U_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ are two admissible coverings of $\psi^{-1}(W)$ such that for any $i$, $U_i \subset\subset_W V_i$, then $\varphi^{-1}(U_i) \subset\subset_W \varphi^{-1}(V_i)$.

EXAMPLES 1.7. (i) Every proper morphism is without boundary. In particular, finite morphisms and closed immersions are without boundary ([BGR], 9.6).

(ii) Recall (see [P], 2.1) that a morphism $Z \to Y$, with $Z$ separated and $Y$ affinoid, is *Stein* if there exists an affinoid covering $\{Z_m\}_{m \in N}$ of $Z$ such that for any $m$, there is a constant $a_m \in \sqrt{|K^*|}$ with $0 < a_m < 1$ and an affinoid generating system $h_1(m), \dots, h_{r(m)}(m)$ of $\mathcal{O}(Z_m)$ on $\mathcal{O}(Y)$ such that

$$Z_{m-1} = \{x \in Z_m / |h_i(m)(x)| \leqslant a_m \text{ for } i = 1, \dots, r(m)\}.$$

When $Y = \mathrm{Spm}(K)$, we say that $Z$ is a *Stein variety*. A Stein morphism is without boundary. In particular, every Stein variety is without boundary.

PROPOSITION 1.8. (i) *Let $\{X_i\}_{i \in I}$ be an admissible covering of an analytic variety $X$ by open subsets without boundary. Then, $X$ is a variety without boundary.*

(ii) *Let $X$ be a quasi-separated rigid variety without boundary. Then, for any admissible covering $\{U_i\}_{i \in I}$ of $X$, there exists an affinoid covering $\{V_j\}_{j \in J}$ which refines it and such that for any $j$, there exists an affinoid domain $W_j$ of $X$ with $V_j \subset\subset W_j$.*

PROOF (i) Let $\{U_{ij}\}_{j \in J_i}$ and $\{V_{ij}\}_{j \in J_i}$ be two affinoid coverings of $X_i$ such that $U_{ij} \subset\subset V_{ij}$. The family $\{U_{ij}\}_{ij}$ is an admissible covering of $X$ which satisfies, for any couple of indexes $(i, j)$, $U_{ij} \subset\subset V_{ij}$. The claim follows from the fact that both families $\{U_{ij}\}_{ij}$ and $\{V_{ij}\}_{ij}$ are admissible coverings of $X$.

(ii) We can assume that the $U_i$'s are affinoid. Let $\{V_j\}_j$ and $\{W_j\}_j$ two affinoid coverings of $X$ such that, for any $j$, we have $V_j \subset\subset W_j$. The intersection $U_i \cap V_j$ is a finite union of affinoid domains $U_{ij,k}$. Each $U_{ij,k}$ satisfies $U_{ij,k} \subset\subset W_j$ and the family formed of the $U_{ij,k}$'s is an admissible covering of $X$. ∎

LEMMA 1.9. *Let $X_1$ and $X_2$ be two affinoid varieties over the same affinoid variety $Y$ and for each $i = 1, 2$, let $U_i \subset X_i$ be an affinoid domain such that $U_i \subset\subset_Y X_i$. Then*

$$U_1 \times_Y X_2 \subset\subset_{X_2} X_1 \times_Y X_2 \quad \textit{and} \quad U_1 \times_Y U_2 \subset\subset_Y X_1 \times_Y X_2.$$

PROOF. See [BGR], 9.6.2/1. ∎

PROPOSITION 1.10. *Let $\varphi: X \to Y$ be a morphism of analytic varieties and $\{Y_i\}_{i \in I}$ an admissible covering of $Y$. Write $X_i = \varphi^{-1}(Y_i)$. Then*

*$\varphi$ is without boundary if and only if each map $\varphi_i\colon X_i \to Y_i$ is without boundary.*

PROOF. If all the $\varphi_i$'s are without boundary, it is easy to check that $\varphi$ is without boundary. Conversely, if $\varphi$ is without boundary, adopting the notations of Definition 1.5, for a fixed $k$ of $I$ and for any $W_i$ intersecting $Y_k$, the subset $Y_k \cap W_i$ is an admissible open subset and has an admissible affinoid covering $\{U_l\}_l$. We have

$$\varphi^{-1}(U_l) \cap U_{ij} = U_{ij} \times_{W_i} U_l \quad \text{and} \quad \varphi^{-1}(U_l) \cap V_{ij} = V_{ij} \times_{W_i} U_l,$$

so according to the lemma 1.9,

$$\varphi^{-1}(U_l) \cap U_{ij} \subset\subset_{U_l} \varphi^{-1}(U_l) \cap V_{ij}.$$

It follows that $\varphi^{-1}(U_l) \to U_l$ is without boundary. From this, we conclude that $\varphi^{-1}(Y_k) \to Y_k$ is without boundary. ■

COROLLARY 1.11. *Let $\varphi\colon X \to Y$ be a morphism of analytic varieties. The map $\varphi$ is without boundary if and only if for any affinoid domain $V$ (or for any admissible open subset of $Y$) the induced morphism $\varphi^{-1}(V) \to V$ is without boundary.* ■

Recall that a morphism of analytic varieties is called *quasi-compact* if the inverse image of any affinoid domain is a finite union of affinoid domains. This is equivalent to saying that the inverse image of a quasi-compact open subset is quasi-compact.

PROPOSITION 1.12. *Let $\varphi\colon X \to Y$ be a morphism of analytic varieties. Then, $\varphi$ is proper if and only if $\varphi$ is separated, without boundary and quasi-compact.*

PROOF. If the morphism $\varphi$ is proper, it is clear that $\varphi$ is without boundary. Moreover, for any affinoid domain $V$ in $Y$, the induced morphism $\varphi^{-1}(V) \to V$ is proper ([BGR]9.6.2/3). Thus $\varphi^{-1}(V)$ is a finite union of affinoid domains. Conversely, assume $\varphi$ to be without boundary and quasi-compact. Let $V$ be an affinoid domain in $Y$, $\{U_i\}_i$ and $\{V_i\}_i$ two affinoid coverings of $\varphi^{-1}(V)$ such that for any $i$, $U_i \subset\subset_V V_i$. We can extract a finite sub-covering $\{U_{il}\}_l$ of $\{U_i\}_i$. The families $\{U_{il}\}_l$ and $\{V_{il}\}_l$ are finite, formed of affinoid domains and such that for any $i$, $U_{il} \subset\subset_V V_{il}$. So, according to ([BGR]9.6.2/3), the morphism $\varphi$ is proper. ■

PROPOSITION 1.13. *Let $\varphi: X \to Y$ and $\psi: Y \to Z$ be two morphisms of analytic varieties such that $\psi \circ \varphi$ is without boundary and $\psi$ is separated. Then $\varphi$ is without boundary.*

