RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

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Rendiconti del Seminario Matematico della Università di Padova, tome 101 (1999), p. 63-82

<http://www.numdam.org/item?id=RSMUP_1999__101__63_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 101 (1999)

Approximation of Functions of Two Variables by Some Operators in Weighted Spaces.

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ABSTRACT - We study some linear positive operators $T_{m,n}^{\{i\}}$ of the Szasz-Mirakjan type in polynomial and exponential weighted spaces of continuous functions of two variables. We give theorems on the degree of approximation, and theorems of the Voronovskaja and Bernstein type for these operators. Similar results for functions of one variable are given in [3]-[8].

1. - Notation.

1.1. We take the following notation: $N := \{1, 2, \dots\}$, $N_0 := N \cup \{0\}$, $R_0 := [0, +\infty)$, $R_0^2 = R_0 \times R_0$ and, for $p \in N_0$ and $x \in R_0$, let

(1) $$w_0(x) := 1\,, \qquad w_p(x) := (1 + x^p)^{-1} \qquad \text{if } p \geq 1\,.$$

For fixed $p_1, p_2 \in N_0$ let

(2) $$w_{p_1, p_2}(x, y) := w_{p_1}(x)\, w_{p_2}(y)\,, \qquad (x, y) \in R_0^2,$$

and let $C_{1; p_1, p_2}$ be the space of all real-valued functions f defined on R_0^2 such that $w_{p_1, p_2}(\cdot, \cdot)\, f(\cdot, \cdot)$ is uniformly continuous and bounded on R_0^2 and the norm is given by

(3) $$\|f\|_{1; p_1, p_2} := \sup_{(x, y) \in R_0^2} w_{p_1, p_2}(x, y)\, |f(x, y)|\,.$$

$C_{1; p_1, p_2}$ with the norm (3) is called the polynomial weighted space ([1]).

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For $f \in C_{1;\, p_1, p_2}$ we define the modulus of continuity

$$\omega(f, C_{1;\, p_1, p_2}; t, s) := \sup_{\substack{0 \le h \le t \\ 0 \le \delta \le s}} \|\Delta_{h, \delta} f(\cdot, \cdot)\|_{1;\, p_1, p_2}, \qquad t, s \ge 0, \tag{4}$$

where $\Delta_{h, \delta} f(x, y) = f(x+h, y+\delta) - f(x, y)$. Moreover, for fixed $m, p_1, p_2 \in N_0$, let $C^m_{1;\, p_1, p_2}$ be the space of all functions $f \in C_{1;\, p_1, p_2}$ having the partial derivatives of the order $\le m$ belonging also to $C_{1;\, p_1, p_2}$.

1.2. Similarly as in [2] we define the exponential weighted space $C_{2;\, q_1, q_2}$. Let for a fixed $q > 0$

$$v_q(x) := e^{-qx}, \qquad x \in R_0, \tag{5}$$

and, for fixed $q_1, q_2 > 0$, let

$$v_{q_1, q_2}(x, y) := v_{q_1}(x)\, v_{q_2}(y), \qquad (x, y) \in R_0^2. \tag{6}$$

We denote by $C_{2;\, q_1, q_2}$, $q_1, q_2 > 0$, the space of all real-valued functions f defined on R_0^2 for which $v_{q_1, q_2}(\cdot, \cdot)\, f(\cdot, \cdot)$ is uniformly continuous and bounded on R_0^2 and the norm is given by

$$\|f\|_{2;\, q_1, q_2} := \sup_{(x, y) \in R_0^2} v_{q_1, q_2}(x, y)\, |f(x, y)|. \tag{7}$$

Analogously as above we define the modulus of continuity $\omega(f, C_{2;\, q_1, q_2}; \cdot, \cdot)$ of $f \in C_{2;\, q_1, q_2}$ and the class $C^m_{2;\, q_1, q_2}$.

1.3. In this paper we introduce the following operators in the space $C_{1;\, p_1, p_2}$ and $C_{2;\, q_1 q_2}$:

$$T^{\{1\}}_{m,n}(f; x, y) := \frac{1}{1 + \sinh ny} \sum_{j=0}^{\infty} a_{m,j}(x)\, f\left(\frac{2j}{m}, 0\right) + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x)\, b_{n,k}(y)\, f\left(\frac{2j}{m}, \frac{2k+1}{n}\right), \tag{8}$$

$$T^{\{2\}}_{m,n}(f; x, y) := \frac{1}{1 + \sinh ny} \sum_{j=0}^{\infty} a_{m,j}(x)\, \frac{m}{2} \int_{2j/m}^{(2j+2)/m} f(u, 0)\, du + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{m,j}(x)\, b_{n,k}(y)\, \frac{mn}{4} \int_{2j/m}^{(2j+2)/m} \int_{(2k+1)/n}^{(2k+3)/n} f(u, v)\, du\, dv, \tag{9}$$

for all $m, n \in N$ and $(x, y) \in R_0^2$, where

(10) $$a_{m,j}(x) := \frac{1}{\cosh mx} \frac{(mx)^{2j}}{(2j)!}, \qquad j \in N_0,$$

(11) $$b_{n,k}(y) := \frac{1}{1 + \sinh ny} \frac{(ny)^{2k+1}}{(2k+1)!}, \qquad k \in N_0,$$

and $\cosh x$, $\sinh x$, $\tanh x$ are the elementary hyperbolic functions.

From (8)-(11) we deduce that $T_{m,n}^{\{i\}}$ $m, n \in N$, $n \in N$, $i = 1, 2$, are linear positive operators well-defined in every space $C_{1;\,p_1,\,p_2}$ and $C_{2;\,q_1,\,q_2}$. Moreover,

(12) $$T_{m,n}^{\{i\}}(1; x, y) = 1, \qquad \text{for } m, n \in N,\ (x, y) \in R_0^2,\ i = 1, 2.$$

In Section 2 we shall give some auxiliary properties of $T_{m,n}^{\{i\}}$. In Sect. 3 we shall give some approximation theorems, the Voronovskaja theorem and the Bernstein inequality for these operators.

In this paper we shall denote by $M_k(a, b)$, $k = 1, 2, \dots$, the suitable positive constants depending only on indicated parameters a, b.

1.4. The operators $T_{m,n}^{\{i\}}$ are some analogues of the operators $L_n^{\{i\}}$ considered for function f of one variable in the papers [3]-[8], i.e.

