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Elasticity of Initially Stressed Bodies with Voids. Weak Solutions.

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ABSTRACT - In our study, the general results from the theory of elliptic equations are applied in order to obtain the existence and uniqueness of the generalizated solutions for the boundary value problems in Elasticity of initial stressed bodies with voids.

1. Introduction.

The theories of the bodies with voids do not represent a material length scale, but are quite sufficient for a large number of the solid mechanics applications. Our present paper is dedicated to the behavior of the porous solids in which the matrix material is elastic and the interstices are voids of material. The intended applications of this theory are to the geological materials, like rocks and soils and to the manufactured porous materials. First, we write down the basic equations and conditions of the mixed boundary value problem whitin the context of linear theory of initially stressed bodies with voids, as in the paper [4]. Next we use some general results from paper [2], in order to obtain the existence and uniqueness of a weak solution of the problem. For convenience, the notations chosen are almost identical to those of [3], [4].

2. Basic equations.

Let B be an open region of Euclidian three-dimensional space and bounded by the piece-wise smooth surface ∂B. A fixed system of rectan-

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gular Cartesian axes is used and we adopt the Cartesian tensor notation. The points in B are denoted by (x_i) or (x). The variable t is the time and $t \in [0, t_0)$. We shall employ the usual summation over repeated subscrips while subscripts preceded by a comma denote the partial differentiation with respect to the spatial argument. We also use a superposed dot to denote the partial differentiation with respect to t. The Latin indices are understood to range over the integers $(1, 2, 3)$. The behavior of initially stressed bodies with voids is characterized by the following kinematic variables:

$$u_i = u_i(x, t), \quad \varphi_{jk} = \varphi_{jk}(x, t), \quad \sigma = \sigma(x, t), \quad (x, t) \in B \times [0, t_0). \tag{1}$$

Our paper is concerned with an anisotropic and nonhomogeneous material. We restricte our considerations to the Elastostatics so that the basic equations become

– the equations of equilibrium

$$\begin{cases} (\tau_{ij} + \eta_{ij})_{,j} + \varrho F_i = 0\,, \\ \mu_{ijk,i} + \eta_{jk} + u_{j,i} M_{ik} + \varphi_{ki} M_{ji} - \varphi_{kr,i} N_{ijr} + \varrho G_{jk} = 0\,; \end{cases} \tag{2}$$

– the balance of the equilibrated forces

$$h_{i,i} + g + \varrho L = 0\,; \tag{3}$$

– the constitutive equations

$$\begin{cases} \tau_{ij} = u_{j,k} P_{ki} + C_{ijmn} \varepsilon_{mn} + G_{mnij} \gamma_{mn} + F_{mnrij} \chi_{mnr} + a_{ij} \sigma + d_{ijk} \sigma_{,k}\,, \\ \eta_{ij} = -\varphi_{jk} M_{ik} + \varphi_{jk,r} N_{rik} + G_{ijmn} \varepsilon_{mn} + \\ \qquad\qquad + B_{ijmn} \gamma_{mn} + D_{ijmnr} \chi_{mnr} + b_{ij} \sigma + e_{ijk} \sigma_{,k}\,, \\ \mu_{ijk} = u_{j,r} N_{irk} + F_{ijkmn} \varepsilon_{mn} + \\ \qquad\qquad + D_{mnijk} \gamma_{mn} + A_{ijkmnr} \chi_{mnr} + c_{ijk} \sigma + f_{ijkm} \sigma_{,m}\,, \\ h_i = d_{mni} \varepsilon_{mn} + e_{mni} \gamma_{mn} + f_{mnri} \chi_{mnr} + d_i \sigma + g_{ij} \sigma_{,j}\,, \\ g = -a_{ij} \varepsilon_{ij} - b_{ij} \gamma_{ij} - c_{ijk} \chi_{ijk} - d_i \sigma_{,i} - \xi \sigma\,; \end{cases} \tag{4}$$

– the geometrical equations

$$\varepsilon_{ij} = \frac{1}{2}(u_{j,i} + u_{i,j}), \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}, \quad \chi_{ijk} = \varphi_{jk,i}, \quad \sigma = \nu - \nu_0. \tag{5}$$