PROOF. Let $W$ be an affinoid domain of $Z$. Since $\psi \circ \varphi$ is without boundary, there exists two admissible coverings $\{U_i\}_i$ and $\{V_i\}_i$ of $(\psi \circ \varphi)^{-1}(W) = \varphi^{-1}(\psi^{-1}(W))$ such that for any $i$, $U_i \subset\subset_W V_i$. Let $\{Y_j\}_j$ be an affinoid covering of $\psi^{-1}(W)$. Lemma 1.9 tells us that

$$U_i \times_W Y_j \subset\subset_{Y_j} V_i \times_W Y_j .$$

By a change of basis, the morphism $X \xrightarrow{\delta} X \times_Z Y$ is a closed immersion, so

$$\delta^{-1}(U_i \times_W Y_j) \subset\subset_{Y_j} \delta^{-1}(V_i \times_W Y_j) .$$

The assertion results from the fact that the $\delta^{-1}(U_i \times_W Y_j)$'s form an admissible covering of $Y_j$. ■

PROPOSITION 1.14. *Let $\varphi: X \to Y$ and $\psi: Y' \to Y$ be two morphisms of analytic varieties such that $\varphi$ is without boundary. Then the morphism $X \times_Y Y' \xrightarrow{\varphi'} Y'$ is without boundary.*

PROOF. Let $W$ (resp. $V$) be an affinoid domain of $Y$ (resp. $Y'$) such that $\psi(V) \subset W$ and consider two admissible coverings $\{U_i\}_i$ and $\{V_i\}_i$ of $\varphi^{-1}(W)$ such that $U_i \subset\subset_W V_i$. Lemma 1.9 gives us

$$U_i \times_W V \subset\subset_V V_i \times_W V .$$

When $V$ runs through all the affinoid domains of $\psi^{-1}(W)$, the

$$(U_i \times_W V \subset\subset_V V_i \times_W V)\text{'s}$$

form an admissible covering of $\varphi'^{-1}$. Thus, $\varphi'$ is without boundary. ■

REMARK 1.15. It is not clear that the composition of two morphisms without boundary is again without boundary. But when the morphisms are separated and locally quasi-compact between quasi-separated and paracompact analytic varieties, as shall see later on, the formal approach enables us to show that this is true (4.8).

## 2. Topological properties of quasi-paracompact formal $R$-schemes.

Recall that an $R$-algebra without torsion is called *admissible* if it is a quotient $R\{T_1, \dots, T_n\}/\mathfrak{J}$, where $\mathfrak{J}$ is a finitely generated ideal of $R\{T_1, \dots, T_n\}$. A formal $R$-scheme is said to be *admissible* if it is locally $R$-isomorphic to the affine formal scheme Spf$(A)$ of an admissible $R$-algebra $A$ which is admissible.

DEFINITIONS 2.1. (i) A sheaf $\mathfrak{J}$ of ideals of an admissible formal $R$-scheme $(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ is called *open* if it contains an ideal of definition (i.e. for every open affine subset $\mathcal{U}$ of $\mathcal{X}$ there exists an integer $n$ such that $\pi^{n+1}\mathcal{O}_{\mathcal{X}}(\mathcal{U})$ is contained in $\mathfrak{J}(\mathcal{U})$.

(ii) Let $(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ be an admissible formal $R$-scheme and $\mathfrak{J}$ a coherent open ideal of $\mathcal{O}_{\mathcal{X}}$. *The blowing-up* of the ideal $\mathfrak{J}$ in $\mathcal{X}$ defined as

$$\mathcal{X}' := \varinjlim_{\lambda} \operatorname{Proj} \bigoplus_{n=0}^{\infty} (\mathfrak{J}^n \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}} / \pi^{\lambda+1});$$

it is an admissible formal $R$-scheme. The projection $\varphi: \mathcal{X}' \to \mathcal{X}$ is called *admissible formal blowing-up* of center $\mathcal{Y}$, where $\mathcal{Y}$ is the formal subscheme of $\mathcal{X}$ corresponding to the sheaf of ideal $\mathfrak{J}$ of $\mathcal{O}_{\mathcal{X}}$.

DEFINITION 2.2. We say that a topological space $X$ is of *quasi-compact type (q.c.t.) if it admits* a covering $\{X_i\}_{i\in I}$ of finite type by quasi-compact open subsets (i.e. $X$ is the union of the $X_i$'s and for all $i$ in $I$, $X_i$ meets only a finite number of $X_j$'s). We say that an admissible formal $R$-scheme is of q.c.t. if his underlying topological space is of q.c.t.

REMARKS 2.3. (i) Let $\mathcal{X}$ be an admissible formal $R$-scheme of q.c.t and $\mathfrak{J}$ a coherent open ideal of $\mathcal{O}_{\mathcal{X}}$. If $\mathcal{X}' \to \mathcal{X}$ is the admissible formal blowing-up of $\mathfrak{J}$ in $\mathcal{X}$, then $\mathcal{X}'$ is also an admissible formal $R$-scheme of q.c.t. In fact, the blowing-up operation is of local nature. Thus, if the family $\{\mathcal{Y}_i\}_{i\in I}$ is a covering of $X$ by quasi-compact open subsets and if, for each $i$, we consider the admissible formal blowing-up $\mathcal{Y}_i' \to \mathcal{Y}_i$ of $\mathfrak{J}_{|\mathcal{Y}_i}$ in $\mathcal{Y}_i$, we view $\mathcal{Y}_i'$ as a quasi-compact open sub-scheme of $\mathcal{X}$. The subsets $\mathcal{Y}_i'$ covers $\mathcal{X}'$. Moreover, $\mathcal{Y}_i' \cap \mathcal{Y}_j' = \emptyset$ if $\mathcal{Y}_i$ does not meet $\mathcal{Y}_j$. Thus, we see that $\{\mathcal{Y}_i'\}_{i\in I}$ is a covering of finite type of $\mathcal{X}'$ if $\{\mathcal{Y}_i\}_{i\in I}$ is of finite type.

(ii) Let $X$ be a topological space of q.c.t. Since any closed subset of a quasi-compact space is also quasi-compact, every closed subset of $X$ is of q.c.t. It follows that the connected components and the irreducible components of $X$ are of q.c.t. Assume that $X$ has the following property: the intersection of any two quasi-compact open subsets is also quasi-compact. Then every covering $(X_i)_i$, of $X$ by quasi-compact open subsets, admits a refinement $(X_i')_i$, which is a covering of finite type by quasi-compact open subsets. In fact, let $(X_k)_k$ be a covering of finite type of $X$ by quasi-compact open subsets. Each open subset $X_k$ is covered by a finite number of $X_j$'s. For each $k$, the intersection of $X_j$ and $X_k$ is quasi-compact and they form a covering of finite type of $X$.