(13) $$L_n^{\{1\}}(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x) f\left(\frac{2k}{n}\right),$$

(14) $$L_n^{\{2\}}(f; x) := \sum_{k=0}^{\infty} a_{n,k}(x) \frac{n}{2} \int_{2k/n}^{(2k+2)/n} f(t)\, dt,$$

(15) $$L_n^{\{3\}}(f; x) := \frac{f(0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} b_{n,k}(x) f\left(\frac{2k+1}{n}\right),$$

(16) $$L_n^{\{4\}}(f; x) := \frac{f(0)}{1 + \sinh nx} + \sum_{k=0}^{\infty} b_{n,k}(x) \frac{n}{2} \int_{(2k+1)/n}^{(2k+3)/n} f(t)\, dt,$$

$x \in R_0$, $n \in N$. The operators $L_n^{\{i\}}$ are examined in [3]-[8] for functions blonging to a polynomial or exponential weighted space ($L_n^{\{i\}}$, $i = 1, 2$, were introducend in [3]; $L_n^{\{i\}}$, $i = 3, 4$, were defined in [5]).

By (10), (11) and (13)-(16) we have

(17) $$L_n^{\{i\}}(1;x)=1 \quad \text{for all } n\in N,\ x\in R_0,\ 1\leq i\leq 4.$$

2. - Auxiliary results.

2.1. First we shall give some properties of the operators $L_n^{\{i\}}$ proved in [3]-[8].

LEMMA 1 ([3], [5]). *For all $n\in N$, $x\in R_0$ and $1\leq i\leq 4$ we have*

$$|L_n^{\{i\}}(t-x;x)|\leq \frac{5}{n}, \qquad L_n^{\{i\}}((t-x)^2;x)\leq 11\,\frac{x+1}{n}. \quad \blacksquare$$

LEMMA 2 ([7]). *For every fixed $x_0\in R_0$ there exists a positive constant $M_1(x_0)$ such that for all $n\in N$ and $1\leq i\leq 4$*

$$L_n^{\{i\}}((t-x_0)^4;x_0)\leq M_1(x_0)\cdot n^{-2}. \quad \blacksquare$$

LEMMA 3 ([7]. *For every fixed $x_0\in R_0$ we have*

$$\lim_{n\to\infty} nL_n^{\{i\}}(t-x_0;x_0)=\begin{cases}0 & \text{if } i=1,3,\\ 1 & \text{if } i=2,\end{cases}$$

$$\lim_{n\to\infty} nL_n^{\{4\}}(t-x_0;x_0)=\begin{cases}1 & \text{if } x_0>0,\\ 0 & \text{if } x_0=0,\end{cases}$$

$$\lim_{n\to\infty} nL_n^{\{i\}}((t-x_0)^2;x_0)=x_0 \qquad \text{for } 1\leq i\leq 4. \quad \blacksquare$$

LEMMA 4 ([3], [5]). *For every fixed $p\in N_0$ there exist positive constants $M_2(p)$ and $M_3(p)$ such that for all $x\in R_0$, $n\in N$ and $1\leq i\leq 4$*

$$w_p(x)\,L_n^{\{i\}}\left(\frac{1}{w_p(t)};x\right)\leq M_2(p),$$

$$w_p(x)\,L_n^{\{i\}}\left(\frac{(t-x)^2}{w_p(t)};x\right)\leq M_3(p)\,\frac{x+1}{n}. \quad \blacksquare$$

LEMMA 5 ([4], [6]). *For every fixed $q > 0$ and $r > q$ there exist positive constants $M_k(q, r)$, $k = 4, 5$, and a natural number $n_0 > q(\ln(r/q))^{-1}$ such that for all $n > n_0$, $x \in R_0$ and $1 \leq i \leq 4$*

$$v_r(x)\, L_n^{\{i\}}\left(\frac{1}{v_q(t)}\,;\, x\right) \leq M_4(q, r),$$

$$v_r(x)\, L_n^{\{i\}}\left(\frac{(t-x)^2}{v_q(t)}\,;\, x\right) \leq M_5(q, r)\,\frac{x+1}{n}. \quad \blacksquare$$

LEMMA 6 ([8]). *For every fixed $p, s \in N_0$ there exist positive constants $M_6(s)$ and $M_7(p, s)$ such that for all $n \in N$ we have*

$$\left|\frac{d^s}{dx^s}\frac{1}{1+\sinh nx}\right| \leq M_6(s)\,\frac{n^s}{1+\sinh nx}, \qquad x \in R_0,$$

$$\sup_{x \in R_0} w_p(x) \sum_{k=0}^{\infty} \left|\frac{d^s}{dx^s} a_{n,k}(x)\right| \frac{1}{w_p(2k/n)} \leq M_7(p, s)\, n^s,$$

$$\sup_{x \in R_0} w_p(x) \sum_{k=0}^{\infty} \left|\frac{d^s}{dx^s} b_{n,k}(x)\right| \frac{1}{w_p((2k+1)/n)} \leq M_7(p, s)\, n^s. \quad \blacksquare$$

LEMMA 7 ([8]). *For every fixed $s \in N_0$ and $r > q > 0$ there exist a positive constant $M_8(q, r, s)$ and a natural number $n_0 > q(\ln(r/q))^{-1}$ such that for all $n > n_0$*

$$\sup_{x \in R_0} v_r(x) \sum_{k=0}^{\infty} \left|\frac{d^s}{dx^s} a_{n,k}(x)\right| \frac{1}{v_q(2k/n)} \leq M_8(q, r, s)\, n^s,$$

$$\sup_{x \in R_0} v_r(x) \sum_{k=0}^{\infty} \left|\frac{d^s}{dx^s} b_{n,k}(x)\right| \frac{1}{v_q((2k+1)/n)} \leq M_8(q, r, s)\, n^s, \quad \blacksquare$$

2.2. In this part we shall give some basic properties of the operators $T_{m,n}^{\{i\}}$. From (8)-(16) we deduce that if $f \in C_{1;\,p_1, p_2}$, $p_1, p_2 \in N_0$, or $f \in C_{2;\,q_1, q_2}$, $q_1, q_2 > 0$, and $f(x, y) = f_1(x)\, f_2(y)$ for $(x, y) \in R_0^2$, then

$$T_{m,n}^{\{1\}}(f; x, y) = L_m^{\{1\}}(f_1; x)\, L_n^{\{3\}}(f_2; y), \tag{18}$$

$$T_{m,n}^{\{2\}}(f; x, y) = L_m^{\{2\}}(f_1; x)\, L_n^{\{4\}}(f_2; y), \tag{19}$$

for all $(x, y) \in R_0^2$ and $m, n \in N$.

Applying Lemmas 1-7 and (18), (19) and (1)-(6), we immediately obtain the following two lemmas.

