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DIEGO AVERNA

SALVATORE A. MARANO

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Existence of Solutions for Operator Inclusions: a Unified Approach (*).

DIEGO AVERNA (**) - SALVATORE A. MARANO (***)

ABSTRACT - For a class of operator inclusions with multivalued terms fulfilling mixed semicontinuity hypotheses, the existence of solutions is established by chiefly using Bressan and Colombo's directionally continuous selection theorem as well as Ky Fan's fixed point principle. An application to the Cauchy problem is then performed.

Introduction.

Let I be a compact real interval, let F be a closed-valued multifunction from $I \times \mathbb{R}^n$ into $\mathbb{R}^n$, and let $(t_0, x_0) \in I \times \mathbb{R}^n$. A function $u\colon I \to \mathbb{R}^n$ is called a solution of the multivalued Cauchy problem

$$\text{(P}'\text{)} \qquad \begin{cases} x' \in F(t, x) & \text{in } I, \\ x(t_0) = x_0 \end{cases}$$

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(**) Indirizzo dell'A.: Dipartimento di Matematica ed Applicazioni, Facoltà di Ingegneria, Università di Palermo, Viale delle Scienze, 90128 Palermo, Italy. E-mail: averna@mbox.unipa.it
(***) Indirizzo dell'A.: Dipartimento di Matematica, Università di Catania, Viale A. Doria 6, 95125 Catania, Italy. E-mail: marano@dipmat.unict.it

provided it is absolutely continuous, $u'(t) \in F(t, u(t))$ for almost every $t \in I$, and $u(t_0) = x_0$.

The literature concerning (P′) varies considerably in its assumptions about regularity of F; see [1,18,10] and the references given there. Nevertheless, the main existence theorems essentially are as follows. Suppose the multifunction F satisfies some kind of integrable boundedness (which may change from one paper to another). Then problem (P′) has a solution if either

(h_1) F takes convex values, $F(\cdot, x)$ is measurable, while $F(t, \cdot)$ is upper semicontinuous, or

(h_2) F is measurable in (t, x) and lower semicontinuous with respect to x.

A number of attempts have been made to unify these two approaches, for instance, by introducing mixed semicontinuity conditions [17,13,12,19]. Himmelberg and Van Vleck's result [12, Theorem A] extends the pioneering work by Olech [17] and requires that

(h_3) $F(\cdot, x)$ is measurable, while $F(t, \cdot)$ has a closed graph. Moreover, for each (t, x), either $F(t, x)$ is convex or $F(t, \cdot)$ is lower semicontinuous at x.

The global version of Lojasiewicz's result [13, Theorem 1] (see also the paper by Tolstonogov [19]) assumes that

(h_4) F is measurable in (t, x). Furthermore, for almost every t and all x, either $F(t, x)$ is convex and $F(t, \cdot)$ has a closed graph at x or $F(t, \cdot)$ is lower semicontinuous on some neighbourhood of x.

Obviously, (h_3) includes (h_1) but is independent of (h_2), whereas (h_4) generalizes (h_2) and is independent of (h_1). It should then be noted that the proofs of such results are rather involved.

For autonomous F, another contribution has recently been performed by Deimling [10, Corollary 6.4] through simpler arguments based on a technique, previously developed by Bressan in [7], which employs directionally continuous selections from lower semicontinuous multifunctions. The same author asked [10, Problem 4, p. 75] (see besides [18, p. 148]) whether it is possible to establish the above-mentioned theorems in the spirit of his easier proof.

The present paper tries to place and solve the question within a more abstract framework to which Bressan's idea can still be adapted. Accordingly, here, we consider a problem of the type

$$(\mathrm{P}_F) \qquad \begin{cases} u \in U\,, \\ \Psi(u)(t) \in F(t,\, \Phi(u)(t)) \quad \text{in } I\,, \end{cases}$$

where U is a nonempty set, F now denotes a closed-valued multifunction from $I \times X$ into Y, X and Y being separable real Banach spaces, while Φ and Ψ indicate operators defined on U and taking relevant Bochner integrable functions as values. A point $u \in U$ is called a solution of (P_F) provided $\Psi(u)(t) \in F(t,\, \Phi(u)(t))$ for almost every $t \in I$.

Supposing that F fulfils mixed semicontinuity conditions, we first construct a convex closed-valued integrably bounded multifunction $(t, x) \mapsto G(t, x)$ with the following properties: $G(\cdot, x)$ is measurable; $G(t, \cdot)$ has a closed graph; each solution to the problem

$$(\mathrm{P}_G) \qquad \begin{cases} u \in U\,, \\ \Psi(u)(t) \in G(t,\, \Phi(u)(t)) \quad \text{in } I\,, \end{cases}$$

satisfies (P_F) as well. From a technical point of view, it probably represents the most difficult step of the work and is achieved in Theorem 2.1. Next, assuming the space Y finite-dimensional and using a modified version (Theorem 2.2) of a result by Naselli Ricceri and Ricceri [16, Theorem 1], which chiefly goes back to Ky Fan's fixed point principle, we solve (P_F) through (P_G); see Theorem 2.3.

Since the hypotheses about Φ and Ψ are general enough to comprise in (P_F) both (P') and several other known problems, this result meaningfully specializes whenever we make suitable choices of the above operators. Specifically, concerning (P'), Theorem 2.3 yields at once a result (Theorem 2.4) that improves the global version of [13, Theorem 1] and, in non-mixed situations, requires either (h_1) or (h_2) only. Moreover, from Theorem 2.3 it is possible to deduce Theorem A in [12] and hence the result of [17, p. 190]; see Theorem 2.5.

Naturally, Theorem 2.1 could also be used together with other existence results for specific problems having a right-hand side like G.

The paper is organized into four sections, including the Introduction. Notations, definitions, and preliminary results are collected in Section 1. Basic assumptions and statements of the main theorems, as well as some special cases sufficiently interesting to be explicitly considered, are presented and discussed in Section 2. Finally, Section 3 contains the proofs of Theorems 2.1, 2.2, and 2.5.