In these equations we have used the following notations: ϱ the constant mass density; u_i the components of the displacement field; φ_{jk} the components of the dipolar displacement field; ν the volume distribution function which in the reference state is ν_0; σ a measure of the volume change of the bulk material which results from void compaction or distention; F_i the components of the body forces; G_{jk} the components of the dipolar body forces; L the extrinsic equilibrated body force; g the intrinsic equilibrated force; τ_{ij}, η_{ij}, μ_{ijk} the components of the stress tensors; h_i the components of the equilibrated stress; ε_{ij}, γ_{ij}, χ_{ijk} kinematic characteristic of the strain; A_{ijkmnr}, a_{ij}, B_{ijmn}, b_{ij}, …, ξ the characteristic prescribed functions of material and they obey the following symmetries

$$\begin{cases} C_{ijmn} = C_{mnij} = C_{jimn}\,, \quad B_{ijmn} = B_{mnij}\,, \quad G_{ijmn} = G_{ijnm}\,, \quad g_{ij} = g_{ji}\,, \\ A_{ijkmnr} = A_{mnrijk}\,, \quad F_{ijkmn} = F_{ijknm}\,, \quad a_{ij} = a_{ji}\,, \quad k_{ij} = k_{ji}\,, \quad P_{ij} = P_{ji}\,. \end{cases} \tag{6}$$

The physical significances of the functions L and h_i are presented in the works of Goodman and Cowin, [1], and Nunziato and Cowin, [5]. The prescribed functions P_{ij}, M_{ij} and N_{ijk}, from (1) and (3), satisfy the following equations

$$(P_{ij} + M_{ij})_{,j} = 0\,, \qquad N_{ij,\,i} + M_{jk} = 0\,.$$

3. Existence and uniqueness theorems.

In this section we use some results from the theory of elliptic equations in order to derive the existence and uniqueness of a weak solution of the mixed boundary-value problem in the context of initially stressed bodies with voids. Throughout this section we assume that B is a Lipschitz region of the Euclidian three-dimensional space. We use the notations:

$$\boldsymbol{W} = [W^{1,2}(B)]^{13}\,, \qquad \boldsymbol{W}_0 = [W_0^{1,2}(B)]^{13}\,, \tag{7}$$

with the convention that $A^{13} = \underbrace{A \times A \times \ldots \times A}$ and where $W^{k,m}$ are the familiar Sobolev spaces. With other words, $\boldsymbol{W}$ is defined as the spaces of all $\boldsymbol{u} = (u_i, \varphi_{ij}, \sigma)$, where u_i, φ_{ij}, $\sigma \in W^{1,2}(B)$ with the norm

$$|\boldsymbol{u}|^2_{\boldsymbol{W}} = |\sigma|^2_{W^{1,2}(B)} + \sum_{i=1}^{3} |u_i|^2_{W^{1,2}(B)} + \sum_{j=1}^{3}\left(\sum_{i=1}^{3} |\varphi_{ij}|^2_{W^{1,2}(B)}\right). \tag{8}$$

Let $\partial B = S_u \cup S_t \cup C$ be a disjoint decomposition of ∂B where C is a set of surface measure and S_u and S_t are either empty or open in ∂B. Assume the following boundary conditions

(9) $$\begin{cases} u_i = \tilde{u}_i\,, \quad \varphi_{jk} = \tilde{\varphi}_{jk}\,, \quad \sigma = \tilde{\sigma} \ \text{ on } S_u\,, \\ t_i \equiv (\tau_{ij} + \eta_{ij})\, n_j = \tilde{t}_i\,, \quad \mu_{jk} \equiv \mu_{ijk} n_i = \tilde{\mu}_{jk}\,, \quad h \equiv h_i n_i = \tilde{h} \ \text{ on } S_t\,, \end{cases}$$

where $\tilde{u}_i$, $\tilde{\varphi}_{jk}$, $\tilde{\sigma} \in W^{1,2}(S_u)$ and $\tilde{t}_{ij}$, $\tilde{\mu}_{jk}$, $\tilde{h} \in L_2(S_t)$. Also we define $\boldsymbol{V}$ as a subspace of $\boldsymbol{W}$ of all $\boldsymbol{u} = (u_i, \varphi_{jk}, \sigma)$ which satisfy the boundary conditions:

(10) $$u_i = 0\,, \quad \varphi_{jk} = 0\,, \quad \sigma = 0 \ \text{ on } S_u\,.$$

On $\boldsymbol{W} \times \boldsymbol{W}$ we define the bilinear form $A(\boldsymbol{v}, \boldsymbol{u})$ by