PROPOSITION 2.4. *Let $X$ be a topological space.*

(i) *Let $\{X_i\}_{i\in I}$ be a covering of finite type of $X$ by quasi-compact open subsets. Then, the system $\{X_i\}_{i\in I}$ has the following property*:

(*) *There exists a family $\{Z_i\}_{i\in I}$ of quasi-compact closed subsets of $X$ such that*
(P) *for any $i$ of $I$, $X_i \subset Z_i$ and for any $j$ of $I$ such that $X_i \cap X_j = \emptyset$, $Z_i$ does not meet $X_j$.*

(ii) *If $X$ is of q.c.t., then every quasi-compact open subset is contained in a quasi-compact closed subset.*

PROOF. Let $U$ be a quasi-compact open subset of $X$ and $\{X_i\}_{i\in I}$ a covering of finite type. For any $i$, we write $U_i = U \cap X_i$. The open subset $U$ is covered by a finite number of $U_i$'s, therefore $U$ is contained in a finite union of $X_i$. Thus, (i) implies (ii). Write $Y_j = X\backslash X_j$. For any fixed index $i$, the intersection $Z_i$ of the seubsets $Y_j$, such that $X_j$ does not meet $X_i$, is a closed subset contained in the union of $X_i'$ of the $X_k$'s which meet $X_i$. In fact, the complementary of $Z_i$ is the union of the $X_j$'s which not meet $X_i$. As the covering $\{X_i\}_{i\in I}$ is of finite type, $X_i'$ is quasi-component. Since $Z_i$ is a closed subset contained in $X_i'$, it is also quasi-compact. According to the definition of $Y_j$, we have $X_i \subset Y_j$ and therefore $X_i$ is contained in $Z_i$. It is clear that the system $\{Z_i\}_{i\in I}$ satisfies $(P)$. ■

PROPOSITION 2.5. *Let $X$ be a topological space of quasi-compact type. Then,*

(i) *If $X$ is irreducible, it is quasi-compact.*

(ii) *The irreducible closed subsets of $X$ are quasi-compact. In particular, any irreducible component of $X$ is quasi-compact.*

PROOF. (i) If $X$ is irreducible, then every non empty open subset of $X$ is dense. Let $U$ be a non empty open subset of $X$. According to the Proposition 2.4, there exists a quasi-compact closed subset which contains $U$, which is inevitably $X$. Hence, $X$ is quasi-compact.

(ii) If $F$ is an irreducible closed subset of $X$, then $F$ is of q.c.t. and according to (i) it is quasi-compact. ■

COROLLARY 2.6. (i) *A topological space of q.c.t. having a finite number of irreducible components is quasi-compact.*

(ii) *The irreducible components of a locally Noetherian topological space of q.c.t. are quasi-compact and form a locally finite family.*

PROOF. Part (i) is immediate. For (ii), we know from ([BOU] 2.2.4.10), that a Noetherian topological space has a finite number of irreducible components. By (Loc.cit 2.1.4.7) the set of irreducible components of locally Noetherian space is locally finite. ■

COROLLARY 2.7. *Let $X$ be a topological space of q.c.t whose set of irreducible components is locally finite. Then the space $X$ is locally connected and every connected component $C$ of $X$ is a space of q.c.t. Moreover, the components $C$ are caracterized by the following properties:*

($P_1$) *$C$ is a space whose set of irreducible components is locally finite.*

($P_2$) *Any irreducible component which meets $C$ is contained in $C$.*

($P_3$) *If $F$ and $F'$ are two irreducible components of $C$, then there exists a finite sequence $(F_i)_{1 \leqslant i \leqslant n}$ of irreducible components of $C$ such that $F = F_1$, $F_n = F'$ and, $F_{i-1}$ meets $F_i$, for any $i$, $1 \leqslant i \leqslant n$.*

PROOF. The space $X$ is locally connected by ([EGA]0.2.1.5). Let $C$ be a connected component of $X$, $C$ is a closed subset of $X$; therefore of q.c.t. Since any irreducible component is connected, we see that if $F$ is an irreducible component which meets $C$, then $C$ contains $F$. The component $C$ is an union of irreducible components of $X$. Thus, the irreducible components of $C$ form a locally finite set. Property ($P_3$) results from ([EGA] 0.2.1.10).

Conversely, let $C'$ be a subset of $X$ satisfying ($P_1$), ($P_2$) and ($P_3$). According to (loc. cit.), $(P_1)$ and $(P_3)$ imply $C'$ connected. Property $(P_2)$ states that $C'$ is a connected component of $X$. ■

LEMMA 2.8. *Let $X$ be a topological space whose irreducible components are quasi-compact and form a locally finite set. Then,*

(i) *the irreducible components of $X$ form a set of finite type, and*

(ii) *every quasi-compact open subset has a finite number of irreducible components.*

PROOF. For each point $x$ of $X$, we consider an open subset $U_x$ which meets only a finite number of irreducible components of $X$. Let $F$ be an irreducible component of $X$. Since $F$ is quasi-compact, there exists some points $x_1, \ldots, x_n$ of $F$ such that $F$ is contained in the union of the $U_{x_i}$'s. Each $U_{x_i}$ meets only a finite number of irreducible components, therefore, so does $F$. Now, let $U$ be a quasi-compact open subset of $X$, there exists some points $x_1, \ldots, x_m$ of $U$ such that $U$ is contained in the union of all the $U_{x_i}$'s. According to ([BOU], 2.4.7), $U$ has only a finite number of irreducible components. ■

PROPOSITION 2.9. *Let $X$ be a topological space of quasi-compact type whose set of irreducible components is locally finite. Then*

(i) *Every irreducible component meets only a finite number of irreducible components.*

(ii) *Every quasi-compact open subset has only a finite number of irreducible components.*

PROOF. Proposition 2.5 implies that every irreducible component of $X$ is quasi-compact. The claim, then, follows from Lemma 2.8. ■

DEFINITION 2.10. We say that a topological space $X$ *is quasi-paracompact* if any open covering of $X$ has a locally finite refinement. A formal $R$-scheme is quasi-paracompact if its underlying topological space is quasi-paracompact.

Let $\{X_i\}_{i\in I}$ be a family of subsets of a set $X$. Assume that $\{X_i\}_{i\in I}$ covers $X$. For each $i$ in $I$, let $B_i$ be a finite family of subsets of $X_i$, such that $B_i$ covers $X_i$. The union of the $B_i$ is of finite type in the following two cases:

(i) The system $\{X_i\}_{i\in I}$ is of finite type. In fact, a fixed element of $B_i$ can meet an element of $B_j$ only if $X_j$ meets $X_i$. The statement follows by fact that the number of these $X_j$'s is finite.

(ii) For any index $i$ of $I$, each element of $B_i$ meets only a finite number of elements of $\{X_i\}_{i\in I}$.