LEMMA 8. *For every fixed* $p_1, p_2 \in N_0$ *there exists a positive constant* $M_9(p_1, p_2)$ *such that for all* $m, n \in N$ *and* $i = 1, 2$

$$\left\| T^{\{i\}}_{m,n}\left(\frac{1}{w_{p_1,p_2}(t,z)}; \cdot, \cdot \right) \right\|_{1;p_1,p_2} \leqslant M_9(p_1, p_2).$$

Consequently,

$$\|T^{\{i\}}_{m,n}(f; \cdot, \cdot)\|_{1;p_1,p_2} \leqslant M_9(p_1, p_2) \|f\|_{1;p_1,p_2}$$

for every $f \in C_{1;p_1,p_2}$, $m, n \in N$ *and* $i = 1, 2$. *This inequality and* (8)-(11) *show that* $T^{\{i\}}_{m,n}$, $m, n \in N$, $i = 1, 2$, *is a linear positive operator from the space* $C_{1;p_1,p_2}$ *into* $C_{1;p_1,p_2}$, $p_1, p_2 \in N_0$. ■

LEMMA 9. *For every fixed* $q_1, q_2 > 0$ *and* $r_1 > q_1$, $r_2 > q_2$ *there exist a positive constant* $M^*_{10} \equiv M_{10}(q_1, q_2, r_1, r_2)$ *and natural numbers* m_0 *and* n_0,

$$m_0 > q_1 \left(\ln \frac{r_1}{q_1} \right)^{-1}, \qquad n_0 > q_2 \left(\ln \frac{r_2}{q_2} \right)^{-1}, \tag{20}$$

such that for all $m > m_0$, $n > n_0$ *and* $i = 1, 2$

$$\left\| T^{\{i\}}_{m,n}\left(\frac{1}{v_{q_1,q_2}(t,z)}; \cdot, \cdot \right) \right\|_{2;r_1,r_2} \leqslant M^*_{10}.$$

Consequently, for every $f \in C_{2;q_1,q_2}$, $m > m_0$, $n > n_0$ *and* $i = 1, 2$, *we have*

$$\|T^{\{i\}}_{m,n}(f; \cdot, \cdot)\|_{2;r_1,r_2} \leqslant M^*_{10} \|f\|_{2;q_1,q_2}.$$

From this and by (8)-(11) *it follows that* $T^{\{i\}}_{m,n}$, $i = 1, 2$, *is a linear positive operator from the space* C_2; q_1, q_2 *into* $C_{2;r_1,r_2}$ *provided that* $m > m_0$ *and* $n > n_0$. ■

Applying the above lemmas, we shall prove the following

LEMMA 10. *Suppose that* (x_0, y_0) *is a fixed point in* R_0^2 *and* φ *is a given function with some space* $C_{1;p_1,p_2}$, $p_1, p_2 \in N_0$, *and* $\varphi(x_0, y_0) = 0$. *Then*

$$\lim_{n\to\infty} T_{n,n}^{\{i\}}(\varphi(t, z); x_0, y_0) = 0, \qquad i = 1, 2. \tag{21}$$

PROOF. Let $i = 1$. By (8) we have for every $n \in N$

$$\begin{aligned} (22)\quad & w_{p_1,p_2}(x_0, y_0)\,|T_{n,n}^{\{1\}}(\varphi(t, z); x_0, y_0)| \leqslant \\ & \leqslant \frac{w_{p_1,p_2}(x_0, y_0)}{1+\sinh ny_0} \sum_{j=0}^{\infty} a_{n,j}(x_0) \left|\varphi\left(\frac{2j}{n}, 0\right)\right| + \\ & + w_{p_1,p_2}(x_0, y_0) \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a_{n,j}(x_0)\, b_{n,k}(y_0) \left|\varphi\left(\frac{2j}{n}, \frac{2k+1}{n}\right)\right| := \\ & := A_n(x_0, y_0) + B_n(x_0, y_0). \end{aligned}$$

We shall prove that $\lim_{n\to\infty} A(x_0, y_0) = 0 = \lim_{n\to\infty} B_n(x_0, y_0)$. First let $x_0 > 0$ and $y_0 > 0$. Choose $\varepsilon > 0$. By the properties of φ there exist two positive constant M_{11} and $\delta \equiv \delta(\varepsilon)$ such that

$$w_{p_1,p_2}(t, z)\,|\varphi(t, z)| \leqslant M_{11} \quad \text{for all } (t, z) \in R_0^2, \tag{23}$$

$$w_{p_1,p_2}(t, z)\,|\varphi(t, z)| \leqslant \frac{\varepsilon}{4M_9^*} \quad \text{for } |t - x_0| < \delta,\ |z - y_0| < \delta, \tag{24}$$

where $M_9^* \equiv M_9(p_1, p_2)$ is a fixed positive constant given in Lemma 8.

By (23), (2), (13) and Lemma 4 we get

$$\begin{aligned} A_n(x_0, y_0) & \leqslant \frac{w_{p_1,p_2}(x_0, y_0)}{1+\sinh ny_0} \sum_{j=0}^{\infty} a_{n,j}(x_0) \frac{1}{w_{p_1}(2j/n)} \leqslant \\ & \leqslant \frac{M_{11} w_{p_1}(x_0)}{1+\sinh ny_0} L_n^{\{1\}}\left(\frac{1}{w_{p_1}(t)}; x_0\right) \leqslant \frac{M_{11} M_2(p_1)}{1+\sinh ny_0} \quad \text{for } n \in N, \end{aligned}$$

which gives $\lim_{n\to\infty} A_n(x_0, y_0) = 0$. In the case B_n we write

$$B_n(x_0, y_0) = w_{p_1, p_2}(x_0, y_0)\left\{\sum_{|2j/n - x_0| < \delta} \sum_{|(2k+1)/n - y_0| < \delta} + \right.$$

$$\left. + \sum_{|2j/n - x_0| < \delta} \sum_{|(2k+1)/n - y_0| \geqslant \delta} + \sum_{|2j/n - x_0| \geqslant \delta} \sum_{|(2k+1)/n - y_0| < \delta} + \sum_{|2j/n - x_0| \geqslant \delta} \sum_{|(2k+1)/n - y_0| \geqslant \delta}\right\} \cdot$$

$$\cdot a_{n,j}(x_0)\, b_{n,k}(y_0) \left|\varphi\left(\frac{2j}{n}, \frac{2k+1}{n}\right)\right| := S_1 + S_2 + S_3 + S_4.$$

Using (24) and Lemma 8, we get

$$S_1 < \frac{\varepsilon}{4M_9^*} w_{p_1, p_2}(x_0, y_0) \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{n,j}(x_0)\, b_{n,k}(y_0) \left(w_{p_1, p_2}\left(\frac{2j}{n}, \frac{2k+1}{n}\right)\right)^{-1} \leqslant$$

$$\leqslant \frac{\varepsilon}{4M_9^*} \left\| T_{m,n}^{\{1\}}\left(\frac{1}{w_{p_1, p_2}(t, z)}; \cdot, \cdot\right)\right\|_{1; p_1, p_2} \leqslant \frac{\varepsilon}{4} \quad \text{for all } n \in N.$$