(11) $$A(\boldsymbol{v}, \boldsymbol{u}) =$$

$$= \int_B \{ C_{ijmn}\, \varepsilon_{mn}(\boldsymbol{u})\, \varepsilon_{ij}(\boldsymbol{v}) + G_{mnij}[\varepsilon_{ij}(\boldsymbol{v})\, \gamma_{mn}(\boldsymbol{u}) + \varepsilon_{ij}(\boldsymbol{u})\, \gamma_{mn}(\boldsymbol{v})] +$$

$$+ F_{mnrij}[\varepsilon_{ij}(\boldsymbol{v})\, \chi_{mnr}(\boldsymbol{u}) + \varepsilon_{ij}(\boldsymbol{u})\, \gamma_{mnr}(\boldsymbol{v})] + B_{ijmn}\, \gamma_{ij}(\boldsymbol{v})\, \gamma_{mn}(\boldsymbol{u}) +$$

$$+ D_{ijmnr}[\gamma_{ij}(\boldsymbol{v})\, \chi_{mnr}(\boldsymbol{u}) + \gamma_{ij}(\boldsymbol{u})\, \gamma_{mnr}(\boldsymbol{v})] + A_{ijkmnr}\, \chi_{ijk}(\boldsymbol{v})\, \chi_{mnr}(\boldsymbol{u}) +$$

$$+ P_{ki}\, u_{j,k}\, \psi_{j,i} - M_{ik}(u_{j,i}\, \psi_{jk} + v_{j,i}\, \varphi_{jk}) + N_{rik}(u_{j,k}\, \psi_{jk,r} + v_{j,k}\, \varphi_{jk,r}) +$$

$$+ a_{ij}[\varepsilon_{ij}(\boldsymbol{v})\sigma + \varepsilon_{ij}(\boldsymbol{u})\, \gamma] + b_{ij}[\gamma_{ij}(\boldsymbol{v})\, \sigma + \gamma_{ij}(\boldsymbol{u})\gamma] +$$

$$+ c_{ijk}[\chi_{ijk}(\boldsymbol{v})\, \sigma + \chi_{ijk}(\boldsymbol{u})\gamma] + d_{ijk}[\varepsilon_{ij}(\boldsymbol{v})\, \sigma_{,k} + \varepsilon_{ij}(\boldsymbol{u})\gamma_{,k}] +$$

$$+ e_{ijk}[\gamma_{ij}(\boldsymbol{v})\, \sigma_{,k} + \gamma_{ij}(\boldsymbol{u})\, \gamma_{,k}] + f_{ijkm}[\chi_{ijk}(\boldsymbol{v})\, \sigma_{,m} + \chi_{ijk}(\boldsymbol{u})\, \gamma_{,m}] +$$

$$+ d_i[\sigma\gamma_{,i} + \gamma\sigma_{,i}] + g_{ij}\, \sigma_{,i}\gamma_{,j} + \xi\sigma\gamma \}\, dV\,,$$

where

$$\boldsymbol{u} = (u_i, \varphi_{jk}, \sigma), \qquad \boldsymbol{v} = (v_i, \psi_{jk}, \gamma), \qquad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\,(u_{j,i} + u_{i,j}),$$

$$\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}\,(v_{j,i} + v_{i,j}), \qquad \gamma_{ij}(\boldsymbol{u}) = u_{j,i} - \varphi_{ij}\,, \qquad \gamma_{ij}(\boldsymbol{v}) = v_{j,i} - \psi_{ij}\,,$$

$$\chi_{ijk}(\boldsymbol{u}) = \varphi_{jk,i}\,, \qquad \chi_{ijk}(\boldsymbol{v}) = \psi_{jk,i}\,.$$

We assume that the constitutive coefficients are bounded measurable functions in B which satisfy (6). From (11) and (6) we deduce

$$A(\boldsymbol{v}, \boldsymbol{u}) = A(\boldsymbol{u}, \boldsymbol{v}), \tag{12}$$

