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A priori Inequalities in $L^\infty(\Omega)$ for Solutions of Elliptic Equations in Unbounded Domains.

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ABSTRACT - We prove some a priori inequalities in $L^\infty(\Omega)$ for subsolutions of elliptic equations in divergence form, with Dirichlet's boundary conditions, in unbounded domains.

1. Introduction.

In an open subset Ω of $\mathbb{R}^n$, not necessarily bounded, we consider a linear uniformly elliptic second order operator in variational form with discontinuous coefficients, associated to the bilinear form

$$a(u, v) = \int_\Omega \left\{ \sum_{i,j=1}^n a_{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n (b_i u_{x_i} v + d_i u v_{x_i}) + cuv \right\} dx \tag{1}$$

If $u \in H^1(\Omega)$ is a solution of the inequality

$$a(u, v) \leq \int_\Omega \left\{ f_0 v + \sum_{i=1}^n f_i v_{x_i} \right\} dx \qquad \forall v \in C_0^1(\Omega), \ v \geq 0 \text{ in } \Omega , \tag{2}$$

we can consider the problem of determining the minimal hypotheses on the coefficients b_i, d_i, c of the bilinear form (1) and on the known functions f_i ($i = 0, 1, \ldots, n$) for the subsolution u to be (essentially) bounded from above in Ω. Such a problem was already studied e.g. in [2] and [3],

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where an inequality of the kind

$$\text{(3)} \qquad \operatorname*{ess\,sup}_{\Omega} u \leq \max(0, \max_{\partial\Omega} u) + K_1\left\{\|f_0\|_{L^{p/2}(\Omega)} + \sum_{i=1}^{n}\|f_i\|_{L^p(\Omega)}\right\} + K_2\|u\|_{L^2(\Omega)}$$

was proved, by supposing Ω bounded and f_i, $d_i \in L^p(\Omega)$ $(i = 1, 2, \ldots, n)$, f_0, $c \in L^{p/2}(\Omega)$, $p > n$.

The aim of the present work is to extend these results first of all allowing the set Ω to be unbounded and relaxing the hypotheses on the functions f_0, f_i, b_i, d_i, c $(i = 1, 2 \ldots, n)$. Finally, the constants in the a priori inequality (3) are explicitly evaluated.

2. Notations and Hypotheses.

Let Ω be an open subset (bounded or unbounded) of $\mathbb{R}^n$. Let $a_{ij} \in L^\infty(\Omega)$ $(i, j = 1, 2, \ldots, n)$, $\sum_{i,j=1}^{n} a_{ij} t_i t_j \geq \nu |t|^2$ $\forall t \in \mathbb{R}^n$ a.e. in Ω, where ν is a positive constant. Let $c^+ := \max(c, 0)$, $c^- := \min(c, 0)$ and suppose that $c^+ \in L^{2n/(n+2)}(\Omega')$ for any Ω' bounded, $\Omega' \subset \Omega$. Let us define the spaces

$$\text{(4)} \qquad X^p(\Omega) := \{f \in L^p_{\text{loc}}(\Omega) \colon \omega(f, p, \delta) < +\infty \ \forall \delta > 0\}$$

$$\text{(5)} \qquad X^p_0(\Omega) := \{f \in X^p(\Omega) \colon \lim_{\delta \to 0^+} \omega(f, p, \delta) = 0\}$$

where

$$\text{(6)} \qquad \omega(f, p, \delta) := \sup\{\|f\|_{L^p(E)} \colon E \text{ measurable}, \ E \subset \Omega, \text{meas}(E) \leq \delta\}.$$

Remark 1. If $f \in L^p_{\text{loc}}(\Omega)$, we define, for $k > 0$,

$$\text{(7)} \qquad \phi(f, p, k) := \inf\{\text{meas}(E) \colon E \text{ measurable}, \ E \subset \Omega, \|f\|_{L^p(E)} \geq k\},$$

and we have

$$\text{(8)} \qquad f \in X^p(\Omega) \quad \text{if and only if} \quad \exists k_0 > 0 \text{ such that } \phi(f, p, k_0) > 0,$$

$$\text{(9)} \qquad f \in X^p_0(\Omega) \quad \text{if and only if} \quad \phi(f, p, k) > 0 \quad \forall k > 0.$$