Since $|(2k+1)/n - y_0| \geqslant \delta$ implies $1 \leqslant ((2k+1)/n - y_0)^2 \delta^{-2}$, so we get by (23), (2) and (18)

$$S_2 \leqslant M_{11} w_{p_1, p_2}(x_0, y_0) \sum_{|2j/n - x_0| < \delta} \sum_{|(2k+1)/n - y_0| \geqslant \delta} a_{n,j}(x_0)\, b_{n,k}(y_0) \cdot$$

$$\cdot \left(w_{p_1, p_2}\left(\frac{2j}{n}, \frac{2k+1}{n}\right)\right)^{-1} \leqslant \frac{M_{11}}{\delta^2} \left\{w_{p_1}(x_0) \sum_{j=0}^{\infty} a_{n,j}(x_0) \left(w_{p_1}\left(\frac{2j}{n}\right)\right)^{-1}\right\} \cdot$$

$$\cdot \left\{w_{p_2}(y_0) \sum_{|(2k+1)/n - y_0| \geqslant \delta} b_{n,k}(y_0) \left(\frac{2k+1}{n} - y_0\right)^2 \left(w_{p_2}\left(\frac{2k+1}{n}\right)\right)^{-1}\right\} \leqslant$$

$$\leqslant \frac{M_{11}}{\delta^2} \left\{w_{p_1}(x_0)\, L_n^{\{1\}}\left(\frac{1}{w_{p_1}(t)}; x_0\right)\right\} \left\{w_{p_2}(y_0)\, L_n^{\{3\}}\left(\frac{(z - y_0)^2}{w_{p_2}(z)}; y_0\right)\right\}.$$

Using Lemma 4, we obtain for $n \in N$

$$S_2 \leqslant M_{11} M_2(p_1)\, M_3(p_2) \frac{y_0 + 1}{n\delta^2} \equiv M_{12}(p_1, p_2) \frac{y_0 + 1}{n\delta^2}.$$

Analogously we obtain

$$S_3 \leqslant M_{13}(p_1, p_2)\,\frac{x_0+1}{n\delta^2},$$

$$S_4 \leqslant M_{14}(p_1, p_2)\,\frac{(x_0+1)(y_0+1)}{n^2\delta^4}, \qquad n \in N.$$

It is obvious that for a fixed $(x_0, y_0) \in R_0^2$ and for given positive numbers ε, δ, $M_k(p_1, p_2)$ with $12 \leqslant k \leqslant 14$ there exist natural numbers n_1, n_2, n_3 such that

$$M_{12}(p_1, p_2)\,\frac{y_0+1}{n\delta^2} < \frac{\varepsilon}{4} \qquad \text{for } n > n_1,$$

$$M_{13}(p_1, p_2)\,\frac{x_0+1}{n\delta^2} < \frac{\varepsilon}{4} \qquad \text{for } n > n_2,$$

$$M_{14}(p_1, p_2)\,\frac{(x_0+1)(y_0+1)}{n^2\delta^4} < \frac{\varepsilon}{4} \qquad \text{for } n > n_3.$$

Hence there exists a natural number n_4, $n_4 \geqslant \max\{n_1, n_2, n_3\}$, such that

$$S_k < \frac{\varepsilon}{4} \qquad \text{for all } n > n_4 \text{ and } 1 \leqslant k \leqslant 4,$$

which implies

$$B_n(x_0, y_0) < \varepsilon \qquad \text{for all } n > n_4,$$

i.e. $\lim_{n\to\infty} B_n(x_0, y_0) = 0$. Combining these, we obtain (21) for $i = 1$ and $x_0 > 0$, $y_0 > 0$ from (22). Now let $x_0 \cdot y_0 = 0$. From the assumption on φ and from (10), (11) it follows that

$$A_n(0, 0) = 0 = B_n(0, 0) \qquad \text{for } n \in N,$$

and for $x_0 = 0$, $y_0 > 0$

$$A_n(0, y_0) = \frac{w_{p_2}(y_0)}{1+\sinh ny_0}\,|\varphi(0, 0)|,$$

$$B_n(0, y_0) = w_{p_2}(y_0)\sum_{k=0}^{\infty} b_{n,k}(y_0)\left|\varphi\left(0, \frac{2k+1}{n}\right)\right|.$$

If $x_0 > 0$, $y_0 = 0$, then $B_n(x_0, 0) = 0$ and

$$A_n(x_0, 0) = w_{p_1}(x_0)\sum_{j=0}^{\infty} a_{n,j}(x_0)\left|\varphi\left(\frac{2k}{n}, 0\right)\right|$$

for $n \in N$. Now arguing as in the case $B_n(x_0, y_0)$ with $x_0 > 0, y_0 > 0$, we obtain

$$\lim_{n\to\infty} A_n(x_0, y_0) = 0\,, \qquad \lim_{n\to\infty} B_n(x_0, y_0) = 0\,,$$

for every (x_0, y_0) such that $x_0 \cdot y_0 = 0$.

Thus the proof of (21) for $i = 1$ and for every fixed $(x_0, y_0) \in R_0^2$ is completed.

The proof of (21) for $i = 2$ is identical.

Similar we can prove the following

LEMMA 11. *Suppose that (x_0, y_0) is a fixed point in R_0^2 and φ is a given function belonging to some space $C_{2;\,q_1,\,q_2}$, $q_1, q_2 > 0$, and $\varphi(x_0, y_0) = 0$. Then the assertion* (21) *holds.*

3. – Main results.

3.1. In this section we shall estimate the degree of approximation of functions belonging to $C_{1;\,p_1,\,p_2}$ or $C_{2;\,q_1,\,q_2}$ by the operators $T_{m,\,n}^{\{i\}}$.