1. Preliminaries.

Given a complete metric space (Z, d), the symbol $\mathcal{B}(Z)$ indicates the Borel σ-algebra of Z. A function ψ from a real interval I into Z is said to be Lipschitz continuous at the point $t \in I$ when there exist a neighbourhood V_t of t and a constant $k_t \geq 0$ such that $d(\psi(\tau), \psi(t)) \leq k_t |\tau - t|$ for every $\tau \in I \cap V_t$. Let W be a subset of Z. We denote by $\overline{W}$ the closure of W. If W is nonempty, $z_0 \in Z$, and $\varepsilon \geq 0$, we write $d(z_0, W) := \inf_{w \in W} d(z_0, w)$ as well as

$$B(W, \varepsilon) := \{z \in Z : d(z, W) \leq \varepsilon\}, \qquad B^0(W, \varepsilon) := \{z \in Z : d(z, W) < \varepsilon\},$$

$$B(z_0, \varepsilon) := B(\{z_0\}, \varepsilon), \qquad B^0(z_0, \varepsilon) := B^0(\{z_0\}, \varepsilon), \qquad B^0(z_0, +\infty) := Z.$$

When W is bounded, the nonnegative real number

$$\alpha(W) := \inf\{\delta > 0 : W \text{ has a finite cover of sets with diameter smaller than } \delta\}$$

is called Kuratowski's measure of noncompactness of W. One evidently has $\alpha(W) = 0$ if and only if W is relatively compact. Further helpful properties may for instance be found in [10, Proposition 9.1].

Now, let $(Z, \|\cdot\|_Z)$ be a Banach space. The symbols $co(W)$ and $\overline{co}(W)$ respectively indicate the convex hull and the closed convex hull of the set W. We denote by $M(I, Z)$ the family of all (equivalence classes of) strongly Lebesgue measurable functions from I into Z. Given any $p \in [1, +\infty]$, we write p' for the conjugate exponent of p besides $L^p(I, Z)$ for the space of $u \in M(I, Z)$ satisfying $\|u\|_{L^p(I, Z)} < +\infty$, where

$$\|u\|_{L^p(I,Z)} := \begin{cases} \left(\int_I \|u(t)\|_Z^p \, d\mu \right)^{1/p} & \text{if } p < +\infty, \\ \operatorname*{ess\,sup}_{t \in I} \|u(t)\|_Z & \text{if } p = +\infty, \end{cases}$$

and μ is the Lebesgue measure on I. Finally, $\mathbb{R}^n$ indicates the real Euclidean n-space while $W^{j,p}(I, \mathbb{R}^n)$, $j \in \mathbb{N}$, represents the family of all $u \in C^{j-1}(I, \mathbb{R}^n)$ such that $u^{(j-1)}$ is absolutely continuous and $u^{(j)} \in L^p(I, \mathbb{R}^n)$.

Let X be a nonempty set and let H be a multifunction from X into Z (briefly, $H: X \to 2^Z$), namely a function which assigns to each point $x \in X$ a nonempty subset $H(x)$ of Z. If $V \subseteq X$ we write $H(V) := \cup_{x \in V} H(x)$ and $H|_V$ for the restriction of H to V. The graph of H is the set $\{(x, z) \in X \times$

$\times Z\colon z\in H(x)\}$. When $W\subseteq Z$, we define $H^-(W) := \{x\in X\colon H(x)\cap W\neq\emptyset\}$. If $(X,\mathcal{F})$ is a measurable space and $H^-(W)\in\mathcal{F}$ for any open subset W of Z, we say that H is $\mathcal{F}$-measurable, or simply measurable as soon as no confusion can arise. When X has a complete σ-finite measure defined on $\mathcal{F}$, Z is separable, and $H(x)$ is closed for all $x\in X$, the $\mathcal{F}$-measurability of the multifunction H is equivalent to the $\mathcal{F}\otimes\mathcal{B}(Z)$-measurability of its graph; see [11, Theorem 3.5]. Using this we immediately infer the following

PROPOSITION 1.1. *Let E be a nonempty Lebesgue measurable subset of $\mathbb{R}$ and let Z be a separable Banach space.*

(i) *Suppose $m\colon E\to\mathbb{R}_0^+$ is measurable and $z_0\in Z$. Then the multifunction $t\mapsto B(z_0, m(t))$, $t\in E$, is measurable.*

(ii) *Suppose H_1 and H_2 are closed-valued measurable multifunctions from E into Z complying with $H_1(t)\cap H_2(t)\neq\emptyset$, $t\in E$. Then the multifunction $t\mapsto H_1(t)\cap H_2(t)$, $t\in E$, is measurable.*

Let X be a metric space. We say that H is upper semicontinuous at the point $x_0\in X$ if to every open set $W\subseteq Z$ satisfying $H(x_0)\subseteq W$ there corresponds a neighbourhood V_0 of x_0 such that $H(V_0)\subseteq W$. The multifunction H is called upper semicontinuous when it is upper semicontinuous at each point of X. In such a case its graph is clearly closed in $X\times Z$ provided that $H(x)$ is closed for all $x\in X$. We say that H has a (sequentially) closed graph at x_0 if the conditions $\{x_k\}\subseteq X$, $\{z_k\}\subseteq Z$, $\lim_{k\to\infty} x_k = x_0$, $\lim_{k\to\infty} z_k = z_0$, $z_k\in H(x_k)$, $k\in\mathbb{N}$, imply $z_0\in H(x_0)$.

The result below is an easy consequence of Ky Fan's fixed point theorem; see for instance [6, Theorem 2.1].

THEOREM 1.1. *Let X be a metrizable locally convex topological vector space and let V be a weakly compact convex subset of X. Suppose H is a multifunction from V into itself with nonempty convex values and weakly sequentially closed graph. Then there exists $x_0\in V$ such that $x_0\in H(x_0)$.*

We say that H is lower semicontinuous at the point x_0 if to every open set $W\subseteq Z$ fulfilling $H(x_0)\cap W\neq\emptyset$ there corresponds a neighbourhood V_0 of x_0 such that $H(x)\cap W\neq\emptyset$, $x\in V_0$. The multifunction H is called lower semicontinuous when it is lower semicontinuous at each point of X. A selection from H is a function $h\colon X \to Z$ with the property $h(x) \in H(x)$ for all $x \in X$.

Let M be a positive real number and let $(X, \|\cdot\|_X)$ be a Banach space. According to [7, 8], we define

$$\Gamma^M := \{(t, x) \in \mathbb{R}_0^+ \times X\colon \|x\|_X \leq Mt\}.$$

If $\Omega \subseteq \mathbb{R} \times X$ is nonempty and $h\colon \Omega \to Z$, we say that h is Γ^M-continuous (or simply directionally continuous) at the point $(t_0, x_0) \in \Omega$ when, given $\varepsilon > 0$, one may find $\delta > 0$ such that $(t, x) \in \Omega$, $t_0 < t < t_0 + \delta$, $(t, x) - (t_0, x_0) \in \Gamma^M$ imply $\|h(t, x) - h(t_0, x_0)\|_Z < \varepsilon$. The function h is called Γ^M-continuous if it is Γ^M-continuous at each point of Ω.