PROPOSITION 2.11. *Let $X$ be a locally Noetherian topological space. Then $X$ is quasi-paracompact if and only if $X$ is of q.c.t.*

PROOF. Assume that $X$ is quasi-paracompact space. As it is locally Noetherian space, there exists a covering of $X$ by Noetherian open subsets. This covering has a locally finite refinement $(X_i)_i$. Each point $x$ of $X$ belongs to a finite number of $X_j$'s. We can therefore find a quasi-compact open neighbourhood $V_x$ which is contained in each $X_j$ and does not meet the other $X_i$'s. Since each $X_i$ of the system $(X_i)_i$ is quasi-compact, it is covered by a finite number of $V_x$'s. Denote by $B_i$ the finite family composed by these $V_x$'s. The previous remark shows that union of the $B_i$'s is a system of finite type. Hence, we conclude that $X$ is of q.c.t. Conversely, if $X$ is of q.c.t., there exists a covering of finite type $(X_i)_i$ of $X$, of quasi-compact open subsets. If $(Y_j)_j$ is another covering of $X$, then each $X_i$ is covered by finite number of $Y_{i_k}$'s. Write $X_{i_k}'$ for the intersection of $Y_{i_k}$ and $X_i$. By the previous remark, the family of $X_{i_k}'$'s is of finite type and refines the covering $(Y_j)_j$. Thus locally finite. ∎

Let $X$ be a quasi-paracompact formal $R$-scheme. The underlying topological space of any formal affine open sub-scheme of $X$ is Noetherian. Consequently, the underlying topological space of $X$ is locally Noetherian, and $X$ is of q.c.t. Let $(X_i)_i$ be a covering of finite type of $X$ by quasi-compact open subsets. As intersection of two affine open subschemes is quasi-compact, the intersection of two quasi-compact open subsets is also quasi-compact. Thus, any covering $(X_i')_i$ of $X$ by quasi-compact open subsets has a refinement of finite type by quasi-compact open subsets. Every quasi-compact open subset of $X$ is contained in a quasi-compact closed subset and the other way round. The covering $(X_i)_i$ has the property (*) of Proposition 2.4. Every irreducible closed subset of $X$ is quasi-compact, and in particular, each irreducible component of $X$ is quasi-compact and meets only a finite number of irreducible components. Finally, the space $X$ is locally connected and each connected component is characterized by the properties $(P_1)$, $(P_2)$ and $(P_3)$ of Corollary 2.7.

## 3. Extension of a theorem of Raynaud.

In the following all blowing-ups will be formal and admissible (see [BL]).

There exists a functor

$$\mathrm{rig}: (\text{admissible formal } R\text{-schemes}) \to (\text{analytic varieties})$$
$$\mathfrak{X} \to \mathfrak{X}_{\mathrm{rig}},$$

where, $\mathfrak{X}_{\mathrm{rig}}$ is the generic fibre of a formal $R$-scheme $\mathfrak{X}$.

REMARKS 3.1 (see [BL]4.1.*a* and *b*).

(i) The functor rig turns admissible formal blowing-ups into isomorphisms.

(ii) The functor rig is faithful.

DEFINITION 3.2. Let $X$ be an analytic variety. Any admissible formal $R$-scheme $\mathfrak{X}$ whose the generic fibre is isomorphic to $X$, is called an *R-model* (or *model*) of $X$.

Let $X$ be an analytic variety. Does there exist an $R$-model $\mathfrak{X}$ for X? If $X$ is an affinoid variety $\mathrm{Spm}(A_K)$, with $A_K = K\{T_1, \ldots, T_n\}/\mathfrak{J}$, the answer is yes: we take for $\mathfrak{X}$ the formal spectrum $\mathrm{Spf}(A)$ where $A$ is the quotient algebra $R\{T_1, \ldots, T_n\}/\mathfrak{J} \cap R\{T_1, \ldots, T_n\}$. If $X$ is quasi-separated and quasi-compact, the answer is given by a theorem of Raynaud (see [BL] 4.1). In the following, we show that there exists an $R$-model for $X$, for $X$ quasi-separated and paracompact.

LEMMA 3.3. *Let $\mathfrak{X}$ be a quasi-paracompact admissible formal $R$-scheme and $\{\mathfrak{X}_i\}_{i \in I}$ a covering of finite type of $\mathfrak{X}$ by formal open sub-schemes. Let us be given some coherent open ideals $\mathfrak{J}_i$ of $\mathcal{O}_{\mathfrak{X}}$ with $\mathfrak{J} \mid \mathfrak{X}_j = \mathcal{O}_{\mathfrak{X} \mid \mathfrak{X}_j}$ for all $j$ such that, $\mathfrak{X}_j$ does not meet $\mathfrak{X}_i$. Then, there exists an unique ideal $\mathfrak{J}$ of $\mathcal{O}_{\mathfrak{X}}$ such that, for each i of I, $\mathfrak{J}_{\mid \mathfrak{X}_i}$ is the product $\mathfrak{J}_i'$ of $\mathfrak{J}_{j \mid \mathfrak{X}_i}$, where $\mathfrak{X}_j$ meets $\mathfrak{X}_i$. The ideal $\mathfrak{J}$ is open and coherent.*

PROOF. If $\mathfrak{J}$ exists, by local considerations it is unique, open and coherent. Now, to prove the existence of $\mathfrak{J}$, it remains to check that for any couple $(i, j)$ of $I^2$ such that $\mathfrak{X}_i$ meets $\mathfrak{X}_j$, we have

$$\mathfrak{J}_i' \mid_{\mathfrak{X}_i \cap \mathfrak{X}_j} = \mathfrak{J}_j' \mid_{\mathfrak{X}_i \cap \mathfrak{X}_j}.$$

But it is easy to see that for any open subset $U$ of $\mathfrak{X}_j \cap \mathfrak{X}_i$, $\mathfrak{J}'_{i(U)} = \mathfrak{J}'_{j(U)}$. The ideals $\mathfrak{J}'_{i(U)}$ and $\mathfrak{J}'_{j(U)}$ are simply the products of $\mathfrak{J}_{k(U)}$ where $\mathfrak{X}_k$ meets both $\mathfrak{X}_j$ and $\mathfrak{X}_i$. ■

LEMMA 3.4. *Let $\mathfrak{X}$ be a quasi-paracompact formal R-scheme and $\{\mathfrak{U}_i\}_{i\in I}$ a covering of finite type of $\mathfrak{X}$ by quasi-compact open sub-schemes. For any i, let $\mathfrak{J}_i$ be a coherent open ideal of $\mathcal{O}_{\mathfrak{U}_i}$ and $\varphi_i\colon \mathfrak{Y}'_i \to \mathfrak{U}_i$ be the admissible formal blowing of $\mathfrak{J}_i$ in $\mathfrak{U}_i$. Then, each $\varphi_i$ extends to an admissible formal blowing-up $\psi_i\colon \mathfrak{X}'_i \to \mathfrak{X}$. Furthermore, there is an admissible formal blowing-up $\psi\colon \mathfrak{X}' \to \mathfrak{X}$ which extends each $\psi_i$.*

PROOF. Proposition 2.4 exhibits a family $\{F_i\}_{i\in I}$ of quasi-compact closed subsets of $\mathfrak{X}$, such that, for each index $i$ of $I$, $\mathfrak{U}_i \subset F_i$ and for any index $j$ of $I$ such that $\mathfrak{U}_i \cap \mathfrak{U}_j = \emptyset$, $F_i$ does not meet $\mathfrak{U}_j$. For any $i$, denote by $\mathfrak{V}_i$ the open subset $F_i^c$.