THEOREM 1. *Suppose that $g \in C_{1;\,p_1,\,p_2}^1$ with some $p_1, p_2 \in N_0$. Then there esists a positive constant $M_{15}(p_1, p_2)$ such that for all $(x, y) \in R_0^2$, $m, n \in N$ and $i = 1, 2$*

$$w_{p_1,\,p_2}(x, y)\,|T_{m,\,n}^{\{i\}}(g;\, x, y) - g(x, y)| \leq \tag{25}$$

$$\leq M_{15}(p_1, p_2)\left\{\left\| \frac{\partial g}{\partial x} \right\|_{1;\,p_1,\,p_2} \sqrt{\frac{x+1}{m}} + \left\| \frac{\partial g}{\partial y} \right\|_{1;\,p_1,\,p_2} \sqrt{\frac{y+1}{n}}\right\}.$$

PROOF. We shall prove (25) only for $i = 1$ because the proof of (25) for $i = 2$ is analogous. Let $i = 1$ and let (x, y) be a fixed point in R_0^2. Then, for every $(t, z) \in R_0^2$ and $g \in C_{1;\,p_1,\,p_2}^1$, we can write

$$g(t, z) - g(x, y) = \int_x^t \frac{\partial}{\partial u} g(u, z)\, du + \int_y^z \frac{\partial}{\partial v} g(x, v)\, dv\,.$$

From this and by (12) we get for $m, n \in N$

$$T_{m,\,n}^{\{1\}}(g(t, z);\, x, y) - g(x, y) =$$

$$= T_{m,\,n}^{\{1\}}\left(\int_x^t g_u'(u, z)\, du;\, x, y\right) + T_{m,\,n}^{\{1\}}\left(\int_y^z g_v'(x, v)\, dv;\, x, y\right)$$

and consequently

$$(26)\qquad w_{p_1,p_2}(x,y)\,|T^{\{1\}}_{m,n}(g(t,z);x,y)-g(x,y)|\leqslant$$

$$\leqslant w_{p_1,p_2}(x,y)\left\{T^{\{1\}}_{m,n}\left(\left|\int_x^t g'_u(u,z)\,du\right|;x,y\right)+\right.$$

$$\left.T^{\{1\}}_{m,n}\left(\left|\int_y^z g'_v(x,v)\,dv\right|;x,y\right)\right\}:=S_1+S_2.$$

By (3) it follows that

$$\left|\int_x^t \frac{\partial}{\partial u}g(u,z)\,du\right| = \left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\left|\int_x^t \frac{du}{w_{p_1,p_2}(u,z)}\right|\leqslant$$

$$\leqslant \left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\left(\frac{1}{w_{p_1,p_2}(t,z)}+\frac{1}{w_{p_1,p_2}(x,z)}\right)|t-x|$$

and analogously

$$\left|\int_y^z \frac{\partial}{\partial v}g(x,v)\,dv\right|\leqslant \left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\left(\frac{1}{w_{p_1,p_2}(x,z)}+\frac{1}{w_{p_1,p_2}(x,y)}\right)|z-y|\,,$$

which by (1), (2), (8) and (18) imply

$$S_1\leqslant \left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2} w_{p_1,p_2}(x,y)\cdot$$

$$\cdot\left\{T^{\{1\}}_{m,n}\left(\frac{|t-x|}{w_{p_1,p_2}(t,z)};x,y\right)+T^{\{1\}}_{m,n}\left(\frac{|t-x|}{w_{p_1,p_2}(x,z)};x,y\right)\right\}\leqslant$$

$$\leqslant \left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2} w_{p_2}(y)\,L^{\{3\}}_n\left(\frac{1}{w_{p_2}(z)};y\right)\cdot$$

$$\cdot\left\{w_{p_1}(x)\,L^{\{1\}}_m\left(\frac{|t-x|}{w_{p_1}(t)};x\right)+L^{\{1\}}_m(|t-x|;x)\right\},$$

$$S_2\leqslant \left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\left\{w_{p_2}(y)\,L^{\{3\}}_n\left(\frac{|z-y|}{w_{p_2}(z)};y\right)+L^{\{3\}}_n(|z-y|;y)\right\}.$$

Using the Hölder inequality, Lemma 1 and Lemma 4, we get by (13), (15) and (17)

$$L_{mm}^{\{1\}}(|t-z|;x) \leq \{L_m^{\{1\}}((t-x)^2;x)\}^{1/2}\{L_m^{\{1\}}(1;x)\}^{1/2} \leq 4\sqrt{\frac{x+1}{m}},$$

$$w_{p_1}(x)\,L_m^{\{1\}}\left(\frac{|t-x|}{w_{p_1}(t)};x\right) \leq w_{p_1}(x)\left\{L_m^{\{1\}}\left(\frac{1}{w_{p_1}(t)};x\right)\right\}^{1/2}\cdot$$

$$\cdot\left\{L_m^{\{1\}}\left(\frac{(t-x)^2}{w_{p_1}(t)};x\right)\right\}^{1/2} \leq M_{16}(p_1)\sqrt{\frac{x+1}{m}},$$

and analogously

$$L_n^{\{3\}}(|z-y|;y) \leq 4\sqrt{\frac{y+1}{n}},$$

$$w_{p_2}(y)\,L_n^{\{3\}}\left(\frac{|z-y|}{w_{p_2}(z)};y\right) \leq M_{17}(p_2)\sqrt{\frac{y+1}{n}}.$$

Consequently, for $m, n \in N$, we obtain

$$S_1 \leq M_{18}(p_1, p_2)\left\|\frac{\partial g}{\partial x}\right\|_{1;p_1,p_2}\sqrt{\frac{x+1}{m}},$$

$$S_2 \leq M_{19}(p_1, p_2)\left\|\frac{\partial g}{\partial y}\right\|_{1;p_1,p_2}\sqrt{\frac{y+1}{n}}.$$

From these inequalities and (26), we obtain (25) for $m, n \in N$, $(x, y) \in R_0^2$ and $i = 1$. Thus the proof is completed. ■

Arguing as in the proof of Theorem 1 and applying Lemma 5, we can prove the following

THEOREM 2. *Suppose that* $g \in C^1_{2;q_1,q_2}$ *with some* $q_1, q_2 > 0$ *and* $r_1 > q_1$, $r_2 > q_2$. *Then there exist a positive constant* $M_{20}^* \equiv$

$\equiv M_{20}(q_1, q_2, r_1 r_2)$ *and natural numbers* m_0 *and* n_0 *satisfying the conditions* (20) *such that for all* $(x, y) \in R_0^2$, $m > m_0$, $n > n_0$ *and* $i = 1, 2$

$$v_{r_1, r_2}(x, y) \,|\, T^{\{i\}}_{m,n}(g(t, z); x, y) - g(x, y) \leqslant$$

$$\leqslant M^*_{20}\left\{\left\| \frac{\partial g}{\partial x} \right\|_{2;\, q_1, q_2} \sqrt{\frac{x+1}{m}} + \left\| \frac{\partial g}{\partial y} \right\|_{2;\, q_1, q_2} \sqrt{\frac{y+1}{n}}\right\}. \qquad \blacksquare$$

Applying the above theorems, we shall prove the main two theorems.

