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Totally Inert Groups.

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Let G be any group and H be a subgroup of G. Then H is called an *inert subgroup* of G if $|H: H \cap H^g| < \infty$ for all $g \in G$.

Every finite subgroup, every normal subgroup and the group itself are the trivial examples of inert subgroups in a group. Moreover if $G = GL(n, \mathbb{Q})$, then $SL(n, \mathbb{Z})$ is inert in G see [7, page 55]. In a barely transitive group G the stabilizer H of a point and any group containing H are inert subgroups of G see [5].

A group G is called *totally inert group* (*TIN-group*) if every subgroup of G is inert.

Clearly every FC-group is a TIN-group. But there exist totaly inert non-FC- groups. The groups constructed by Olsanskiĭ are the examples of non-FC TIN-groups. The following is an easier example of a non-FC, TIN-group.

Let A be an infinite abelian $2'$-group. Let t be an involutory automorphism of A such that $a^t = a^{-1}$ for all $a \in A$. Then $G = A \rtimes \langle t \rangle$ is not an FC-group but it is a TIN-group. Because for any $H < G$ the group $H \cap A$ is a normal subgroup of G.
Clearly the property of being a TIN-group is a natural generalization of being an FC-group. The above example shows that the class of TIN-groups is larger than the class of FC-groups.

One may think that every FC by finite group is TIN-group. But the following example shows that this is not true.

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Let $G = A \rtimes \langle t \rangle$ where $A = \langle a_1 \rangle \times \langle a_1^t \rangle \times \langle a_2 \rangle \times \langle a_2^t \rangle \times \ldots$ and the order of t is 2. Then for the subgroups $C = \langle a_1 \rangle \times \langle a_2 \rangle \times \ldots$ and $C^t = \langle a_1^t \rangle \times \times \langle a_2^t \rangle \times \ldots$ we have $C \cap C^t = 1$. This group G is an FC by finite group but it is not a TIN-group.

It is clear that any subgroup of a TIN-group is a TIN-group and any homomorphic image of a TIN-group is a TIN-group. But the following example shows that direct product of two TIN-groups is not necessarily a TIN-group.

Let A be an infinite abelian $2'$-group and t be an involutory automorphism of A such that $a^t = a^{-1}$ for all $a \in A$. Let $G = A \times (A \rtimes \langle t \rangle)$ and $H = \{(a, a): a \in A\}$. Let $(1, t)$ be an element of G. Then $H^{(1,t)} = = \{(a, a^t): a \in A\} = \{(a, a^{-1}): a \in A\}$. Then $x \in H \cap H^{(1,t)}$ implies that $x = (a, a) = (a, a^{-1})$. Hence $a^2 = 1$, but this implies $a = 1$ as A is an abelian $2'$-group. Hence $x = 1$ and H is not an inert subgroup of G.

What are the structures of locally finite *TIN*-groups?

The groups constructed by Olsanskiĭ [6] are examples of simple *TIN*-groups. But of course they are not locally finite. Does there exist an infinite locally finite simple TIN-group? In this article we answer this question negatively.

THEOREM 1. *There exists no infinite simple locally finite TIN-group.*

Basic Properties of Inert Subgroups.

LEMMA 1. (i) *If H is an infinite simple inert subgroup of a group G, then $H \trianglelefteq G$.*

(ii) *Let G be a simple group and H be a proper inert subgroup of G. Then H is residually finite.*

(iii) *Homomorphic image of a TIN-group is a TIN-group.*

(iv) *If H is an inert subgroup of G and $N \trianglelefteq G$, then HN is inert in G.*

PROOF. (i) Trivial.

(ii) $\bigcap_{g \in G} H^g \lhd G$. Since G is a simple group we have $\cap_{g \in G} H^g = 1$. Hence the result follows.

(iii) and (iv) are trivial.

LEMMA 2. *Let P be a locally and residually finite p-group for some prime p. If P/P' is finite, then P is a finite group.*

PROOF. Since P/P' is finite, there exists a finite subset $K \subseteq G$ such that $\langle K, P' \rangle = P$. Assume if possible that $\langle K \rangle \neq P$. Let $g \in P - \langle K \rangle$. Then $\langle K, g \rangle$ is finite and there exists a normal subgroup $N \lhd P$ such that $|P: N| < \infty$ and $N \cap \langle K, g \rangle = 1$. Let $\overline{P} = P/N, \overline{K} = KN/N$. Since $\langle K, P' \rangle = P$ we get $\langle \overline{K}, \overline{P'} \rangle = \overline{P}$, but for a finite p-group the commutator subgroup lies inside the Frattini subgroup therefore it is a non generator. Hence $\langle \overline{K} \rangle = \overline{P}$. But $\bar{g} = gN \notin \overline{K}$. So we get $P = \langle K \rangle$. It follows that P is a finite group.