Let $i \in I$ and $n$ an integer such that $\pi^{n+1} \in \mathfrak{J}_i$, denote by $\mathfrak{J}'_i$ the reduction of $\mathfrak{J}_i$ modulo $\pi^{n+1}$ and $X_i$ the ordinary scheme after reduction. Note that the underlying topological space of $\mathfrak{X}_i$ is the same as the underlying topological space of $\mathfrak{X}$; moreover it is locally Noetherien. We extend $\mathfrak{J}'_i$ to the disjoint union of $\mathfrak{U}_i$ and $\mathfrak{V}_i$ by $\mathcal{O}_{\mathfrak{X}_i}$ on $\mathfrak{V}_i$. The statement of ([EGA], I.6.9.7) enables us to extend $\mathfrak{J}'_i$ to a coherent ideal over $\mathfrak{X}_i$. Thus, $\mathfrak{J}_i$ extends to a coherent open ideal $\mathfrak{J}_i^{\wedge} := h^{-1}(\mathfrak{J}'_i)$ of $\mathfrak{X}$, where the map $h\colon \mathcal{O}_{\mathfrak{X}} \to \mathcal{O}_{\mathfrak{X}_i}$ is the canonical morphism. Since blowing-up is of local nature, the admissible formal blowing-up $\psi_i\colon \mathfrak{X}'_i \to \mathfrak{X}$ of $\mathfrak{J}_i^{\wedge}$ extends the blowing-up $\varphi_i$. Lemma 3.3 gives us the existence of a coherent open ideal $\mathfrak{J}$, such that $\mathfrak{J}|_{U_i}$ is the product of the $\mathfrak{J}_j^{\wedge}|_{U_i}$, for the open subsets $\mathfrak{U}_j$ which meet $\mathfrak{U}_i$. Hence the blowing-up $\psi\colon \mathfrak{X}' \to \mathfrak{X}$ of $\mathfrak{J}$ in $\mathfrak{X}$ extends all the $\psi_i$'s. ■

PROPOSITION. 3.5. *Let $X$ be a quasi-compact analytic variety and $\{U_i\}_{i\in I}$ a finite family of quasi-compact admissible open subsets of $X$.*

(i) *There exists a quasi-compact R-model $\mathfrak{X}$ of $X$ and a finite family $\{\mathfrak{U}_i\}_{i\in I}$ of quasi-compact formal open sub-schemes of $\mathfrak{X}$ such that the family $\{\mathfrak{U}_{i,\mathrm{rig}}\}_{i\in I}$ coincides with $\{U_i\}_{i\in I}$. If $\{U_i\}_{i\in I}$ covers $X$ then $\{\mathfrak{U}_i\}_{i\in I}$ covers $\mathfrak{X}$.*

(ii) *Let $\mathfrak{X}$ be an R-model of $X$. If $\{\mathfrak{U}_i\}_{i\in I}$ is a finite family of admissible formal R-schemes such that each $\mathfrak{U}_i$ is an R-model of $U_i$, then there exists a blowing-up $\mathfrak{X}' \to \mathfrak{X}$ of $\mathfrak{X}$ and a finite family $\{\mathfrak{V}_i\}_{i\in I}$ of* *quasi-compact formal open sub-schemes of $\mathcal{X}'$, such that $\mathcal{V}_i$ is a blowing-up of $\mathcal{U}_i$, for each $i \in I$.*

PROOF. Raynaud's theorem gives existence of an $R$-model $\mathcal{Y}$ for $X$. The space $\mathcal{X}$ in (i) is obtained by blowing-up in $\mathcal{Y}$ (see [BL]4.4). We now prove (ii): the [BL] 4.4 gives a blowing-up $\mathcal{X}''$ of $\mathcal{X}$ and a family $\{\mathcal{W}_i\}_{i \in I}$ of formal open sub-schemes of $\mathcal{X}''$ sent by the functor rig onto $\{U_i\}_{i \in I}$. For each $i \in I$, we can, by [BL] 4.1.c, dominate $\mathcal{U}_i$ and $\mathcal{W}_i$ by a third model $\mathcal{V}_i$. Considering $\mathcal{V}_i$ as the blowing-up of $\mathcal{W}_i$, the Lemma 2.6 of [BL] applied to $\mathcal{X}''$ shows that there exists a blowing-up $\mathcal{X}'$ of $\mathcal{X}$ and we can consider the $\mathcal{V}_i$'s as formal open sub-schemes of $\mathcal{X}'$. ■

LEMMA 3.6. *Let $X$ be a quasi-separated and paracompact analytic variety and $\{X_i\}_{i \in I}$ an admissible covering of finite type of $X$ by quasi-compact open subsets. Then,*

(i) *there exists an $R$-model $\mathcal{X}$ for $X$ which is a quasi-paracompact admissible formal scheme, and*

(ii) *there is a covering of finite type $\{\mathcal{X}_i'\}_{i \in I}$ of $\mathcal{X}$ by open sub-schemes, such that the family $\{\mathcal{X}'_{i,\mathrm{rig}}\}_{i \in I}$ coincides with $\{X_i\}_{i \in I}$.*

PROOF. To build the model $\mathcal{X}$ we proceed as follows:

(i) For each $i$, we can find a model $\mathcal{X}_i$ of $X_i$ and formal open sub-schemes $\mathcal{X}_{ij}$ of $\mathcal{X}_i$ (for each $j$ such that $X_i \cap X_j \neq \emptyset$) such that $\mathcal{X}_{ij}$ is a model of $X_{ij} = X_i \cap X_j$ (Proposition 3.5).

(ii) For $i$ and $j$ fixed, we can find blowing-ups $\mathcal{X}_{ij}'$ of $\mathcal{X}_{ij}$ and $\mathcal{X}_{ji}'$ of $\mathcal{X}_{ji}$ such that $\mathcal{X}_{ij}' = \mathcal{X}_{ji}'$ ([BL]4.1.c).

(iii) For $i, j$ fixed, as in (i) we construct a model $\mathcal{X}_{ij}''$ of $X_{ij}$ and a formal open sub-scheme $\mathcal{X}_{ijk}$ of $\mathcal{X}_{ij}'$ such that $\mathcal{X}_{ij}''$ is a blowing-up of $\mathcal{X}_{ij}'$ and $\mathcal{X}_{ijk}$ is a model of $X_{ijk} = X_i \cap X_j \cap X_k$ (Proposition 3.5 and [BL]4.1.c).

(iv) As in (ii), for $i, j, k$, fixed, we can blow-up $\mathcal{X}_{ijk}$ to $\mathcal{X}_{ijk}'$ in order to obtain $\mathcal{X}_{ijk}' = \mathcal{X}_{ikj}'$.