Remark 2. If G is a measurable subset of Ω such that $\text{meas}(G) \leq \phi(f, p, k)$, then it turns out that $\|f\|_{L^p(G)} \leq k$. In fact, if not there would exist a subset G_0 of G with positive measure but so small that

$$\|f\|_{L^p(G\backslash G_0)} > k$$

which is in contradiction with the definition of ϕ, since meas$(G\backslash G_0) <$ $<$ meas(G). ■

REMARK 3. If $1 \leq q < p$ it turns out $X^p(\Omega) \subset X_0^q(\Omega)$.
In fact, if $E \subset \Omega$, meas$(E) \leq \delta$, $f \in X^p(\Omega)$ we have

$$\|f\|_{L^q(E)} \leq \|f\|_{L^p(E)}[\text{meas}(E)]^{(p-q)/pq} \leq \omega(f, p, \delta)\delta^{(p-q)/pq}$$

whence

$$\omega(f, q, \delta) \leq \omega(f, p, \delta)\delta^{(p-q)/pq}. \qquad ■$$

We denote by S the constant in the Sobolev inequality

$$\|g\|_{L^{2n/(n-2)}(\mathbb{R}^n)} \leq S\|g_x\|_{L^2(\mathbb{R}^n)} \qquad \forall g \in C_0^1(\mathbb{R}^n).$$

It is a well known fact (see e.g. [4]) that S is given by the following formula:

$$S = [n(n-2)\pi]^{-1/2}\Gamma(n)^{1/n}\Gamma(n/2)^{-1/n}. \tag{10}$$

LEMMA. *Let $u \in H_0^1(\Omega)$, $B \subset \Omega$, $u = 0$ in B. Then there exists a sequence $\{u_j\}_{j\in\mathbb{N}} \subset H_0^1(\Omega)$ such that $u_j = 0$ in B, u_j has compact support in Ω $(j = 1, 2, \ldots)$, $\lim_j \|u - u_j\|_{H^1(\Omega)} = 0$.*

PROOF. It follows from the results of [3] that $u^+ := \max(u, 0)$, $u^- := \min(u, 0)$ both belong to $H_0^1(\Omega)$, therefore we may assume without loss of generality that $u \geq 0$ in Ω. By definition of $H_0^1(\Omega)$, there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset C_0^1(\Omega)$ such that $\lim_j \|u - \phi_j\|_{H^1(\Omega)} = 0$; we may assume $\phi_j \geq 0$ in Ω $(j = 1, 2, \ldots)$. Consider the functions $u_j :=$ $= \min(u, \phi_j)$ $(j = 1, 2, \ldots)$. These functions are in $H_0^1(\Omega)$ and they vanish on B and where $\phi_j = 0$. Furthermore it is easy to verify that $|(u - u_j)_x| \leq |(u - \phi_j)_x|$ where all the derivatives exist (i.e. almost everywhere in Ω), whence

$$\|(u - u_j)_x\|_{L^2(\Omega)} \leq \|(u - \phi_j)_x\|_{L^2(\Omega)} \qquad (j = 1, 2, \ldots). \tag{11}$$

Therefore the sequence $\{u_j\}_{j\in\mathbb{N}}$ has the required properties. ■

3. Main result.

THEOREM. *In addition to the hypotheses mentioned above, we assume*: $p > n$, $c^- \in X_0^{np/(n+p))}(\Omega)$, $b_i \in X_0^n(\Omega)$, $d_i \in X_0^p(\Omega)$, $f_i \in X^p(\Omega)$ $(i = 1, 2, \ldots, n)$, $f_0 \in X^{np/(n+p)}(\Omega)$, $u \in H^1_{\text{loc}}(\Omega)$,

$$(12)\qquad a(u, v) \leq \int_\Omega \left\{ f_0 v + \sum_{i=1}^n f_i v_{x_i} \right\} dx \qquad \forall v \in C_0^1(\Omega), \quad v \geq 0 \ \textit{in}\ \Omega .$$

Furthermore suppose that there exists a nonnegative real number m *such that* $\max(u - m, 0) \in H_0^1(\Omega)$.