THEOREM 3. *For every fixed* $p_1, p_2 \in N_0$ *there exists a positive constant* $M_{21}(p_1, p_2)$ *such that for every* $f \in C_{1;\, p_1, p_2}$, $(x, y) \in R_0^2$, $m, n \in N$ *and* $i = 1, 2$ *there holds true*

$$(27) \qquad w_{p_1, p_2}(x, y) \,|\, T^{\{i\}}_{m,n}(f(t, z); x, y) - f(x, y) \,| \leqslant$$

$$\leqslant M_{21}(p_1, p_2)\, \omega\left(f, C_{1;\, p_1, p_2}; \sqrt{\frac{x+1}{m}}, \sqrt{\frac{y+1}{n}}\right).$$

PROOF. Let $f_{h,\delta}$ be Steklov means of function $f \in C_{1;\, p_1, p_2}$ defined by the formula

$$(28) \quad f_{h,\delta}(x, y) := \frac{1}{h\delta} \int_0^h \int_0^\delta f(x+u, y+v)\, du\, dv\,, \quad (x, y) \in R_0^2,\ h, \delta > 0\,.$$

From this it follows that

$$f_{h,\delta}(x, y) - f(x, y) = \frac{1}{h\delta} \int_0^h \int_0^\delta \varDelta_{u,v} f(x, y)\, du\, dv\,,$$

$$\frac{\partial}{\partial x} f_{h,\delta}(x, y) = \frac{1}{h\delta} \int_0^\delta \varDelta_{h,0} f(x, y+v)\, dv\,,$$

$$\frac{\partial}{\partial y} f_{h,\delta}(x, y) = \frac{1}{h\delta} \int_0^h \varDelta_{0,\delta} f(x+u, y)\, du\,,$$

which imply $f_{h,\delta} \in C^1_{1;p_1,p_2}$ for every fixed $h, \delta > 0$ and moreover

(29) $$\|f_{h,\delta} - f\|_{1;p_1,p_2} \leqslant \omega(f, C_{1;p_1,p_2}; h, \delta),$$

(30) $$\left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{1;p_1,p_2} \leqslant 2h^{-1}\omega(f, C_{1;p_1,p_2}; h, \delta),$$

(31) $$\left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{1;p_1,p_2} \leqslant 2\delta^{-1}\omega(f, C_{1;p_1,p_2}; h, \delta).$$

By (8)-(12) and (28)-(31) we can write

(32) $$w_{p_1,p_2}(x, y)\,|T^{\{i\}}_{m,n}(f; x, y) - f(x, y)| \leqslant$$

$$\leqslant w_{p_1,p_2}(x, y)\{|T^{\{i\}}_{m,n}(f(t, z) - f_{h,\delta}(t, z); x, y)| +$$

$$+ |T^{\{i\}}_{m,n}(f_{h,\delta}(t, z); x, y) - f_{h,\delta}(x, y)| + |f_{h,\delta}(x, y) - f(x, y)|\} :=$$

$$:= A_1 + A_2 + A_3,$$

for every fixed $(x, y) \in R^2_0$, $m, n \in N$, $\delta > 0$ and $i = 1, 2$.

Using Lemma 8 and (29), we get

$$A_1 \leqslant M_9(p_1, p_2)\|f - f_{h,\delta}\|_{1;p_1,p_2} \leqslant M_9(p_1, p_2)\,\omega(f, C_{1;p_1,p_2}; h, \delta).$$

By Theorem 1 and (29)-(31) we have

$$A_2 \leqslant M_{15}(p_1, p_2)\left\{\left\| \frac{\partial f_{h,\delta}}{\partial x} \right\|_{1;p_1,p_2} \sqrt{\frac{x+1}{m}} + \left\| \frac{\partial f_{h,\delta}}{\partial y} \right\|_{1;p_1,p_2} \sqrt{\frac{y+1}{n}}\right\} \leqslant$$

$$\leqslant 2M_{15}(p_1, p_2)\,\omega(f, C_{1;p_1,p_2}; h, \delta)\left\{h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right\},$$

$$A_3 \leqslant \omega(f, C_{1;p_1,p_2}; h, \delta).$$

Hence, from (32) it follows that

$$w_{p_1,p_2}(x, y)\,|T^{\{i\}}_{m,n}(f; x, y) - f(x, y)| \leqslant$$

$$\leqslant M_{22}(p_1, p_2)\,\omega(f, C_{1;p_1,p_2}; h, \delta)\left\{1 + h^{-1}\sqrt{\frac{x+1}{m}} + \delta^{-1}\sqrt{\frac{y+1}{n}}\right\}.$$

Now, for every fixed $(x, y) \in R^2_0$, $m, n \in N$ and $i = 1, 2$, setting

$h=\sqrt{(x+1)/m}$ and $\delta=\sqrt{(y+1)/n}$ we obtain the desired estimations (27). ■

THEOREM 4. *Suppose that* $f\in C_{2;\,q_1,\,q_2}$ *with some* $q_1, q_2>0$ *and let* $r_1>q_1$, $r_2>q_2$. *Then there exist a positive constant* $M_{23}^*\equiv\equiv M_{23}(q_1, q_2, r_1, r_2)$ *and natural numbers* m_0 *and* n_0 *satisfying the conditions* (20) *such that for all* $(x, y)\in R_0^2$, $m>m_0$, $n>n_0$ *and* $i=1, 2$

(33) $$v_{r_1,\,r_2}(x, y)\,|T_{m,\,n}^{\{i\}}(f;\,x, y)-f(x, y)|\leqslant$$
$$\leqslant M_{23}^*\,\omega\left(f,\,C_{2;\,q_1,\,q_2};\,\sqrt{\frac{x+1}{m}},\,\sqrt{\frac{y+1}{n}}\right).\qquad ■$$

PROOF. Analogously as in the proof of Theorem 3 we use the Steklov means $f_{h,\,\delta}$ of $f\in C_{2;\,q_1,\,q_2}$ defined by (28). By (5)-(7) and (28) and by our assumptions, we have

(34) $$\|f-f_{h,\,\delta}\|_{2;\,r_1,\,r_2}\leqslant\|f-f_{h,\,\delta}\|_{2;\,q_1,\,q_2}\leqslant\omega(f,\,C_{2;\,q_1,\,q_2};\,h,\,\delta),$$

(35) $$\left\|\frac{\partial f_{h,\,\delta}}{\partial x}\right\|_{2;\,q_1,\,q_2}\leqslant 2h^{-1}\omega(f,\,C_{2;\,q_1,\,q_2};\,h,\,\delta),$$