(v) Fix $i$, $j$ again and we blow-up $\mathcal{X}_{ijk}''$ to $\mathcal{X}_{ijk}'''$ in order to see the $\mathcal{X}_{ijk}'$'s as a formal open sub-schemes of $\mathcal{X}_{ij}'''$ (Lemma 3.4). Similarly, for each $i$, blow-up $\mathcal{X}_i$ to $\mathcal{X}_i'$ in order to see the $\mathcal{X}_{ij}'''$'s as formal open sub-schemes of $\mathcal{X}_i'$. The space $\mathcal{X}$, obtained by glueing the $\mathcal{X}_i'$'s along the $\mathcal{X}_{ij}'''$'s, is a quasi-paracompact admissible formal $R$-scheme. It is clear that the functor rig applied to $\mathcal{X}$ gives the analytic variety $X$. Furthermore the covering $\{\mathcal{X}_i'\}_{i\in I}$ is of finite type and is mapped by the functor rig onto $\{X_i\}_{i\in I}$. ■

PROPOSITION 3.7. *Let $X$ be a paracompact and quasi-separated analytic variety and $\mathcal{X}$ an $R$-model for $X$. Let $\{X_i\}_{i\in I}$ be a locally finite family composed of quasi-compact open subsets of $X$ and $\{\mathcal{Y}_i\}_{i\in I}$ be a family of quasi-compact formal $R$-schemes sent onto $\{X_i\}_{i\in I}$ by the functor rig. Then there exists a blowing-up $\mathcal{X}'$ of $\mathcal{X}$ and a family of formal sub-schemes $\{\mathcal{Y}_i'\}_{i\in I}$ of $\mathcal{X}'$ such that for each $i\in I$, $\mathcal{Y}_i'$ is a blowing-up of $\mathcal{Y}_i$. If $\{X_i\}_{i\in I}$ is a covering of finite type, so the family $\{\mathcal{Y}_i'\}_{i\in I}$.*

PROOF. Choose a family of finite type which contains $\{X_i\}_{i\in I}$. In order to get the blowing-up $\mathcal{X}'$ of $\mathcal{X}$, we proceed as for the proof of Lemma 3.6, taking in step (i), $\mathcal{X}_i$ as a blowing-up of the formal $R$-scheme $\mathcal{Y}_i$. ■

REMARK 3.8. Let $X$ be a quasi-separated analytic variety that is not paracompact but has a covering of finite type by quasi-compact open subsets. As in lemma 3.6, there exists a quasi-paracompact admissible formal scheme $\mathcal{X}$ such that $\mathcal{X}_{\mathrm{rig}} = X$. Thus there exists admissible formal schemes which are quasi-paracompact but whose generic fibre is not paracompact. For example: Let $(D_n)_n$ be a countable family of copies of the unit disc and $X$ the glueing of all the $D_n$'s along the same open disc of radius less than 1. Then $X$ admits a covering of finite type but it is not paracompact.

DEFINITION 3.9. Let $\mathcal{X}$ be an admissible formal $R$-scheme and $\{\mathcal{X}_i\}_{i\in I}$ be a covering of $\mathcal{X}$ by quasi-compact formal open sub-schemes. Say that $\{\mathcal{X}_i\}_{i\in I}$ is *admissible* if for any blowing-up $\mathcal{X}'\to\mathcal{X}$ and any quasi-compact formal open sub-scheme $\mathcal{U}$ of $\mathcal{X}'$ there exists a commutative diagram

$$\begin{tikzcd}
\mathcal{U} \arrow[r, hook] & \mathcal{X}' \arrow[r] & \mathcal{X} \\
\mathcal{U}'' \arrow[u] \arrow[r, hook] & \mathcal{X}_{\mathcal{U}} \arrow[r, hook] & \mathcal{X}'' \arrow[u] \arrow[r, "="] & \mathcal{X}'' \arrow[u]
\end{tikzcd}$$

where $\mathcal{X}''$ is an admissible formal $R$-scheme which dominates both $\mathcal{X}$ and $\mathcal{X}'$, $\mathcal{U}''$ is a blowing-up of $\mathcal{U}$ and $\mathcal{X}_{\mathcal{U}}$ is a formal open sub-scheme of $\mathcal{X}''$ which is a blowing-up of a finite union of $\mathcal{X}_i$.

DEFINITION 3.10. We say that a quasi-paracompact admissible formal $R$-scheme is *generically paracompact* if it admits a blowing-up with an admissible covering of finite type.

LEMMA 3.11. *Let $X$ be a quasi-separated analytic variety admitting a covering of finite type $\{X_i\}_{i\in I}$ by quasi-compact open subsets. Let $\mathcal{X}$ be a model of $X$ and $\{\mathcal{X}_i\}_{i\in I}$ a covering of $\mathcal{X}$ by formal open sub-schemes such that, for each index $i$, $\mathcal{X}_i$ is a model of $X_i$. Then $\{X_i\}_{i\in I}$ is an admissible covering if and only if $\{\mathcal{X}_i\}_{i\in I}$ is an admissible covering.*

PROOF. If $\{X_i\}_{i\in I}$ is an admissible covering, then so is $\{\mathcal{X}_i\}_{i\in I}$. In fact, if $\mathcal{U}'$ is a quasi-compact formal open sub-scheme of a blowing-up $\mathcal{X}'$ of $\mathcal{X}$, then $\mathcal{U}'_{\mathrm{rig}} = U$ is a quasi-compact open subset of $X$. As $X$ is quasi-separated, we can assume $\mathcal{U}$ to be a finite union of $X_i$'s. Therefore Proposition 3.7 infers that $\{\mathcal{X}_i\}_{i\in I}$ is an admissible covering. Conversely, assume that $\{\mathcal{X}_i\}_{i\in I}$ is an admissible covering. Then, according to Proposition 3.7, for any affinoid domain $V$ of $X$, there exists a model $\mathcal{V}$ of $V$ and a blowing-up $\mathcal{X}'$ of $\mathcal{X}$ such that $\mathcal{V}$ is a formal open sub-scheme of $\mathcal{X}'$. Since $\{\mathcal{X}_i\}_{i\in I}$ is an admissible covering, there exists a blowing-up $\mathcal{V}'\to\mathcal{V}$ with $\mathcal{V}'_{\mathrm{rig}}$ contained in a finite union of $X_i$'s. Therefore, $\{X_i\}_{i\in I}$ is an admissible covering. ∎

REMARK 3.12. It follows from Lemma 3.6 and 3.11 that the blowing-up of a generically paracompact admissible formal $R$-scheme is also generically paracompact.

For our next statement, we need the notion of localization of a category:

Let $\mathcal{C}$ be a category and $M$ be a class of morphisms of $\mathcal{C}$. The localization of $\mathcal{C}$ with respect to $M$ is a category $\mathcal{C}_M$, together with a functor $\mathcal{C}\to\mathcal{C}_M$ such that, for any functor $\mathcal{C}\to\mathcal{C}'$ that turns morphisms of $M$ onto isomorphisms of $\mathcal{C}'$, there is an unique functor $\mathcal{C}_M\to\mathcal{C}'$ that makes the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{C}_M \arrow[dl] \\ & \mathcal{C}' & \end{tikzcd}$$

commutative. The category $\mathcal{C}_M$ is unique. Of course, uniqueness and commutativity are meant up to equivalence.

REMARK 3.13. (i) If $M$ is a system of morphisms in $\mathcal{C}$, there is a natural equivalence between the localization of $\mathcal{C}$ with respect to $M$, and the localization of $\mathcal{C}$ with respect to the system of finite compositions of morphisms in $M$.

(ii) One can prove (see [B.L]2.5) that if $\mathcal{X}$ is a quasi-compact formal $R$-scheme, the composition of two admissible blowing-ups is an admissible blowing-up. It seems that this result is no longer true when we substitute "quasi-paracompact" for "quasi-compact", but no-counter example is known to the author.