A basic fact about lower semicontinuous multifunctions is established by the next result; see [8, Theorems 1 and 2].

THEOREM 1.2. *Let X, Z be two Banach spaces, let $\Omega \subseteq \mathbb{R} \times X$ be nonempty, and let $M > 0$. Then any closed-valued lower semicontinuous multifunction H from Ω into Z admits a Γ^M-continuous selection.*

Finally, for $\Omega \subseteq \mathbb{R} \times X$ and $(t_0, x_0) \in \mathbb{R} \times X$, we write $\Omega_{t_0} := \{x \in X\colon (t_0, x) \in \Omega\}$ as well as $\Omega^{x_0} := \{t \in \mathbb{R}\colon (t, x_0) \in \Omega\}$. Moreover, $\mathrm{proj}_X(\Omega)$ indicates the projection of Ω onto X. We say that a multifunction $H\colon \Omega \to 2^Z$ has the lower Scorza Dragoni property if to every $\varepsilon > 0$ there corresponds a closed subset I_ε of I such that $\mu(I \setminus I_\varepsilon) < \varepsilon$ and $H|_{(I_\varepsilon \times X) \cap \Omega}$ is lower semicontinuous.

2. Basic assumptions and statements of the main results.

From now on, I denotes a compact real interval with the Lebesgue measure structure $(\mathcal{L}, \mu)$, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are separable real Banach spaces of zero vectors 0_X and 0_Y respectively, $p, q, s \in [1, +\infty]$, $q < +\infty$, and $q \leq p \leq s$.

Let D be a nonempty closed subset of X, let $A \subseteq I \times D$, and let $C := (I \times D) \setminus A$. We always suppose that the set A complies with

(a_1) $A \in \mathcal{L} \otimes \mathcal{B}(X)$ and A_t is open in D for every $t \in I$.

Moreover, let F be a closed-valued multifunction from $I \times D$ into Y, let $m \in L^s(I, \mathbb{R}_0^+)$, and let $N \in \mathcal{L}$ with $\mu(N) = 0$. The conditions below will be assumed in what follows.

(a_2) $F|_A$ has the lower Scorza Dragoni property.

(a_3) $F(t, x) \cap B^0(0_Y, m(t)) \neq \emptyset$ whenever $(t, x) \in A \cap [(I \backslash N) \times D]$.

(a_4) The set $\{x \in \mathrm{proj}_X(C)$: $F|_C(\cdot, x)$ is measurable$\}$ is dense in $\mathrm{proj}_X(C)$.

(a_5) For every $(t, x) \in C \cap [(I \backslash N) \times D]$ the set $F(t, x)$ is convex, $F(t, \cdot)$ has a closed graph at x, and $F(t, x) \cap B(0_Y, m(t)) \neq \emptyset$.

(a_6) If $t \in I \backslash N$ then there exists $k_t \geq 0$ such that $\alpha(F(t, L) \cap \cap B(0_Y, m(t))) \leq k_t \alpha(L)$ for any bounded subset L of D.

Finally, let U be a nonempty set and let Φ: $U \to M(I, X)$, Ψ: $U \to \to L^s(I, Y)$ be two operators. We will make the hypotheses:

(a_7) To each $\varrho \in L^s(I, \mathbb{R}_0^+)$ there corresponds $\varrho^* \in M(I, \mathbb{R}_0^+)$ so that if $u \in U$ and $\|\Psi(u)(t)\|_Y \leq \varrho(t)$ a.e. in I, then $\Phi(u)$ is Lipschitz continuous with constant $\varrho^*(t)$ at almost all $t \in I$.

(a_8) Ψ is bijective and for any $v \in L^s(I, Y)$ and any sequence $\{v_k\} \subseteq L^s(I, Y)$ weakly converging to v in $L^q(I, Y)$ there exists a subsequence of $\{\Phi(\Psi^{-1}(v_k))\}$ which converges a.e. in I to $\Phi(\Psi^{-1}(v))$. Furthermore, a nondecreasing function $\varphi: [0, +\infty[\to [0, +\infty]$ can be defined in such a way that

$$\operatorname*{ess\,sup}_{t \in I} \|\Phi(u)(t)\|_X \leq \varphi(\|\Psi(u)\|_{L^p(I, Y)}), \qquad u \in U.$$

REMARK 2.1. The above inequality trivially holds whenever $\varphi(r) := = +\infty$ on $[0, +\infty[$. Nevertheless, different choices of φ sometimes might be more convenient.

Significant couples of operators Φ, Ψ fulfil conditions (a_7) and (a_8). Here are three typical situations; for some more general cases we refer to [5, 14].

EXAMPLE 2.1. Pick $t_0 \in I$, $x_0 \in \mathbb{R}^n$, $X := Y := \mathbb{R}^n$, $U := \{u \in \in W^{1,s}(I, \mathbb{R}^n)$: $u(t_0) = x_0\}$. If $u \in U$, define $\Phi(u) := u$, $\Psi(u) := u'$. An easy verification ensures that both (a_7) and (a_8), with $\varphi(r) = \|x_0\|_X + \mu(I)^{1/p'} r$, come true.

EXAMPLE 2.2. Let $I := [a, b]$, $X := \mathbb{R}^n \times \mathbb{R}^n$, $Y := \mathbb{R}^n$, $U := \{u \in \in W^{2,s}(I, \mathbb{R}^n)$: $u(a) = u(b) = 0\}$. Set, for $u \in U$, $\Phi(u) := (u, u')$ and $\Psi(u) := u''$. Elementary computations yield (a_7) while, reasoning as in the proof of [15, Theorem 2.1], we conclude that (a_8) holds.

EXAMPLE 2.3. Put $X := Y := \mathbb{R}$, $U := L^s(I, \mathbb{R})$, and choose $\lambda: I \times \times I \to \mathbb{R}$ satisfying $\lambda(0, \cdot) \in L^{s'}(I, \mathbb{R})$. Suppose there is $\lambda_0 \in L^{s'}(I, \mathbb{R}_0^+)$ such that

$$|\lambda(t', \tau) - \lambda(t'', \tau)| \leq \lambda_0(\tau)\, |t' - t''|$$

for all $t', t'' \in I$ and almost all $\tau \in I$. If $u \in U$, define $\Phi(u)(t) := = \int_I \lambda(t, \tau)\, u(\tau)\, d\mu$, $\Psi(u)(t) := u(t)$, $t \in I$. The operators Φ and Ψ clearly comply with (a_7) and (a_8).