Then there exist constants K_1, K_2, K_3, *depending on the coefficients of* $a(\cdot, \cdot)$, *on* n *and* p, *such that*

$$(13)\qquad \operatorname*{ess\,sup}_\Omega u \leq K_1 \|\max(u - m, 0)\|_{L^2(\Omega)} + 2^{np/(p-n)} m +$$
$$+ K_2 \left\{ S\omega(f_0, np/(p+n), K_3) + \sum_{i=1}^n \omega(f_i, p, K_3) \right\}$$

where:

S is the Sobolev constant (10),
$K_1 = (4/3)^{np/(p-n)} + 2^{np/(p-n)} K_3^{-1/2}$,
$K_2 = (3S/\nu)[2^{np/(p-n)} - 1]$,
$K_3 = \min\{1, \phi(b_i, n, \nu/(6Sn)), \phi(d_i, p, \nu/(6Sn)), \phi(c^-, np/(p+n), \nu/(6S^2))$
$(i = 1, 2, \ldots, n)\}$

PROOF. First of all we notice that if $t \geq m$ obviously the function $u_t := \max(u - t, 0)$ is in $H_0^1(\Omega)$ as well. Moreover, it is easy to check that (12) is verified also by nonnegative functions $v \in H_0^1(\Omega)$ with compact support contained in Ω. In fact, let A be an open bounded set containing the support of v, such that $\overline{A} \subset \Omega$. It is easy to find a sequence $\{v_j\}_{j \in \mathrm{N}} \subset C_0^1(A)$ which converges to v in the norm of $H^1(A)$. We can write (12) with v_j instead of v and let j go to infinity, taking into account Hölder's and Sobolev's inequalities and the fact that $u \in H^1(A)$ by hypothesis (and also $u \in L^{2n/(n-2)}(A)$). So, (12) is true if $v \in H_0^1(\Omega)$ with compact support contained in Ω. Then from the lemma above we can find a sequence of functions $\{u_j\}_{j \in \mathrm{N}} \subset H_0^1(\Omega)$ having compact support in Ω, vanishing where $u_t = 0$ (i.e. where $u \leq t$), and converging to u_t in the norm of $H^1(\Omega)$. As before, we can write (12) with u_j instead of v and let j go to infinity, because u_t and u_j are different from zero only in a (fixed) set of finite measure, in which $u = u_t + t$, thus allowing again the use of Hölder's and Sobolev's inequalities. We conclude that (12) can be written with v replaced by u_t (where it is always $t \geq m$). Let us denote for brevity

$$\Omega_t := \{x \in \Omega : u(x) > t\}.$$

By using Hölder's and Sobolev's inequalities, and taking into account our previous hypotheses, we deduce

$$\nu \|(u_t)_x\|^2_{L^2(\Omega_t)} \leq \sum_{i=1}^n \int_{\Omega_t} a_{ij} u_{x_i} (u_t)_{x_j} dx,$$

$$\left| \int_\Omega \sum_{i=1}^n b_i u_{x_i} u_t \, dx \right| \leq \sum_{i=1}^n \int_{\Omega_t} |b_i (u_t)_{x_i} u_t| \, dx \leq S \sum_{i=1}^n \|b_i\|_{L_n(\Omega_t)} \|(u_t)_x\|^2_{L^2(\Omega_t)},$$

$$\left| \int_\Omega \sum_{i=1}^n d_i u (u_t)_{x_i} dx \right| \leq \sum_{i=1}^n \int_{\Omega_t} |d_i u_t (u_t)_{x_i}| \, dx + t \sum_{i=1}^n \int_{\Omega_t} |d_i (u_t)_{x_i}| \, dx \leq$$

$$\leq S \sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} (\text{meas}\, \Omega_t)^{(p-n)/np} \|(u_t)_x\|^2_{L^2(\Omega_t)} +$$

$$+ t \sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} (\text{meas}\, \Omega_t)^{(p-2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)},$$