DEFINITION 1. *Let X and Y be two subgroups of a group G. We say that X and Y are* commensurable *if $|X: X \cap Y| < \infty$ and $|Y: Y \cap X| < \infty$.*

LEMMA 3. *If X is an inert subgroup of a group G and X is commensurable with the subgroup Y of G, then Y is an inert subgroup of G.*

PROOF. Let $g \in G$. By assumption $|X: X \cap X^g| < \infty$ and $|Y: X \cap \cap Y| < \infty$ and $|X: X \cap Y| < \infty$. Then $|X^g: X^g \cap X^{g^2}| < \infty$. It follows that

$$|X \cap X^g: X \cap X^g \cap X^{g^2}| < \infty .$$

Since the group X is inert we have $|X \cap Y: X \cap X^g \cap Y| < \infty$. Hence $|Y: X \cap X^g \cap Y| < \infty$ and $|X: X \cap X^g \cap Y| < \infty$. This gives $|X^g: X^g \cap \cap X^{g^2} \cap Y^g| < \infty$. Then $|X \cap X^g: X \cap X^g \cap X^{g^2} \cap Y^g| < \infty$. Then we get $|X: X \cap X^g \cap X^{g^2} \cap Y^g| < \infty$ and $|X: X \cap X^g \cap Y| < \infty$. Moreover $|X: X \cap X^g \cap X^{g^2} \cap Y \cap Y^g| < \infty$. Then $|Y: Y \cap X| < \infty$ implies $|Y: Y \cap Y^g| < \infty$.

LEMMA 4 [1, Corollary 2.6]. *Let G be a simple TIN-group and $1 \neq \neq K \lhd H$ be subgroups of G. Then H/K is an FC-group.*

LEMMA 5. *Let G be a simple TIN-group. Then*

(i) *for all non identity elements x and y in G, the groups $C_G(x)$ and $C_G(y)$ are commensurable.*

(ii) *if $C_G(x)$ is infinite for a non identity torsion element x in G, then G is locally finite.*

(iii) *if G is locally finite, $H < G$, then either H is an FC-group or $F(H) = 1$ where $F(H)$ is the Hirsh-Plotkin radical of H.*

PROOF. (i). Let $1 \neq x$ be an element of G. Then $N = \{y \in G: |C_G(x): C_G(x) \cap C_G(y)| < \infty\}$ is a normal subgroup of G. It is clear that product of two elements are in N. So we show the normality of N. Let $y \in N$ and $t \in G$. Then $|C_G(x): C_G(x) \cap C_G(y)| < \infty$. It follows that $|C_G(x)^t: C_G(x)^t \cap C_G(y)^t| < \infty$. Then $|C_G(x) \cap C_G(x)^t: C_G(x) \cap C_G(y)^t \cap C_G(x)^t| < \infty$. Since $C_G(x)$ is an inert subgroup of G we have $|C_G(x): C_G(x) \cap C_G(y^t) \cap C_G(x^t)| < \infty$. Hence $|C_G(x): C_G(x) \cap C_G(y^t)| < \infty$. But G is simple and $x \in N$ implies that $N = G$. Hence for all x and y in G the groups $C_G(x)$ and $C_G(y)$ are commensurable.
It follows that for any $x \in G$ the group $C_G(x)$ is an FC-group.

(ii) Let $1 \neq x$ be a fixed element of G such that $C_G(x)$ is infinite. Let $T(G) = \{g \in G$: order of g is finite$\}$ be the set of torsion elements of G. Then $T(G)$ is a normal subgroup of G. Indeed if K is a finite subset of $T(G)$, then by (i) $|C_G(x): C_G(x) \cap C_G(K)| < \infty$. Hence $C_G(K)$ is an infinite group. Let $a \in C_G(K)$. Then $K \leqslant C_G(a)$ and $C_G(a)$ is an FC-group. Then by Dietzmann Lemma $\langle K \rangle \leqslant \langle K^{C_G(a)} \rangle$ is finite. Hence $\langle K \rangle$ is a finite subgroup. Hence $T(G) = G$ and G is locally finite.

(iii) Let x and y be nonidentity elements of H. Then $K_1 = \langle x^H \rangle$ and $K_2 = \langle y^H \rangle$ are commensurable.

Indeed, for any $i = 1, 2$ by Lemma 4 H/K_i are FC-groups. Let $\bar{y} = yK_1 \in H/K_1$. Then $\langle \bar{y}^{\bar{H}} \rangle = K_2K_1/K_1 \cong K_2/(K_1 \cap K_2)$ which is finite. Similarly $K_1/(K_1 \cap K_2)$ is finite.
Let $F(H)$ be the Hirsh-Plotkin radical. Assume if possible that H is not an FC-group. Then by [2, Theorem 1.4] $F(H)$ is a p-group and by [2, Corollary 2.2 and Theorem 4.5] $FC(F(H)) = 1$. Since there exists no FC-element every conjugacy class is infinite. Let $x \in F(H) \backslash \{1\}$ and $\langle x^{F(H)} \rangle = L$. Since the group L is an infinite residually finite p-group we get $L > L'$. Let $y \in L'$. Then $\langle y^{F(H)} \rangle \lhd L'$ and by Lemma 2 $|L: \langle y^{F(H)} \rangle|$ is infinite. But any two normal subgroups of $F(H)$ are commensurable by the above paragraph. Hence we obtain a contradiction.