Next, for $H: I \times D \to 2^Y$, consider the problem

$$(\mathrm{P}_H) \qquad \begin{cases} u \in U, \\ \Psi(u)(t) \in H(t, \Phi(u)(t)) \text{ in } I. \end{cases}$$

The point $u \in U$ is called a solution to (P_H) provided that $\Phi(u)(t) \in D$ and $\Psi(u)(t) \in H(t, \Phi(u)(t))$ for almost every $t \in I$.

THEOREM 2.1. *Assume F, Φ, and Ψ satisfy hypotheses* (a_2)-(a_7). *Then there exists a convex closed-valued multifunction G from $I \times D$ into Y having the properties*:

(i_1) *$G(t, x) \subseteq B(0_Y, m(t))$ for all $(t, x) \in I \times D$.*

(i_2) *$\alpha(G(t, L)) \leq k_t\, \alpha(L)$ whenever $t \in I \setminus N$ and $L \subseteq D$ is bounded.*

(i_3) *The set $\{x \in D\colon G(\cdot, x)$ is measurable$\}$ is dense in D.*

(i_4) *For every $t \in I$ the graph of $G(t, \cdot)$ is closed.*

(i_5) *Any solution of* (P_G) *is also a solution to* (P_F).

REMARK 2.2. The proof of the above result (see Section 3) actually shows that $G(\cdot, x)$ is measurable for each $x \in D$ as soon as the same holds regarding $F|_C(\cdot, x)$, $x \in \mathrm{proj}_X(C)$. So, bearing in mind Example 2.2 and taking $C = \emptyset$, we deduce that Theorem 1 in [3] is a special case of Theorem 2.1.

Before establishing the existence of solutions to (P_F) we state the following result, which remains valid even if Y has not finite dimension but is only reflexive; see [4]. However, in that setting we have to assume also (i_2) whereas, for finite-dimensional Y, (i_1) evidently forces (i_2).

THEOREM 2.2. *Let Y have finite dimension, let Φ and Ψ be like in* (a_8), *let $r > 0$ be such that $\|m\|_{L^p(I, \mathbb{R})} \leq r$, and let $D = B(0_X, \varphi(r))$. Sup-*

pose G is a convex closed-valued multifunction from $I \times D$ into Y complying with (i_1), (i_3), *and* (i_4) *of Theorem* 2.1. *Then problem* (P_G) *has at least one solution.*

REMARK 2.3. The preceding result is somewhat similar to [16, Theorem 1] where, in place of (i_1), a weaker condition is assumed. Nevertheless, during the proof of that result one really reduces to a multifunction which fulfils (i_1). Moreover, concerning the operators Φ and Ψ, an hypothesis stronger than (a_8) is there adopted.

Finally, combining Theorems 2.1 and 2.2 immediately yields

THEOREM 2.3. *Let Y be finite-dimensional, let $r > 0$ be such that $\|m\|_{L^p(I,\mathbb{R})} \leq r$, and let $D = B(0_X, \varphi(r))$. Suppose F, Φ, Ψ satisfy* (a_2)-(a_5), (a_7), *and* (a_8). *Then problem* (P_F) *has at least one solution $u \in U$ with $\|\Psi(u)(t)\|_Y \leq m(t)$ a.e. in I.*

This result has a variety of interesting special cases, as the remarks below emphasize.

In the framework of Example 2.1, (P_F) becomes the Cauchy problem

$$(\mathrm{P}') \qquad \begin{cases} u' \in F(t, u) & \text{in } I, \\ u(t_0) = x_0, \end{cases}$$

and from Theorem 2.3 we naturally infer the following simpler and practical result.

THEOREM 2.4. *Let $D = X$ and let* (a_2) *be satisfied. Assume that:*

(b_1) *$F|_C(\cdot, x)$ is measurable whenever $x \in \mathrm{proj}_X(C)$.*

(b_2) *For almost every $t \in I$ and every $x \in C_t$, the set $F(t, x)$ is convex and the multifunction $F(t, \cdot)$ has a closed graph at x.*

(b_3) *There exists $m_0 \in L^s(I, \mathbb{R}_0^+)$ such that*

$$F(t, x) \cap B(0_X, m_0(t)) \neq \emptyset$$

a.e. in I, for all $x \in X$.

Then problem (P') *has a solution $u \in W^{1,s}(I, \mathbb{R}^n)$.*

REMARK 2.4. In Theorem 2.4, different choices of C produce distinct global existence results for (P'). As an example, the case

$(t, x) \mapsto F(t, x)$ convex-valued, measurable with respect to t, and upper semicontinuous in x [10, Section 5] is treated by setting $C = I \times \mathbb{R}^n$, whereas taking $C = \emptyset$ leads to a theorem that comprises some known nonconvex-valued situations [10, Section 6].

REMARK 2.5. The above result improves the global version of [13, Theorem 1], which may be obtained at once from [13, Theorem 1] by replacing the local integrable boundedness hypothesis with a condition like (b_3). This is easily seen as soon as we realize that, because of [13, Propositions 3 and 4], the set

$$A := \{(t, x) \in I \times \mathbb{R}^n : F(t, \cdot)|_V \text{ is lower semicontinuous for some } V \text{ open neighbourhood of } x\}$$

considered in [13, Theorem 1] complies with (a_1). We then note that Theorem 3.1 of [19] requires regularity properties for F stronger than (a_2), (b_1), and (b_2), whereas, in place of (b_3), a weaker condition is assumed. However, during the proof of that result one really reduces to a multifunction fulfilling (b_3).

Theorem 2.4 applies to several concrete cases where the (global) existence results of [17, 13, 12, 19] fail, as the next elementary example shows.