$$\left| \int_\Omega c^- u u_t \, dx \right| \leq \int_{\Omega_t} |c^- u_t^2| \, dx + t \int_{\Omega_t} |c^- u_t| \, dx \leq$$

$$\leq S^2 \|c^-\|_{L^{np/(n+p)}(\Omega_t)} (\text{meas}\, \Omega_t)^{(p-n)/np} \|(u_t)_x\|_{L^2(\Omega_t)} +$$

$$+ tS \|c^-\|_{L^{np/(n+p)}(\Omega_t)} (\text{meas}\, \Omega_t)^{(p-2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)},$$

$$\left| \int_\Omega f_0 u_t \, dx \right| \leq S \|f_0\|_{L^{np/(n+p)}(\Omega_t)} (\text{meas}\, \Omega_t)^{(p-2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)},$$

$$\left| \int_\Omega \sum_{i=1}^n f_i (u_t)_{x_i} dx \right| \leq \sum_{i=1}^n \|f_i\|_{L^p(\Omega_t)} (\text{meas}\, \Omega_t)^{(p-2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)}.$$

Therefore it follows easily from (12)

(14) $$\nu \|(u_t)_x\|^2_{L^2(\Omega_t)} \leq$$

$$\leq t \left[\sum_{i=1}^n \|d_i\|_{L^p(\Omega_t)} + S \|c^-\|_{L^{np/(n+p)}(\Omega_t)} \right] (\text{meas}\, \Omega_t)^{(p-2)/2p} \|(u_t)_x\|_{L^2(\Omega_t)} +$$

$$+\left[S\|f_0\|_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n}\|f_i\|_{L^p(\Omega_t)}\right](\text{meas}\,\Omega_t)^{(p-2)/2p}\|(u_t)_x\|_{L^2(\Omega_t)} +$$

$$+S\left[\sum_{i=1}^{n}\|b_i\|_{L^n(\Omega_t)} + \sum_{i=1}^{n}\|d_i\|_{L^p(\Omega_t)}(\text{meas}\,\Omega_t)^{(p-n)/np}\right]\|(u_t)_x\|^2_{L^2(\Omega_t)} +$$

$$+S^2\|c^-\|_{L^{np/(n+p)}(\Omega_t)}(\text{meas}\,\Omega_t)^{(p-n)/np}\|(u_t)_x\|^2_{L^2(\Omega_t)}.$$

For brevity, let us denote $\alpha(t) := \text{meas}(\Omega_t)$. Then we get

$$\text{(15)}\quad \left\{\nu - S\left[\sum_{i=1}^{n}\|b_i\|_{L^n(\Omega_t)} + \sum_{i=1}^{n}\|d_i\|L^p(\Omega_t)[\alpha(t)]^{(p-n)/np} + \right.\right.$$

$$\left.\left. +S\|c^-\|_{L^{np/(n+p)}(\Omega_t)}[\alpha(t)]^{(p-n)/np}\right]\right\}\|(u_t)_x\|_{L^2(\Omega_t)} \leqslant$$

$$\leqslant \left[S\|f_0\|_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n}\|f_i\|_{L^p(\Omega_t)}\right][\alpha(t)]^{(p-2)/2p} +$$

$$+t\left[\sum_{i=1}^{n}\|d_i\|_{L^p(\Omega_t)} + S\|c^-\|_{L^{np/(n+p)}(\Omega_t)}\right][\alpha(t)]^{(p-2)/2p}.$$

We notice that, when $t \geqslant m$, we have

$$\int_{\Omega_m}(u-m)^2\,dx \geqslant \int_{\Omega_t}(u-m)^2\,dx \geqslant (t-m)^2\alpha(t)$$

that is:

$$\text{(16)}\qquad \alpha(t) \leqslant \frac{\|u_m\|^2_{L_2(\Omega_m)}}{(t-m)^2} \qquad \forall t > m\,.$$

Now we define (see (7))

$$\text{(17)}\quad \delta_0 := \min\{1,\ \phi(b_i, n, \nu/(6Sn)),\ \phi(d_i, p, \nu/(6Sn)),$$

$$\phi(c^-, np/(n+p), \nu/(6S^2)),\ (i = 1, 2, \ldots, n)\}$$

$$\text{(18)}\qquad t_0 := m + \frac{\|u_m\|_{L^2(\Omega)}}{\delta_0^{1/2}}$$

(please note that $\delta_0 > 0$ because of our previous hypotheses and remark 1).