THEOREM 3.14. *The functor rig:* $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{rig}}$ *gives rise to an equivalence between:*

(i) *The category of generically paracompact admissible formal $R$-schemes localized with respect to formal admissible blowing-ups, and*

(ii) *The category of paracompact and quasi-separated analytic varieties.*

PROOF. *Let* $\mathcal{X}$ be a quasi-paracompact formal $R$-scheme. Since the underlying topological space of $\mathcal{X}$ is locally Noetherian, $\mathcal{X}_{\mathrm{rig}}$ is quasi-separated. If furthermore, $\mathcal{X}$ is generically paracompact, then $\mathcal{X}_{\mathrm{rig}}$ is a paracompact quasi-separated analytic variety (Lemma 3.11). Conversely, if $X$ is a paracompact quasi-separated analytic variety, Lemma 3.6 constructs a quasi-paracompact $R$-model $\mathcal{X}$ of $X$ and Lemma 3.11 tells us that this $R$-model is generically paracompact. It remains to prove that the functor rig is fully faithful.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two generically paracompact admissible formal $R$-schemes and $\varphi\colon Y := \mathcal{Y}_{\mathrm{rig}} \to X := \mathcal{X}_{\mathrm{rig}}$ a morphism between associated analytic varieties. We prove that there is a blowing-up $\mathcal{Y}'$ of $\mathcal{Y}$ and a morphism $\psi\colon \mathcal{Y}' \to \mathcal{X}$ of formal schemes such that $\psi_{\mathrm{rig}} = \varphi$. Choose a covering of finite type $(\mathcal{Y}_i)_{i \in I}$ of $\mathcal{Y}$ by quasi-compact open subsets. For any $i \in I$, apply the Raynaud's theorem to $\mathcal{Y}_i$. Then, there exists a blowing-up $\mathcal{Y}_i' \to \mathcal{Y}_i$ together with a morphism $\varphi_i\colon \mathcal{Y}_i' \to \mathcal{X}$ such that $\varphi_{i,\,rig} = \varphi_{|\mathcal{Y}_{i,\,\mathrm{rig}}}$. Lemma 3.4 yields a blowing-up $\psi'\colon \mathcal{Y}' \to \mathcal{Y}$ such that, if we denote by $\psi_i\colon \psi'^{-1}(\mathcal{Y}) \to \mathcal{X}$ the composition of $\varphi_i$ with the unique morphism $\psi'^{-1}(\mathcal{Y}_i) \to \mathcal{Y}_i'$ deduced from blowing-up $\mathcal{Y}_i' \to \mathcal{Y}_i$ and $\psi'^{-1}(\mathcal{Y}_i) \to \mathcal{Y}_i$, then $\psi_i$ satisfies $\psi_{i,\mathrm{rig}} = \varphi_{i,\mathrm{rig}} = \varphi_{|\mathcal{Y}_{i,\mathrm{rig}}}$. Therefore, glueing the morphisms $\psi_i$ gives a morphism $\psi\colon \mathcal{Y}' \to \mathcal{X}$ which satisfies $\psi_{\mathrm{rig}} = \varphi$ (by the Remark 3.1).

Let $\mathcal{C}$ be the category of generically formal $R$-scheme and $\mathcal{C}_M$ the category localized of $\mathcal{C}$ with respect to the system $M$ of finite compositions of admissible formal blowing-ups. The functor rig of $\mathcal{C}_M$, in the category $\mathcal{C}'$ of paracompact quasi-separated analytic varieties, is faithful. This is a direct consequence of Remark 3.1 and of the fact that $\mathrm{Hom}_{\mathcal{C}_M}(\mathcal{X}, \mathcal{Y})$ is the inverse limit of $\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}', \mathcal{Y})$ where $\mathcal{X}'$ runs through all the formal $R$-schemes obtained after a finite number of admissible formal blowing-ups of $\mathcal{X}$ (see [H]3.1). The categories $\mathcal{C}_M$ and $\mathcal{C}'$ are equivalent. The theorem follows from Remark 3.13. ■

## 4. Morphisms without boundary of admissible formal $R$-schemes.

We assume, for technical reasons, that the valuation is discrete.

Recall a result of Lütkebohmert ([L]) needed in Lemma 4.6.

LEMMA 4.1 ([L]2.5). *Let $\varphi\colon \mathcal{V} = \mathrm{Spf}(B) \to \mathcal{W} = \mathrm{Spf}(A)$ be a morphism of affine formal R-schemes and $\mathcal{U}$ a formal open sub-scheme of $\mathcal{V}$. The affinoid domain $\mathcal{U}_{\mathrm{rig}}$ is relatively compact in $\mathcal{V}_{\mathrm{rig}}$ over the affinoid space $\mathcal{W}_{\mathrm{rig}}$ if and only if the Zariski closure of $\mathcal{U}_0$ (the special fibre of $\mathcal{U}$) in $\mathcal{V}_0$ is proper over $\mathcal{W}_0$.* ■

We recall (see [EGA]III.1.(3.4.1)) that a morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of formal $R$-schemes is said to be *proper* if the associated morphism $\varphi_0\colon \mathcal{X}_0 \to \mathcal{Y}_0$, between the ordinary schemes $\mathcal{X}_0$ and $\mathcal{Y}_0$, is proper.

A theorem of W.Lütkebohmert ([L]3.1) states that a morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of quasi-compact admissible formal $R$-schemes is proper if and only if $\varphi_{\mathrm{rig}}$ is proper. We shall extend this result to morphisms without boundary which are locally quasi-compact.

DEFINITION 4.2. Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of admissible formal $R$-schemes. We say that the morphism $\varphi$ is *without boundary* if the associated morphism $\varphi_0\colon \mathcal{X}_0 \to \mathcal{Y}_0$ is universally closed.

REMARKS 4.3. Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of admissible formal $R$-schemes.

(i) The morphism $\varphi$ is separated if and only if the associated morphism $\varphi_0\colon \mathcal{X}_0 \to \mathcal{Y}_0$ is separated (see [EGA]I.10.15.2). From ([BL]4.7) we deduce that the morphism $\varphi_{\mathrm{rig}}\colon \mathcal{X}_{\mathrm{rig}} \to \mathcal{Y}_{\mathrm{rig}}$ is separated if and only if the morphism $\varphi_0\colon \mathcal{X}_0 \to \mathcal{Y}_0$ is separated.

(ii) If $\varphi'\colon \mathcal{X}' \to \mathcal{Y}'$ is another model of the morphism $\varphi_{\mathrm{rig}}\colon \mathcal{X}_{\mathrm{rig}} \to \mathcal{Y}_{\mathrm{rig}}$, then $\varphi$ is without boundary if and only if $\varphi'$ is. In fact, let $\varphi''\colon \mathcal{X}'' \to \mathcal{Y}''$ be a model of $\varphi_{\mathrm{rig}}$ such that $\mathcal{X}''$ (resp. $\mathcal{Y}''$) is a model which dominates $\mathcal{X}$ and $\mathcal{X}'$ (resp. $\mathcal{Y}$ and $\mathcal{Y}'$). It results from ([EGA]II.5.4.(i)) that the morphism $\varphi''$ is proper (resp. without boundary) if and only if $\varphi'$ is proper (resp. without boundary) (and the same holds for the morphism $\varphi$).