EXAMPLE 2.4. Pick $I := [-1, 1]$, $t_0 := -1/2$, $x_0 := -1$. Given any nonmeasurable subset E of I, we define, for $(t, x) \in I \times \mathbb{R}$,

$$F(t, x) := \begin{cases} [0, 1] & \text{if } t > x \text{ and } x \leq -1, \\ [0, 4] & \text{if either } t = x \text{ and } t \notin E \text{ or } -1 < x < t, \\ [0, 5] & \text{if } t = x \text{ and } t \in E, \\ [0, 2] & \text{if } t < x. \end{cases}$$

Obviously, the multifunction $F: I \times \mathbb{R} \to 2^{\mathbb{R}}$ is lower semicontinuous on the set $A := \{(t, x) \in I \times \mathbb{R} : t > x\}$ and (a_1), (a_2), (b_1)-(b_3) hold. So, by Theorem 2.4, problem (P') has a solution belonging to $W^{1,s}(I, \mathbb{R})$. Nevertheless, the results of the above-mentioned papers cannot be used since neither the graph of $F(t, \cdot)$, $t \in]-1, 1]$, is closed at the point x_0 nor F is $\mathcal{L} \otimes \mathcal{B}(\mathbb{R})$-measurable. Let us also observe that solutions to (P') whose graphs exhibit segments lying in $\{(t, x) \in I \times \mathbb{R} : x = -1\}$ and $\{(t, x) \in I \times \mathbb{R} : t = x\}$ might trivially be constructed.

Finally, always in view of Example 2.1, the result below allows one to deduce Theorem A in [12] (and so the result of [17, p. 190]) from Theorem 2.3.

THEOREM 2.5. *Suppose H is a closed-valued multifunction from $I \times \mathbb{R}^n$ into $\mathbb{R}^n$ with the following properties*:

(c_1) *For almost all $t \in I$ and all $x \in \mathbb{R}^n$, either $H(t, x)$ is convex or $H(t, \cdot)$ is lower semicontinuous at x.*

(c_2) *The set $\{x \in \mathbb{R}^n : H(\cdot, x) \text{ is measurable}\}$ is dense in $\mathbb{R}^n$.*

(c_3) *For almost every $t \in I$ the graph of $H(t, \cdot)$ is closed.*

(c_4) *There is $m_1 \in L^s(I, \mathbb{R}_0^+)$ such that $H(t, x) \cap B(0, m_1(t)) \neq \emptyset$ a.e. in I, for all $x \in \mathbb{R}^n$.*

Then there exist a set $A \subseteq I \times \mathbb{R}^n$ and a closed-valued multifunction F from $I \times \mathbb{R}^n$ into $\mathbb{R}^n$ satisfying (a_1)-(a_5). *Moreover, for almost every $t \in I$ and every $x \in \mathbb{R}^n$ one has $F(t, x) \subseteq H(t, x)$.*

Likewise, in the framework of Example 2.2, (P_F) becomes the Dirichlet boundary value problem

$$(\mathrm{P}'') \qquad \begin{cases} u'' \in F(t, u, u') & \text{in } [a, b], \\ u(a) = u(b) = 0, \end{cases}$$

and comments analogous to those made for (P′) come true in the present setting. As an example, through Theorem 2.3 we achieve the following

THEOREM 2.6. *Let $D = X$ and let* (a_2) *be fulfilled. Assume that conditions* (b_1)-(b_3) *of Theorem* 2.4 *hold. Then problem* (P″) *has a solution $u \in W^{2,s}([a, b], \mathbb{R}^n)$.*

We conclude the section by pointing out that, concerning the operators Φ and Ψ, Theorem 2.1 needs hypothesis (a_7) only. This means that it can also be used together with existence results for specific problems where the finite dimension of Y and (a_8) are not required; see for instance [10]. Moreover, the same arguments of [12, 19] show that the multifunction F considered in Theorems 2.3, 2.4, and 2.6 can actually be taken integrably bounded (in the sense of [12, p. 297]).

3. Proofs of Theorems 2.1, 2.2, and 2.5.

Before establishing Theorem 2.1 we formulate the following two technical lemmas.

LEMMA 3.1. *Let F satisfy hypotheses* (a_5) *and* (a_6). *Then, for any $t \in I \setminus N$, the multifunction $x \mapsto F(t, x) \cap B(0_Y, m(t))$, $x \in C_t$, is convex closed-valued and upper semicontinuous.*

PROOF. Fix $t \in I \setminus N$. Owing to (a_5) the set $F(t, x) \cap B(0_Y, m(t))$, $x \in C_t$, is nonempty, convex, and closed. Next, arguing by contradiction, suppose there exist $x \in C_t$, a sequence $\{x_k\} \subseteq C_t$, as well as an open subset Ω of Y fulfilling $\lim_{k \to \infty} x_k = x$, $F(t, x_k) \cap B(0_Y, m(t)) \not\subseteq \Omega$ for all $k \in \mathbb{N}$, $F(t, x) \cap B(0_Y, m(t)) \subseteq \Omega$. Pick $y_k \in F(t, x_k) \cap B(0_Y, m(t)) \setminus \Omega$, $k \in \mathbb{N}$, and note that, on account of (a_6), $\{y_k\}$ exhibits a subsequence $\{y_{k_i}\}$ converging to some $y \in B(0_Y, m(t)) \setminus \Omega$. Since assumption ($a_5$) implies $y \in F(t, x)$, we really obtain $y \in F(t, x) \cap B(0_Y, m(t)) \setminus \Omega$, which is absurd. ∎

LEMMA 3.2. *Let H be a multifunction from $I \times D$ into Y, let $E \in \mathcal{L}$, and let $u \in U$. Suppose $\Psi(u)(t) \in H(t, \Phi(u)(t))$, $t \in E$, and write E^* for the set of all $t \in E$ such that there exists a sequence $\{t_k\} \subseteq E$ having the properties: $t_{k+1} < t_k$, $k \in \mathbb{N}$; $\lim_{k \to \infty} t_k = t$; $\lim_{k \to \infty} \Psi(u)(t_k) = \Psi(u)(t)$. Then $\mu(E^*) = \mu(E)$.*

PROOF. By Lusin's theorem, to every $\varepsilon > 0$ there corresponds $E_\varepsilon \in \mathcal{L}$ with $\mu(E_\varepsilon) > \mu(I) - \varepsilon$ and $\Psi(u)|_{E_\varepsilon}$ continuous. Put $E_1 := E \cap E_\varepsilon$ and observe that $\mu(E_1) > \mu(E) - \varepsilon$. Owing to Lebesgue's density theorem (see for instance [20, Theorem 7.13]) the set $E_2 := \{t \in E_1 \colon t$ is a point of density for $E_1\}$ is a subset of E^* complying with $\mu(E_2) = \mu(E_1)$. Thus, $\mu(E^*) > \mu(E) - \varepsilon$ for any $\varepsilon > 0$, namely $\mu(E^*) = \mu(E)$. ∎