Then if $t \geqslant t_0$ we have

$$\text{(19)}\qquad \alpha(t) \leqslant \alpha(t_0) \leqslant \frac{\|u_m\|^2_{L^2(\Omega)}}{(t_0-m)^2} = \delta_0$$

therefore by the definition of ϕ and remark 2 we deduce

(20) $$\sum_{i=1}^{n} \|b_i\|_{L^n(\Omega_t)} \leq \nu/(6S),$$

(21) $$\sum_{i=1}^{n} \|d_i\|_{L^p(\Omega_t)} \leq \nu/(6S),$$

(22) $$\|c^-\|_{L^{np/(n+p)}(\Omega_t)} \leq \nu/(6S^2).$$

From (16), (17), (19) it follows $\alpha(t) \leq 1$; then from (15), (20), (21), (22) when $t \geq t_0$ we get

(23) $$(\nu/2)\|(u_t)_x\|_{L^2(\Omega_t)} \leq [\alpha(t)]^{(p-2)/2p}\Big[t\Big(\sum_{i=1}^{n}\|d_i\|_{L^p(\Omega_t)} + S\|c^-\|_{L^{np/(n+p)}(\Omega_t)}\Big) + + S\|f_0\|_{L^{np/(n+p)}(\Omega_t)} + \sum_{i=1}^{n}\|f_i\|_{L^p(\Omega_t)}\Big].$$

Let us denote, for brevity,

(24) $$K_4 := (2S/\nu)\Big(\sum_{i=1}^{n}\|d_i\|_{L^p(\Omega_{t_0})} + S\|c^-\|L^{np/(n+p)}(\Omega_{t_0})\Big)$$

(25) $$K_5 := (2S/\nu)\Big(\sum_{i=1}^{n}\|f_i\|_{L^p(\Omega_{t_0})} + S\|f_0\|L^{np/(n+p)}(\Omega_{t_0})\Big)$$

and apply Hölder's and Sobolev's inequalitites to (23), thus obtaining

(26) $$\|u_t\|_{L^1(\Omega_t)} \leq [\alpha(t)]^{(2+n)/2n}\|u_t\|_{L^{2n/(n-2)}(\Omega_t)} \leq [\alpha(t)]^{1+(p-n)/np}(K_4 t + K_5)$$

Now we follow a procedure of [1]. Define

(27) $$\beta(t) := \|u_t\|_{L^1(\Omega_t)}, \qquad t \geq t_0$$

and note that it turns out $\beta(t) = \int_t^{+\infty} \alpha(s)\,ds$. Therefore

(28) $$\beta'(t) = -\alpha(t) \leq 0 \quad \text{a.e. in } [t_0, +\infty).$$

From (26), (28) we get the differential inequality

(29) $$\beta(t) \leq (K_4 t + K_5)[-\beta'(t)]^{1+(p-n)/np} \quad \text{a.e. in } [t_0, +\infty)$$

Suppose now, by contradiction, that $\beta(t) > 0$ $\forall t \geq t_0$ (i.e., by definition of

$\beta(t)$, $\operatorname{ess\,sup}_{\Omega} u = +\infty$). Then in (29) we can divide by $\beta(t)$ obtaining

$$(30) \qquad -\beta'(t)[\beta(t)]^{-np/(np+p-n)} \geqslant (K_4 t + K_5)^{-np/(np+p-n)}$$

Integrating (30) between t_0 and $t^* > t_0$ (suppose for the moment $K_4 > 0$), we obtain

$$(31) \qquad K_4[\beta(t_0)]^{(p-n)/(np+p-n)} - K_4[\beta(t^*)]^{(p-n)/(np+p-n)} \geqslant$$
$$\geqslant (K_4 t^* + K_5)^{(p-n)/(np+p-n)} - (K_4 t_0 + K_5)^{(p-n)/(np+p-n)}$$

which gives a contradiction when t^* tends to $+\infty$.