LEMMA 6. *Let G be a locally finite group. Let A be a normal infinite elementary abelian p-subgroup of G. Assume that every subgroup of A is inert in G. Then for any $x \in G$, there exists a subgroup B of finite index in A such that, for all $b \in B$, $b^x \in \langle b \rangle$.*

PROOF. Assume on the contrary that, there exists $1 \neq x \in G$ such that if $|A: B| < \infty$, then there exists $a_1 \in B$ such that $a_1^x \notin \langle a_1 \rangle$. Then $\langle a_1^x \rangle \cap \langle a_1 \rangle = 1$.

Let $A_1 = \langle a_1 \rangle \times \langle a_1^x \rangle$. Then A_1 is finite and $A = A_1 \times A_1'$.

$$\bigcap_{n \in \mathrm{N}} (A_1')^{x^n} = \bigcap_{g \in \langle x \rangle} (A_1')^g = B_1$$

B_1 is $\langle x \rangle$-invariant and has finite index in A.

By a similar argument there exists $a_2 \in B_1$ such that $\langle a_2^x \rangle \cap \langle a_2 \rangle = 1$. Let $A_2 = \langle a_2^x \rangle \times \langle a_2 \rangle$ and $B_1 = A_2 \times A_2'$

$$\bigcap_{g \in \langle x \rangle} (A_2')^g = B_2.$$

The group B_2 has finite index in B_1 and B_2 is $\langle x \rangle$-invariant. There exists $a_3 \in B_2$ such that $\langle a_3^x \rangle \cap \langle a_3 \rangle = 1$.

Let $A_3 = \langle a_3^x \rangle \times \langle a_3 \rangle$. Continuing like this we obtain an infinite subgroup of A namely, $\langle a_1 \rangle \times \langle a_1^x \rangle \times \langle a_2 \rangle \times \langle a_2^x \rangle \times \ldots$

Let $C = \langle a_1 \rangle \times \langle a_2 \rangle \times \langle a_3 \rangle \times \ldots$ Then $C^x \cap C = 1$. Hence C is not inert in G which is a contradiction.

LEMMA 7. *Let G be a locally finite group and A be a normal infinite elementary abelian p-subgroup of G. If every subgroup of A is inert in G, then for all $x \in G'$, $|[A, x]| < \infty$. I.e G' acts as finitary linear group on A.*

PROOF. Let $x \in G'$. Then $x = x_1 x_2 \ldots x_n$ where $x_i = [y_i, z_i]$ for some $y_i, z_i \in G$, $i = 1, 2, \ldots n$. Clearly $[A, x] \leq [A, x_1][A, x_2] \ldots [A, x_n]$. Hence it is enough to prove that $[A, x_i]$ is finite for all $i = 1, 2, \ldots n$.

By Lemma 6, for y_i and z_i, there exist B_i and B_i' such that $|A : B_i| < \infty$ and $|A : B_i'| < \infty$ and for all $b \in B_i$, $b^{y_i} \in \langle b \rangle$ and for all $b \in B_i'$, $b^{z_i} \in \langle b \rangle$. Let $B = B_i \cap B_i'$. Then $|A : B| < \infty$ and for all $b \in B$, $b^{y_i} \in \langle b \rangle$ and $b^{z_i} \in \langle b \rangle$. This implies $b^{y_i} = b^{n_i}$ and $b^{z_i} = b^{m_i}$ for some n_i and m_i in $\mathbb{Z}$. Since the automorphism group of a cyclic group is abelian we get $b^{[y_i, z_i]} = b$ for all $b \in B$. So $|A : C_A(x_i)| < \infty$ since $B \leq C_A(x_i)$. Hence $[A, x_i] < \infty$.

PROOF OF THE THEOREM 1. Asume that G is a simple locally finite TIN-group. By [4] Theorem 4.4 every locally finite simple group has a local system consisting of countably infinite locally finite simple subgroups. But by Lemma 1 in a simple locally finite TIN-group, an infinite simple subgroup is a normal subgroup, so we may assume that G itself is countable.

By [3, Theorem B Page 190] a countable non-finitary locally finite simple group has maximal subgroups. Let M be a maximal subgroup of G. Then for all $g \in G$ $|M: M \cap M^g| < \infty$ and $|M^g: M \cap M^g| < \infty$. It follows from [1] Lemma 5 that $|\langle M, M^g \rangle: M \cap M^g| < \infty$. As M is a maximal subgroup of G we get $M = M^g$ for all $g \in G$. Hence M is a normal subgroup of G. But G is simple, hence we may assume that G is a countable, finitary locally finite simple group. By [3, page 216 Theorem 8], there exists a prime p such that if H is any proper inert subgroup of G, then $H/O_p(H)$ is locally normal. Consequently if $F(H) = 1$, then H is locally normal. If $F(H) \neq 1$, then by Lemma 5 (iii) H is locally normal. By [3, page 216 Theorem 6] G is isomorphic to an alternating group of finitary permutations on some set Ω. But a stabilizer G_α of $\alpha \in \Omega$ is an infinite proper simple subgroup of G, and this is impossible by Lemma 1 (i).

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