PROOF OF THEOREM 2.1. Define, for every $(t, x) \in A$,

$$\widehat{F}(t, x) := \begin{cases} \overline{F(t, x) \cap B^0(0_Y, m(t))} & \text{if } t \in I \setminus N, \\ B(0_Y, m(t)) & \text{if } t \in N. \end{cases}$$

Obviously, $\widehat{F}(t, x)$ is nonempty and closed. Moreover, $\widehat{F}(t, x) \subseteq B(0_Y, m(t))$. Let us verify that the multifunction $\widehat{F} \colon A \to 2^Y$ has the lower Scorza Dragoni property. To this end, fix $\varepsilon > 0$. Using (a_2), (a_3), and Lusin's theorem gives a compact subset E_ε of $I \setminus N$ such that $\mu(I \setminus E_\varepsilon) < \varepsilon$,

$F|_{(E_\varepsilon \times X)\cap A}$ is lower semicontinuous, $m|_{E_\varepsilon}$ is continuous and positive. So, by [1, Proposition 5, p. 44], the multifunction

$$(t, x) \mapsto F(t, x) \cap B^0(0_Y, m(t)), \qquad (t, x) \in (E_\varepsilon \times X) \cap A,$$

is lower semicontinuous. This immediately leads to the desired conclusion, because for any open set $\Omega \subseteq Y$ one has

$$\widehat{F}|_{(\bar{E}_\varepsilon \times X)\cap A}(\Omega) = \{(t, x) \in A: t \in E_\varepsilon, F(t, x) \cap B^0(0_Y, m(t)) \cap \Omega \neq \emptyset\}.$$

Next, pick $\varrho = m$ in hypothesis (a_7) and write m^* in place of ϱ^*. Since, owing to (a_1), the multifunction $t \mapsto C_t$, $t \in I$, is measurable, Theorem 1 in [2] and standard arguments produce a sequence $\{E_k\}$ of compact subsets of I, no two of which have common points, satisfying the conditions: $\mu(E_0) = 0$, where $E_0 := I \setminus \bigcup_{n \in \mathbb{N}} E_k$; $(E_k \times X) \cap A$ is open in $E_k \times D$; $m^*|_{E_k}$ is continuous, while $\widehat{F}|_{(E_k \times X)\cap A}$ is lower semicontinuous. Let M_k, $k \in \mathbb{N}$, be a real number complying with

$$M_k > \max_{t \in E_k} m^*(t).$$

Then, on account of Theorem 1.2, for each $k \in \mathbb{N}$ there exists a Γ^{M_k}-continuous selection $f_k: (E_k \times X) \cap A \to Y$ from $\widehat{F}|_{(E_k \times X)\cap A}$. Choose $f_0(t, x) \in \widehat{F}(t, x)$, $(t, x) \in (E_0 \times X) \cap A$, and write

$$f(t, x) := \begin{cases} f_k(t, x) & \text{if } (t, x) \in (E_k \times X) \cap A, \quad k \in \mathbb{N}, \\ f_0(t, x) & \text{if } (t, x) \in (E_0 \times X) \cap A. \end{cases}$$

Moreover, for every $(t, x) \in I \times D$, set

$$(1) \qquad F^*(t, x) := \begin{cases} F(t, x) \cap B(0_Y, m(t)) & \text{if } (t, x) \in C \cap [(I \setminus N) \times X], \\ B(0_Y, m(t)) & \text{if } (t, x) \in C \cap (N \times X), \\ \{f(t, x)\} & \text{if } (t, x) \in A, \end{cases}$$

as well as

$$G(t, x) := \bigcap_{\varepsilon > 0} \overline{co}\,(F^*(\{t\} \times [B^0(x, \varepsilon) \cap D])).$$

We claim that the multifunction $G: I \times D \to 2^Y$ fulfils the conclusion of the theorem. Indeed, $G(t, x)$ is a nonempty, convex, closed subset of $B(0_Y, m(t))$. To prove (i_2), fix $t \in I \setminus N$, $L \subseteq D$ bounded, and $\varepsilon > 0$. An elementary computation yields

$$G(t, L) \subseteq \bigcap_{\varepsilon>0} \bigcup_{x\in L} \overline{co}\,(F^*(\{t\}\times[B^0(x,\varepsilon)\cap D])) \subseteq$$

$$\subseteq \overline{co}\,(F^*(\{t\}\times[B^0(L,\varepsilon)\cap D])),$$

while (1), assumption (a_6), and usual properties [10, Proposition 9.1] of Kuratowski's measure of noncompactness α imply

$$\alpha(G(t, L)) \leqslant \alpha(F^*(\{t\}\times[B^0(L,\varepsilon)\cap D])) \leqslant$$

$$\leqslant \alpha((F(\{t\}\times[B^0(L,\varepsilon)\cap D])\cap B(0_Y, m(t))) \leqslant$$

$$\leqslant k_t\,\alpha(B^0(L,\varepsilon)\cap D) \leqslant k_t[\alpha(L)+2\varepsilon].$$

Since ε was arbitrary, we actually have $\alpha(G(t, L)) \leqslant k_t\,\alpha(L)$.

Let us next verify that, for $(t, x)\in I\times D$,

$$(2) \qquad G(t,x) = \begin{cases} F^*(t,x) & \text{if } (t,x)\in C, \\ \bigcap_{\varepsilon>0} \overline{co}\,(f(\{t\}\times[B^0(x,\varepsilon)\cap A_t])) & \text{if } (t,x)\in A. \end{cases}$$

When $(t, x)\in A$ the formula immediately follows from (a_1) and (1). So, suppose $(t, x)\in C$. One obviously has $F^*(t, x)\subseteq G(t, x)$. If $y\in Y\setminus F^*(t, x)$ then $y\notin B(F^*(t, x), \eta)$ as soon as $\eta>0$ is sufficiently small. Bearing in mind that, in view of Lemma 3.1, the multifunction $z\mapsto F^*(t, z)$ is upper semicontinuous at x, we can find $\varepsilon>0$ complying with $F^*(\{t\}\times[B^0(x,\varepsilon)\cap D])\subseteq B^0(F^*(t, x), \eta)$. Therefore, $G(t, x)\subseteq B(F^*(t, x), \eta)$ and consequently $y\notin G(t, x)$. This gives $F^*(t, x) = G(t, x)$.

