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Residually Finite PSP-Groups.

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ABSTRACT - Let G be a group. If there exists an integer $n > 1$ such that for each n-tuple $(H_1, \ldots, H_n)$ of subgroups of G there is a non-identity permutation σ in S_n such that the two sets $H_1 \ldots H_n$ and $H_{\sigma(1)} \ldots H_{\sigma(n)}$ are equal, then G is said to have the property of permutable subgroup products. This is a note to show that a finitely generated residually finite group has the above property if and only if it is finite-by-abelian.

1. - Introduction.

Groups with the property of permutable subgroup products are denoted by PSP. If $n > 1$ is an integer then G is a PSP_n-group if for each n-tuple $(H_1, \ldots, H_n)$ of subgroups of G there is a permutation $\sigma \neq 1$ in S_n such that $H_1 \ldots H_n = H_{\sigma(1)} \ldots H_{\sigma(n)}$. Thus PSP is the union of the classes PSP_n; $n = 2, 3, \ldots$.

It was proved in [1] that a finitely generated residually soluble group G is a PSP-group if and only if G is finite-by-abelian. But the question (**) whether finitely generated residually finite groups have the same property was left open. The result of this note is the following which answers the above question affirmatively.

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THEOREM. *Suppose that G is a finitely generated residually finite group. Then G is a* PSP*-group if and only if G is finite-by-abelian.*

2. - Proof.

We first prove the following Lemma in which we use the notation of [2] for twisted wreath products. We recall the definition of twisted wreath product (compare Neumann [2]). Let A, B be finite groups and H a subgroup of B, and suppose that a faithful action σ of H on A is given. We may form the direct product F of $|B:H|$ copies of A, and define a faithful action of B on F in such a way that B permutes transitively the copies of A and H acts on the first of these copies according to the action of σ. The split extension of F by B with this action is the twisted wreath product of A by B with respect to H; it will be denoted by $A\mathrm{twr}(H)B$ and reference to the action σ will be suppressed. The subgroup F is called the base group of the twisted wreath product. If $H=1$, the group becomes the standard wreath product $A\mathrm{wr}B$ of A by B. We are concerned with the case in which B is cyclic and A is an elementary abelian p-group for some prime p, and H acts faithfully and irreducibly on A.

LEMMA. *Let $G = A\mathrm{twr}(H)B$ be the twisted wreath product of a finite elementary abelian group A and a finite cyclic group B with respect to a subgroup H of B, where H acts on A irreducibly. If G is a* PSP_n*-group, $n>1$ and $|B:H| = m$, then $m < 7^{2n}$.*

PROOF. Let $B = \langle b \rangle$ so that a right transversal to H in B can be chosen to be $T = \{1, b, b^2, \ldots, b^{m-1}\}$. Let $G = BF$, where $F = \{f\colon T \rightarrow A\} = A^T$. For each i, $i = 0, 1, \ldots, m-1$, put $A_i = \{f \in F \mid f(b^j) = 1$ for all $j \neq i\}$. Then for $f \in A_0, f^b(b^i) = 1$ for all $i \neq 1$, since for each $t \in T, f^b(t) = f(t^{b^{-1}})^{h(b,t)}$ where $t^{b^{-1}} = (tb^{-1})^\tau$, $h(b,t) = (tb^{-1})^{\tau-1}$, and τ is the mapping of B to T that maps elements of B onto their coset representatives in T. Thus for $t = b^i$, $0 < i < m$, we have $t^{b^{-1}} = (b^i b^{-1})^\tau = (b^{i-1})^\tau = b^{i-1}$, so that $f^b(t) = f(t^{b^{-1}})^{h(b,t)} = f(b^{i-1})^{h(b,t)} = f(b^{i-1})^{(b^{i-1})^{\tau-1}} = 1$, if $i \neq 1$. Therefore $A_1 = A_0^b$. Similarly $A_i^b = A_{i+1}$ for $i = 1, 2, \ldots, m-1$ and $A_m = A_0$.

Now the proof follows from Lemma 2.1 of [1], since the conditions of that Lemma are satisfied. We include the statement of Lemma 2.1 of [1] for convenience.

LEMMA 2.1 of [1]. *Let G be the split extension of a finite abelian group F of prime exponent p and a finite cyclic group $\langle b \rangle$. Suppose that F is the direct sum $A_0 \oplus \ldots \oplus A_{m-1}$ of m subgroups each normalized by $\langle b^m \rangle$ and $A_i^b = A_{i+1}$ for all $i = 1, 2, \ldots, m-1$ and $A_m = A_0$. If G is a* PSP_n*-group, $n > 1$ then $m < 7^{2n}$.*

PROOF OF THE THEOREM. Let G be a finitely generated residually finite PSP-group. By the Lemma and Theorem 4 of [4], G is soluble-by-finite. By the main theorem of [1] G is finite-by-abelian-by-finite. We can assume that there is a torsion-free abelian normal subgroup A with G/A finite.

Let $g \in G$. Then g acts nilpotently on A by [3], Lemma 2. Since $|gA|$ is finite, it follows that $[A, g] = 1$. Hence $A \leq Z(G)$, $G/Z(G)$ is finite and G' is finite, as required.

The converse follows from the proof of the main result of [3].

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