(iii) Let $\mathcal{X}$ be a separated admissible formal $R$-scheme. If $\mathcal{X}$ is quasi-paracompact, we know that the irreducible components of $\mathcal{X}$ are quasi-compact and form a locally finite system. Furthermore, applying ([EGA]II, 5.4.5 and 5.4.9), shows that $\mathcal{X}$ is without boundary if and only if its irreducible components are proper. Although we do not have a counter example, it seems that if $\mathcal{X}$ is an $R$-scheme whose irreducible components are quasi-compact and form a locally finite system, then $\mathcal{X}$ is not necessarily quasi-paracompact. ■

DEFINITIONS 4.4. (i) We call a rigid morphism $\varphi\colon X \to Y$ *locally quasi-compact*, if for any affinoid domain $W$ of $Y$ and any affinoid domain $V$ of $X$, the open subset $\varphi^{-1}(W) \cap V$ of $X$ is quasi-compact.

(ii) We say that a morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of admissible formal $R$-schemes is *locally quasi-compact* if for any affine open subset $U$ (resp. $V$) of $Y$ (resp. $X$), the open subset $V \cap \varphi^{-1}(U)$ is quasi-compact.

REMARKS 4.5. Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a model of a morphism $\varphi_K\colon X \to Y$ of paracompact quasi-separated analytic varieties.

(i) The morphism $\varphi$ is locally quasi-compact if and only if $\varphi_k$ is locally quasi-compact.

(ii) If $Y'$ (resp. $\mathcal{Y}'$) is a quasi-compact open subset of $Y$ (resp. of $\mathcal{Y}$), then $\varphi_K^{-1}(Y')$ (resp. $\varphi^{-1}(\mathcal{Y}')$) is paracompact.

LEMMA 4.6. *Let $\varphi\colon \mathcal{X} \to \mathcal{Y}$ be a locally quasi-compact separated morphism of quasi-paracompact admissible formal $R$-schemes and $\varphi_K\colon X \to Y$ the associated rigid morphism. If $\varphi$ is without boundary then, for any affinoid domain $W \subset Y$ and affinoid domain $U \subset \varphi^{-1}(W)$, there exists an affinoid domain $V \subset \varphi^{-1}(W)$ such that $U \subset\subset_W V$.*

PROOF. Since $\varphi$ is locally quasi-compact we can assume that $\mathcal{Y}$ is affine. Let $U$ be an affinoid domain in $X$, there exists a blowing-up $\mathcal{X}' \to \mathcal{X}$ and a quasi-compact open subset $\mathcal{U}$ of $\mathcal{X}'$ which is a model of $U$. We have proved in Proposition 2.4 that there exists a quasi-compact closed subset $F$ of $\mathcal{X}'$ which contains $\mathcal{U}$. Thus, the Zariski closure $\mathcal{U}_0'$ of $\mathcal{U}_0$ is quasi-compact. Since the morphism $\mathcal{X}_0' \to \mathcal{Y}_0$ is universally closed and locally finite, the morphism $\mathcal{U}_0' \to \mathcal{Y}_0$ is proper. Let $\mathcal{W}'$ be a quasi-compact open subset of $\mathcal{X}'$ which contains $F$. The induced morphism $\mathcal{W}' \to \mathcal{Y}$ is separated. Then, according to ([L]5.1), there exists a blowing-up $\mathcal{W}'' \to \mathcal{W}'$ and two formal open sub-schemes $\mathcal{U}'' \subset \mathcal{V}'$ of $W''$ such that $\mathcal{U}''$ (resp. $\mathcal{V}'$) is a model for $U$ (resp. for an affinoid domain $V$ in $X$) and the Zariski closure of $\mathcal{U}_0''$ in $\mathcal{W}_0''$ is proper over $\mathcal{Y}_0$. Lemma 4.1 shows that $U \subset\subset_Y V$. ■

THEOREM 4.7. *Let $\varphi_K \colon X \to Y$ be a separated rigid morphism of paracompact quasi-separated varieties. Suppose that $\varphi_K$ is locally quasi-compact and let $\varphi \colon \mathcal{X} \to \mathcal{Y}$ be a model of $\varphi_K$. Then the morphism $\varphi_K$ is without boundary if and only if $\varphi$ is without boundary.*

PROOF. We first assume that $\varphi_K$ is without boundary and prove that $\varphi_0$ is universally closed. Since $\varphi$ is locally quasi-compact, we may assume that $Y$ is affine. The morphism $\varphi_K$ being without boundary, there exists two coverings $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ of $X$ by affinoid domains such that $U_i \subset\subset_Y V_i$. We can assume that the families $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ to be of finite type. Up to a blowing-up of $\mathcal{X}$ (3.7), we can find two coverings $\{\mathcal{U}_i\}_{i\in I}$ and $\{\mathcal{V}_i\}_{i\in I}$ of $\mathcal{X}$ such that for each $i$, $\mathcal{U}_i$ (resp. $\mathcal{V}_i$) is a model of $U_i$ (resp. $V_i$). Since $U_i \subset\subset_Y V_i$, the Zariski closure $\mathcal{U}_{i,0}'$ of $\mathcal{U}_{i,0}$ in $\mathcal{V}_{i,0}$ is proper on $\mathcal{Y}_0$. Using ([EGA]II,5.4.5 and 5.4.9), we conclude that the morphism $\mathcal{X}_0 \to \mathcal{Y}_0$ is universally closed and thereby $\varphi$ is without boundary. Conversely, if $\varphi$ is without boundary, it follows from the previous lemma that $\varphi_K$ is without boundary. ■

REMARK 4.8. Using Theorem 4.7, we obtain that the composition of two separated, locally quasi-compact morphisms without boundary between paracompact and quasi-separated analytic varieties, is without boundary.

PROPOSITION 4.9. *Let $X$ be a paracompact separated analytic variety. If $X$ is without boundary then, for any affinoid domain $U$ of $X$, there exists another affinoid domain $V$ such that $U \subset\subset V$.*

PROOF. The map $X \to \mathrm{Spm}(K)$ is a locally compact morphism and the statement follows from Lemma 4.6. ■

COROLLARY 4.10. *Let $X$ be a paracompact separated analytic variety without boundary and $F$ a Zariski closed subset of $X$. Then $X \backslash F$ is also without boundary.*

PROOF. Corollary 4.9 says that $X$ is without boundary in Lütkebohmert's definition ([L]5.9) (i.e. for all affinoid domain $U$ of $X$, there exists another affinoid domain $V$ such that $U \subset\subset V$). Then, for any Zariski closed subset $F$, the subset $X \backslash F$ is without boundary in Lütkebohmert's definition (5.10) (this result is stated only for quasi-compact varieties but remains true for paracompact varieties) and thereby, it is without boundary with our definition. ■

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