Write $D_1 := \{x\in \mathrm{proj}_X(C)\colon F|_C(\cdot, x)$ is measurable$\}$ and $D_2 := D\setminus \mathrm{proj}_X(C)$. Because of ($a_4$) we get $\overline{D_1\cup D_2} = D$. Thus ($i_3$) is achieved by simply showing that, for any $x\in D_1\cup D_2$ and any $k\in\mathbb{N}$, the multifunction $G(\cdot, x)|_{E_k}$ is measurable. Exploiting the identity $E_k = (E_k\cap C^x)\cup(E_k\cap A^x)$ besides $\mu(N)=0$, it is enough to establish the measurability of $G(\cdot, x)|_{(E_k\setminus N)\cap C^x}$ and $G(\cdot, x)|_{E_k\cap A^x}$. When $x\in D_1$ Proposition 1.1 ensures that the multifunction $t\mapsto F(t, x)\cap B(0_Y, m(t))$, $t\in(E_k\setminus N)\cap C^x$, is measurable. Hence, by (1) and (2), the same holds regarding $G(\cdot, x)|_{(E_k\setminus N)\cap C^x}$. Suppose $x\in D_1\cup D_2$ and choose $y\in Y$, $t\in E_k\cap A^x$. Since the function f_k is Γ^{M_k}-continuous at (t, x), to every $\sigma>0$ there corresponds $\delta > 0$ such that

$$G(\tau, x) = \bigcap_{\varepsilon > 0} \overline{co}\,(f_k(\{\tau\} \times [B^0(x, \varepsilon) \cap A_\tau])) \subseteq B(f_k(t, x), \sigma)$$

for all $\tau \in E_k \cap A^x \cap]t, t+\delta[$. Accordingly,

$$d_Y(y, G(t, x)) - \sigma \leq \|y - f_k(t, x)\|_Y - \|f_k(t, x) - y^*\|_Y \leq \|y - y^*\|_Y$$

for every $y^* \in G(\tau, x)$, which forces

$$d_Y(y, G(t, x)) - \sigma \leq d_Y(y, G(\tau, x)), \qquad \tau \in E_k \cap A^x \cap]t, t+\delta[.$$

This means that the function $d_Y(y, G(\cdot, x))|_{E_k \cap A^x}$ is lower semicontinuous from the right and so measurable. Through [11, Theorem 3.3] we then infer the measurability of $t \mapsto G(t, x)$, $t \in E_k \cap A^x$.
As (i_4) can be verified by using standard arguments (see for instance [1, p. 102]), it remains to prove (i_5). Let $u \in U$ satisfy

$$\Psi(u)(t) \in G(t, \Phi(u)(t)) \quad \text{for every } t \in I \setminus N_1, \tag{3}$$

where $\mu(N_1) = 0$. Taking into account (i_1) and hypothesis (a_7), we obtain a set $N_2 \subseteq I$ of Lebesgue measure zero such that $\Phi(u)$ is Lipschitz continuous with constant $m^*(t)$ at each $t \in I \setminus N_2$. Define $N_0 := N \cup N_1 \cup N_2$. It is evident that $\mu(N_0) = 0$ and $\mu(I \setminus [\cup_{k \in \mathbb{N}}(E_k \setminus N_0)]) = 0$. Moreover, one has

$$\Psi(u)(t) \in F(t, \Phi(u)(t)) \quad \text{a.e. in } E_k \setminus N_0, \quad k \in \mathbb{N}.$$

Indeed, denote by E_k^* the set of all $t \in E_k \setminus N_0$ for which there exists a sequence $\{t_j\} \subseteq E_k \setminus N_0$ enjoying the properties:

$$t_{j+1} < t_j, \ j \in \mathbb{N}; \qquad \lim_{j \to \infty} t_j = t; \qquad \lim_{j \to \infty} \Psi(u)(t_j) = \Psi(u)(t). \tag{4}$$

In view of Lemma 3.2 we are really reduced to see that $\Psi(u)(t) \in F(t, \Phi(u)(t))$ whenever $t \in E_k^*$. So, pick a point t in E_k^*. If $(t, \Phi(u)(t)) \in C$ the above inclusion immediately follows from (1)-(3). Suppose $(t, \Phi(u)(t)) \in A$ and fix $\sigma > 0$. Since $(E_k \times X) \cap A$ is open in $E_k \times D$ and the function f_k is Γ^{M_k}-continuous at $(t, \Phi(u)(t))$, there exists $\delta > 0$ such that the conditions

$$(\tau, z) \in E_k \times D, \qquad t < \tau < t + \delta, \qquad \|z - \Phi(u)(t)\|_X < M_k(\tau - t)$$

imply $(\tau, z) \in A$ as well as

$$\|f_k(\tau, z) - f_k(t, \Phi(u)(t))\|_Y < \frac{\sigma}{2}. \tag{5}$$

Hence, using (4) together with the Lipschitz continuity of $\Phi(u)$ at t yields $\|\Phi(u)(t_j) - \Phi(u)(t)\|_X < M_k(t_j - t)$ for any sufficiently large j. Consequently, by (2) and (5),

$$G(t_j, \Phi(u)(t_j)) = \bigcap_{\varepsilon > 0} \overline{co}\left(f_k(\{t_j\} \times [B^0(\Phi(u)(t_j), \varepsilon) \cap A_{t_j}])\right) \subseteq$$

$$\subseteq B(f_k(t, \Phi(u)(t)), \sigma/2)\,.$$

This clearly forces

$$\|\Psi(u)(t_j) - f_k(t, \Phi(u)(t))\|_Y \leqslant \frac{\sigma}{2}\,,$$

while (4) leads to

$$\|\Psi(u)(t_j) - \Psi(u)(t)\|_Y < \frac{\sigma}{2}$$

for $j \in \mathbb{N}$ large enough. Therefore,

$$\|\Psi(u)(t) - f_k(t, \Phi(u)(t))\|_Y < \sigma\,.$$

As σ was arbitrary, we actually have $\Psi(u)(t) = f_k(t, \Phi(u)(t))$. Thus, $\Psi(u)(t) \in F(t, \Phi(u)(t))$, and the proof is complete. ∎

Theorem 2.2 can be established through reasonings somewhat similar to those employed in [16, Theorem 1]. So, we only present the main ideas of the proof.