(36) $$\left\|\frac{\partial f_{h,\,\delta}}{\partial y}\right\|_{1;\,q_1,\,q_2}\leqslant 2\delta^{-1}\omega(f,\,C_{2;\,q_1,\,q_2};\,h,\,\delta),$$

for $h, \delta>0$. Arguing as in the proof of Theorem 3 and using Lemma 9 and (34)-(36), we obtain for $(x, y)\in R_0^2$, $m>m_0$, $n>n_0$, $h, \delta>0$ and $i=1, 2$

$$v_{r_1,\,r_2}(x, y)\,|(T_{m,\,n}^{\{i\}}(f;\,x, y)-f(x, y)|\leqslant$$
$$\leqslant M_{24}^*\,\omega(f,\,C_{2;\,q_1,\,q_2};\,h,\,\delta)\left\{1+h^{-1}\sqrt{\frac{x+1}{m}}+\delta^{-1}\sqrt{\frac{y+1}{n}}\right\},$$

where $M_{24}^*\equiv M_{24}(q_1, q_2, r_1, r_2)=\text{const}>0$. Setting $h=\sqrt{(x+1)/m}$ and $\delta=\sqrt{(y+1)/n}$ (as in the proof of Theorem 3), we obtain the desired inequality (33). ■

Theorem 3 and Theorem 4 imply the following

COROLLARY 1. *Let* $f\in C_{1;\,p_1,\,p_2}$ *or* $f\in C_{2;\,q_1,\,q_2}$ *with some* $p_1, p_2\in N_0$

and $q_1, q_2 > 0$. *Then for every* $(x, y) \in R_0^2$ *and* $i = 1, 2$

$$\lim_{m, n \to \infty} T_{m,n}^{\{i\}}(f; x, y) = f(x, y). \tag{37}$$

Moreover, the assertion (37) *holds uniformly on every rectangle* $0 \leq x \leq \leq a$, $0 \leq y \leq b$.

3.2. In this part we shall give the Voronovskaja type theorem for the operators $T_{n,n}^{\{i\}}$.

THEOREM 5. *Assume that* $f \in C^2_{1;p_1,p_2}$ *with some* $p_1, p_2 \in N_0$. *Then for every* $(x, y) \in R_+^2 := \{(x, y): x > 0, y > 0\}$ *and* $i = 1, 2$ *we have*

$$\lim_{n \to \infty} n\{T_{n,n}^{\{i\}}(f; x, y) - f(x, y)\} = \tag{38}$$

$$= \frac{x}{2} f''_{xx}(x, y) + \frac{y}{2} f''_{yy}(x, y) + \begin{cases} 0 & \text{if } i = 1, \\ f'_x(x, y) + f'_y(x, y) & \text{if } i = 2. \end{cases}$$

PROOF. Let (x_0, y_0) be a fixed point in R_+^2. Then, by the Taylor formula for $f \in C^2_{1,p_1,p_2}$, we have for every $(t, z) \in R_0^2$

$$f(t, z) = f(x_0, y_0) + f'_x(x_0, y_0)(t - x_0) + f'_y(x_0, y_0)(z - y_0) +$$

$$+ \frac{1}{2}\{f''_{xx}(x_0, y_0)(t - x_0)^2 + 2f''_{xy}(x_0, y_0)(t - x_0)(z - y_0) + f''_{yy}(x_0, y_0)(z - y_0)^2\} +$$

$$+ \psi(t, z; x_0, y_0)\{(t - x_0)^4 + (z - y_0)^4\}^{1/2},$$

where $\psi(\cdot, \cdot; x_0, y_0) \equiv \psi(\cdot, \cdot) \in C_{1;p_1,p_2}$ and $\lim_{(t,z) \to (x_0, y_0)} \psi(t, z; x_0, y_0) = 0$. From this and by (12) we get for every $n \in N$ and $i = 1, 2$

$$T_{n,n}^{\{i\}}(f(t, z); x_0, y_0) = f(x_0, y_0) + f'_x(x_0, y_0)\, T_{n,n}^{\{i\}}(t - x_0; x_0, y_0) +$$

$$+ f'_y(x_0, y_0)\, T_{n,n}^{\{i\}}(z - y_0; x_0, y_0) + \frac{1}{2}\{f''_{xx}(x_0, y_0)\, T_{n,n}^{\{i\}}((t - x_0)^2; x_0, y_0) +$$

$$+ 2f''_{xy}(x_0, y_0)\, T_{n,n}^{\{i\}}((t - x_0)(z - y_0); x_0, y_0) +$$

$$+ f''_{yy}(x_0, y_0)\, T_{n,n}^{\{i\}}((z - y_0)^2; x_0, y_0)\} +$$

$$+ T_{n,n}^{\{i\}}(\psi(t, z)\sqrt{(t - x_0)^4 + (z - y_0)^4}; x_0, y_0).$$

But by (17)-(19) we have for $k \in N$

$$T^{\{i\}}_{n,n}((t-x_0)^k;\, x_0,\, y_0) = \begin{cases} L^{\{1\}}_n((t-x_0)^k;\, x_0) & \text{if } i=1\,, \\ L^{\{2\}}_n((t-x_0)^k;\, x_0) & \text{if } i=2\,, \end{cases}$$

$$T^{\{i\}}_{n,n}((z-y_0)^2;\, x_0,\, y_0) = \begin{cases} L^{\{3\}}_n((z-y_0)^2;\, y_0) & \text{if } i=1\,, \\ L^{\{4\}}_n((z-y_0)^2;\, x_0) & \text{if } i=2\,, \end{cases}$$

$$T^{\{i\}}_{n,n}((t-x_0)(z-y_0);\, x_0,\, y_0) = \begin{cases} L^{\{1\}}(t-x_0;\, x_0)\, L^{\{3\}}_n(z-y_0;\, y_0) & \text{if } i=1\,, \\ L^{\{2\}}_n(t-x_0;\, x_0)\, L^{\{4\}}_n(z-y_0;\, y_0) & \text{if } i=2\,. \end{cases}$$

From these and by Lemma 1 and Lemma 3 we get

$$\lim_{n\to\infty} nT^{\{i\}}_{n,n}(t-x_0;\, x_0,\, y_0) = \begin{cases} 0 & \text{if } i=1\,, \\ 1 & \text{if } i=2\,, \end{cases}$$

$$\lim_{n\to\infty} nT^{\{i\}}_{n,n}(z-y_0;\, x_0,\, y_0) = \begin{cases} 0 & \text{if } i=1\,, \\ 1 & \text{if } i=2\,, \end{cases}$$

$$\lim_{n\to\infty} nT^{\{i\}}_{n,n}((t-x_0)^2;\, x_0,\, y_0) = x_0 \qquad \text{for } i=1,2\,,$$

$$\lim_{n\to\infty} nT^{\{i\}}_{n,n}((z-y_0)^2;\, x_0,\, y_0) = y_0 \qquad \text{for } i=1,2\,,$$

$$\lim_{n\to\infty} nT^{\{i\}}_{n,n}((t-x_0)(z-y_0);\, x_0,\, y_0) = 0 \qquad \text{for } i=1,2\,.$$