Then it must be $\operatorname{ess\,sup}_{\Omega} u < +\infty$. We can rewrite (31) with $t_0 < t^* < \operatorname{ess\,sup}_{\Omega} u$; by letting t^* tend to $\operatorname{ess\,sup}_{\Omega} u$ we get

$$(32) \qquad (K_4 \operatorname{ess\,sup}_{\Omega} u + K_5)^{(p-n)/(np+p-n)} \leqslant$$
$$\leqslant (K_4 t_0 + K_5)^{(p-n)/(np+p-n)} + K_4[\beta(t_0)]^{(p-n)/(np+p-n)}$$

Please note that the constant K_4 is not greater than 2/3 because of (21), (22). From (32) by easy calculations we get

$$(33) \qquad \operatorname{ess\,sup}_{\Omega} u \leqslant (4/3)^{np/(p-n)} \|u_{t_0}\|_{L^1(\Omega_{t_0})} + 2^{np/(p-n)} t_0 + (3/2)[2^{np/(p-n)} - 1] K_5$$

whence, by recalling the definition of t_0 (18) and K_5 (25) one can write

$$(34) \qquad \operatorname{ess\,sup}_{\Omega} u \leqslant 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_0^{-1/2}] \|u_m\|_{L^2(\Omega)} +$$
$$+ (3S/\nu[2^{np/(p-n)} - 1] \left[S \|f_0\|_{L^{np/(p+n)}(\Omega_{t_0})} + \sum_{i=1}^{n} \|f_i\|_{L^p(\Omega_{t_0})} \right].$$

Finally, by taking into account (19), the definition of δ_0 (see (17)) and the functions ϕ, ω, we conclude

$$(35) \qquad \operatorname{ess\,sup}_{\Omega} u \leqslant 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_0^{-1/2}] \|u_m\|_{L^2(\Omega)} +$$
$$+ (3S/\nu)[2^{np/(p-n)} - 1] \left[S\omega(f_0, np/(p+n), \delta_0) + \sum_{i=1}^{n} \omega(f_i, p, \delta_0) \right]$$

with δ_0 given by (17). ■

REMARK 4. If we suppose, in addition to the hypotheses of the previous theorem, that there exists $q \geqslant 1$ such that $u_m \in L^q(\Omega)$, then we can write, instead of (16) and (18)

$$(16') \qquad \alpha(t) \leq \|u_m\|^q_{L^q(\Omega_m)}(t-m)^{-q} \qquad \forall t > m\,,$$

$$(18') \qquad t_0 := m + \|u_m\|_{L^q(\Omega)} \delta_0^{-1/q}$$

and proceeding as before we get to the conclusion in the form

$$(35') \qquad \operatorname*{ess\,sup}_{\Omega} u \leq 2^{np/(p-n)} m + [(4/3)^{np/(p-n)} + 2^{np/(p-n)} \delta_0^{-1/q}] \|u_m\|_{L^q(\Omega)} + \\ + (3S/\nu)[2^{np/(p-n)} - 1]\left[S\omega(f_0, np/(p+n), \delta_0) + \sum_{i=1}^{n} \omega(f_i, p, \delta_0)\right]$$

where δ_0 is always given by (17).

REMARK 5. Suppose the coefficients d_i and c^- of the bilinear form $a(\cdot, \cdot)$ to be identically zero. Then the constant K_4 defined in (24) vanishes, and by integrating (30) we get, more simply,

$$(36) \qquad \operatorname*{ess\,sup}_{\Omega} u \leq t_0 + (np + p - n)/(p-n)\, K_5^{np/(np+p-n)} \|u_{t_0}\|^{(p-n)/(np+p-n)}_{L^2(\Omega)}$$

whence, by taking into account the definitions of t_0, δ_0, ..., and Young's inequality, we deduce

$$(37) \qquad \operatorname*{ess\,sup}_{\Omega} u \leq m + (\delta_0^{-1/2} + 1)\|u_m\|_{L^2(\Omega)} + [np/(p-n)]K_5.$$

This inequality is of the same kind of (35), but the coefficient of m in it is now 1.

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