PROOF OF THEOREM 2.2. Bearing in mind that X is separable and (i_3) holds, we obtain a countable set $D^* \subseteq D$ with the following properties: $\overline{D^*} = D$; $G(\cdot, x)$ is measurable for each $x \in D^*$. If $(t, x) \in I \times D$, we write

$$G_k(t, x) := G(t, B(x, 1/k) \cap D^*) \qquad (k \in \mathbb{N}),$$

$$G_\infty(t, x) := \bigcap_{k \in \mathbb{N}} \overline{G_k(t, x)}, \qquad \widehat{G}(t, x) := co\,(G_\infty(t, x))\,.$$

Owing to (i_4) and the convexity of $G(t, x)$ one has

$$\widehat{G}(t, x) \subseteq G(t, x), \qquad (t, x) \in I \times D\,. \tag{6}$$

Hence, on account of (i_1) and Cantor's theorem, $\widehat{G}(t, x)$ is also nonempty, convex, and compact. Arguing as in [16, p. 263] we then infer that the multifunction $\widehat{G}$ is $\mathcal{L} \otimes \mathcal{B}(D)$-measurable, the graph of $\widehat{G}(t, \cdot)$ is closed for every $t \in I$, while the set

$$K := \{v \in L^q(I, Y) \colon \|v(t)\|_Y \leq m(t) \text{ a.e. in } I\}$$

is convex and weakly compact.

Next, for $v \in K$, assumption (a_8) combined with the inequality $\|m\|_{L^p(I, \mathbb{R}_0^+)} \leq r$ yields

$$\operatorname*{ess\,sup}_{t \in I} \|\Phi(\Psi^{-1}(v))(t)\|_X \leq \varphi(\|v\|_{L^p(I, Y)}) \leq \varphi(\|m\|_{L^p(I, \mathbb{R}_0^+)}) \leq \varphi(r).$$

So, it makes sense to define

$$\Gamma(v) := \left\{w \in K \colon w(t) \in \widehat{G}(t, \Phi(\Psi^{-1}(v))(t)) \text{ a.e. in } I\right\}.$$

The set $\Gamma(v)$ is clearly convex and nonempty; see [16, p. 264]. Let us now verify that the graph of the multifunction $\Gamma \colon K \to 2^K$ is weakly sequentially closed. Pick $v, w \in K$ and choose two sequences $\{v_j\}, \{w_j\} \subseteq K$ fulfilling $w_j \in \Gamma(v_j)$, $j \in \mathbb{N}$, $\lim_{j \to \infty} v_j = v$, $\lim_{j \to \infty} w_j = w$ weakly in $L^q(I, Y)$. Exploiting (a_8) and taking a subsequence if necessary, we may suppose

$$\lim_{j \to \infty} \Phi(\Psi^{-1}(v_j))(t) = \Phi(\Psi^{-1}(v))(t) \tag{7}$$

at almost all $t \in I$. By Mazur's theorem [9, Theorem II.5.2], for each $j \in \mathbb{N}$ there exists $\overline{w}_j \in co(\{w_i \colon i \geq j\})$ such that $\lim_{j \to \infty} \|\overline{w}_j - w\|_{L^q(I, Y)} = 0$; hence

$$\lim_{j \to \infty} \overline{w}_j(t) = w(t) \qquad \text{a.e. in } I, \tag{8}$$

without loss of generality. Let t satisfy (7), (8), and $w_j(t) \in \widehat{G}(t, \Phi(\Psi^{-1}(v_j))(t))$, $j \in \mathbb{N}$. Since $\widehat{G}(t, \cdot)$ is upper semicontinuous at $\Phi(\Psi^{-1}(v))(t)$, to every $\sigma > 0$ there corresponds $\nu \in \mathbb{N}$ such that

$$\widehat{G}(t, \Phi(\Psi^{-1}(v_j))(t)) \subseteq B^0(\widehat{G}(t, \Phi(\Psi^{-1}(v))(t)), \sigma)$$

whenever $j \geq \nu$. Because of (8) this produces

$$w(t) \in B(\widehat{G}(t, \Phi(\Psi^{-1}(v))(t)), \sigma).$$

As σ was arbitrary, we really get $w(t) \in \widehat{G}(t, \Phi(\Psi^{-1}(v))(t))$, namely, $w \in \Gamma(v)$.

We have thus proved that all the hypotheses of Theorem 1.1 hold. So, there is a function $v \in K$ complying with $v \in \Gamma(v)$. Due to (6), the point $u := \Psi^{-1}(v)$ represents a solution of problem (P_G). ■

We conclude this section with the following

PROOF OF THEOREM 2.5. In view of [12, Proposition 3.3] we can suppose that for almost all $t \in I$ and all $x \in \mathbb{R}^n$ either $H(t, x)$ is convex or $H(t, \cdot)$ restricted to some neighbourhood of x is lower semicontinuous. Assumption (c_2) provides a countable subset D^* of $\mathbb{R}^n$ such that $\overline{D^*} = \mathbb{R}^n$ and $H(\cdot, x)$ is measurable whenever $x \in D^*$. If $(t, x) \in I \times \mathbb{R}^n$, we write

$$H_\infty(t, x) := \bigcap_{k \in \mathbb{N}} \overline{H(t, B(x, 1/k) \cap D^*)}.$$

The same arguments made in [16, p. 263] ensure that the multifunction H_∞ is $\mathcal{L} \otimes \mathcal{B}(\mathbb{R}^n)$-measurable. So, owing to [13, Propositions 3 and 4], the set

$$A := \{(t, x) \in I \times \mathbb{R}^n : H_\infty(t, \cdot)|_V \text{ is lower semicontinuous for some } V \text{ open neighbourhood of } x\}$$

satisfies condition (a_1). Next, choose $m \in L^s(I, \mathbb{R}_0^+)$ with $m(t) > m_1(t)$, $t \in I$, and define

$$F(t, x) := \begin{cases} \overline{H_\infty(t, x) \cap B^0(0, m(t))} & \text{if } (t, x) \in A\,, \\ co\,(H_\infty(t, x) \cap B(0, m(t))) & \text{if } (t, x) \in (I \times \mathbb{R}^n) \setminus A\,. \end{cases}$$

It is not difficult to see that the multifunction $F : I \times \mathbb{R}^n \to 2^{\mathbb{R}}$ has properties (a_2)-(a_6). Moreover, by means of standard computations and using (c_1) we obtain $F(t, x) \subseteq H(t, x)$ a.e. in I, for every $x \in \mathbb{R}^n$, which completes the proof. ■

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