Next, using the Hölder inequality, we have for $n \in N$ and $i=1, 2$

$$|T^{\{i\}}_{n,n}(\psi(t,z)\sqrt{(t-x_0)^4+(z-y_0)^4};\, x_0,\, y_0)| \leq$$

$$\leq 2\{T^{\{i\}}_{n,n}(\psi^2(t,z);\, x_0,\, y_0)^{1/2}\{T^{\{i\}}_{n,n}((t-x_0)^4+(z-y_0)^4;\, x_0,\, y_0)\}^{1/2}.$$

It is easily verified that for the function $\varphi(\cdot,\cdot) \equiv \psi^2(\cdot,\cdot)$ we can apply Lemma 10. Hence

$$\lim_{n\to\infty} T^{\{i\}}_{n,n}(\psi^2(t,z;\, x_0,\, y_0);\, x_0,\, y_0) = 0 \qquad \text{for } i=1,2\,.$$

The linearity of $T^{\{i\}}_{m,n}$ and (17)-(19) and Lemma 2 imply that there exists a positive constant $M_{25}(x_0, y_0)$ such that for every $n \in N$ and $i=1, 2$

$$T^{\{i\}}_{n,n}((t-x_0)^4+(z-y_0)^4;\, x_0,\, y_0) \leq M_{25}(x_0, y_0)\, n^{-2}.$$

From the above it follows that

$$\lim_{n\to\infty} nT^{\{i\}}_{n,n}(\psi(t,z;x_0,y_0)\sqrt{(t-x_0)^4+(z-y_0)^4};x_0,y_0)=0 \quad \text{for } i=1,2.$$

Collecting these results, we immediately obtain (38). ■

Reasoning as in the proof of Theorem 5 and using Lemmas 1 ÷ 3 and Lemma 11, we can prove

THEOREM 6. *Let* $f\in C^2_{2;q_1,q_2}$ *with some* $q_1, q_2>0$. *The* (38) *holds for every* $(x,y)\in R^2_+$ *and* $i=1,2$.

3.3. Now we shall give the Bernstein type inequality for the operators $T^{\{i\}}_{m,n}$.

THEOREM 7. *Suppose that* $f\in C_{1;p_1,p_2}$ *with some* $p_1,p_2\in N_0$ *and* $s_1,s_2\in N_0$. *Then there exists a positive constant* $M^*_{26}\equiv \equiv M_{26}(p_1,p_2,s_1,s_2)$ *such that for all* $m,n\in N$ *and* $i=1,2$

$$\left\| \frac{\partial^{s_1+s_2}}{\partial x^{s_1}\partial y^{s_2}} T^{\{i\}}_{m,n}(f;x,y)\right\|_{1;p_1,p_2} \leqslant M^*_{26} m^{s_1} n^{s_2}\|f\|_{1;p_1,p_2}. \tag{39}$$

PROOF. Let $i=1$ and $s_1,s_2\in N_0$. From (8)-(11) we deduce that

$$\left| \frac{\partial^{s_1+s_2}}{\partial x^{s_1}\partial y^{s_2}} T^{\{1\}}_{m,n}(f;x,y)\right| \leqslant$$

$$\leqslant \left|\frac{d^{s_2}}{dx^{s_2}}\frac{1}{1+\sinh ny}\right| \sum_{j=0}^{\infty}\left|\frac{d^{s_1}}{dx^{s_1}}a_{m,j}(x)\right| \left|f\left(\frac{2j}{m},0\right)\right| +$$

$$+\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\left|\frac{d^{s_1}}{dx^{s_1}}a_{m,j}(x)\right|\left|\frac{d^{s_2}}{dy^{s_2}}b_{n,k}(y)\right|\left|f\left(\frac{2j}{m},\frac{2k+1}{n}\right)\right| \leqslant$$

$$\leqslant \|f\|_{1;p_1,p_2}\left\{\left|\left(\frac{1}{1+\sinh ny}\right)^{(s_2)}\right|\sum_{j=0}^{\infty}|(a_{m,j}(x))^{(s_1)}|\frac{1}{w_{p_1}(2j/m)}+\right.$$

$$+\left(\sum_{j=0}^{\infty}|(a_{m,j}(x))^{(s_1)}|\frac{1}{w_{p_1}(2j/m)}\right)\cdot$$

$$\left.\cdot\left(\sum_{k=0}^{\infty}|(b_{n,k}(y))^{(s_2)}|\frac{1}{w_{p_2}((2k+1)/n)}\right)\right\} :=$$

$$:= \|f\|_{1;p_1,p_2}\{Z_1(x,y)+Z_2(x,y)\}, \qquad (x,y)\in R^2_0,\ m,n\in N.$$

But, using Lemma 6 and by (1), (2), we get

$$w_{p_1, p_2}(x, y)\, Z_1(x, y) \leqslant M_{27}(p_1, s_1, s_2)\, m^{s_1} n^{s_2},$$

$$w_{p_1, p_2}(x, y)\, Z_2(x, y) \leqslant M_{28}(p_1, s_1, s_2)\, m^{s_1} n^{s_2},$$

for every $(x, y) \in R_0^2$ and $m, n \in N$. Hence there exists a positive constant $M_{26}^* \equiv M_{26}(p_1, p_2, s_1, s_2)$ such that for all $m, n \in N$ and $(x, y) \in \in R_0^2$

$$w_{p_1, p_2}(x, y) \left| \frac{\partial^{s_1 s_2}}{\partial x^{s_1} \partial y^{s_2}} T_{m, n}^{\{1\}}(f; x, y) \right| \leqslant M_{26}^* m^{s_1} n^{s_2} \|f\|_{1; p_1, p_2},$$

which yields (39) for $i = 1$. The proof of (39) for $i = 2$ is identical. ■

Analogously, applying Lemma 7, we can prove the following

THEOREM 8. *Suppose that* $q_1, q_2, r_1, r_2, s_1, s_2$ *are fixed numbers such that* $q_1 > r_1 > 0$, $q_2 > r_2 > 0$ *and* $s_1, s_2 \in N_0$. *Then exist a positive constant* $M_{29}^* \equiv M_{29}(q_1, q_2, r_1, r_2, s_1, s_2)$ *and natural numbers* m_0 *and* n_0 *satisfying the conditions* (20) *such that*

$$\left\| \frac{\partial^{s_1 + s_2}}{\partial x^{s_1} \partial y^{s_2}} T_{m, n}^{\{i\}}(f; x, y) \right\|_{2; r_1, r_2} \leqslant M_{29}^* m^{s_1} n^{s_2} \|f\|_{2; q_1, q_2}$$

for every $f \in C_{2; q_1, q_2}$ *and for all* $m > m_0$, $n > n_0$ *and* $i = 1, 2$.

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Manoscritto pervenuto in redazione il 9 aprile 1997.