and

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{u}) = &\int_B [C_{ijmn}\varepsilon_{ij}(\boldsymbol{u})\,\varepsilon_{mn}(\boldsymbol{u}) + 2G_{ijmn}\varepsilon_{ij}(\boldsymbol{u})\,\gamma_{mn}(\boldsymbol{u}) + B_{ijmn}\gamma_{ij}(\boldsymbol{u})\,\gamma_{mn}(\boldsymbol{u}) + \\ &+2F_{ijmnr}\varepsilon_{ij}(\boldsymbol{u})\,\chi_{mnr}(\boldsymbol{u}) + 2D_{ijmnr}\gamma_{ij}(\boldsymbol{u})\,\chi_{mnr}(\boldsymbol{u}) + A_{ijkmnr}\chi_{ijk}(\boldsymbol{u})\,\chi_{mnr}(\boldsymbol{u}) + \\ &+P_{ki}u_{j,k}u_{j,i} - 2M_{ik}u_{j,i}\varphi_{jk} + 2N_{rik}u_{j,i}\varphi_{jk,r} + 2a_{ij}\varepsilon_{ij}(\boldsymbol{u})\,\sigma + \\ &+2b_{ij}\gamma_{ij}(\boldsymbol{u})\,\sigma + 2c_{ijk}\chi_{ijk}(\boldsymbol{u})\,\sigma + 2d_{ijk}\varepsilon_{ij}(\boldsymbol{u})\sigma_{,k} + 2e_{ijk}\gamma_{ij}(\boldsymbol{u})\,\sigma_{,k} + \\ &+2f_{ijkm}\chi_{ijk}(\boldsymbol{u})\,\sigma_{,m} + 2d_i\sigma\sigma_{,i} + g_{ij}\sigma_{,i}\sigma_{,j} + \xi\sigma^2]\,dV, \end{aligned} \tag{13}$$

and thus:

$$A(\boldsymbol{u}, \boldsymbol{u}) = 2\int_B U\,dV, \tag{14}$$

where $U = \varrho e$ is the internal energy density associated to $\boldsymbol{u}$ and suppose that U is a positive definite quadratic form, i.e. there exists a positive constant c such that:

$$\begin{aligned} &C_{ijmn}x_{ij}x_{mn} + 2G_{ijmn}x_{ij}y_{mn} + 2F_{ijmnr}x_{ij}z_{mnr} + B_{ijmn}y_{ij}y_{mn} + \\ &+2D_{ijmnr}y_{ij}z_{mnr} + A_{ijkmnr}z_{ijk}z_{mnr} + P_{ki}x_{ji}x_{jk} - 2M_{ik}x_{ji}y_{jk} + 2N_{rik}x_{ji}z_{jkr} + \\ &+2a_{ij}x_{ij}\omega + 2b_{ij}y_{ij}\omega + 2c_{ijk}z_{ijk}\omega + 2d_{ijk}x_{ij}\gamma_k + 2e_{ijk}y_{ij}\gamma_k + \\ &+2f_{ijkm}z_{ijk}\gamma_m + 2d_i\omega\gamma_i + g_{ij}\gamma_i\gamma_j + \xi\omega^2 \geqslant \\ &\geqslant c(x_{ij}x_{ij} + y_{ij}y_{ij} + z_{ijk}z_{ijk} + \gamma_i\gamma_i + \omega^2), \end{aligned} \tag{15}$$

for all x_{ij}, y_{ij}, z_{ijk}, γ_i and ω. Now, we introduce the functionals

$$\begin{cases} f(\boldsymbol{v}) = \displaystyle\int_B \varrho(F_i v_i + G_{jk}\psi_{jk} + L\gamma)\,dV, \\ g(\boldsymbol{v}) = \displaystyle\int_{S_t} (\tilde{t}_i v_i + \tilde{\mu}_{jk}\psi_{jk} + \tilde{h}\gamma)\,dA, \end{cases} \tag{16}$$

where $\boldsymbol{v} = (v_i, \psi_{jk}, \gamma) \in W$ and ϱ, F_i, G_{jk}, $L \in L_2(B)$. Let $\tilde{\boldsymbol{u}} = (\tilde{u}_i, \tilde{\varphi}_{jk}, \sigma) \in W$ be such that $\tilde{u}_i, \tilde{\varphi}_{jk}, \tilde{\sigma}$ on S_u may be obtained by means of the embedding of the $W^{1,2}(B)$ into $L_2(S_u)$. The element $\boldsymbol{u} = (u_i, \varphi_{jk}, \sigma) \in W$ is called *weak* (*or generalizated*) *solution* of the boundary value problem, if

(17)
$$\boldsymbol{u} - \tilde{\boldsymbol{u}} \in V,$$

and

(18)
$$A(\boldsymbol{v}, \boldsymbol{u}) = f(\boldsymbol{v}) + g(\boldsymbol{v})$$

holds for each $\boldsymbol{v} \in V$. It follows from (15) and (13) that

(19)
$$A(\boldsymbol{v}, \boldsymbol{v}) \geqslant 2c \int_B [\varepsilon_{ij}(\boldsymbol{v})\, \varepsilon_{ij}(\boldsymbol{v}) + \gamma_{ij}(\boldsymbol{v})\, \gamma_{ij}(\boldsymbol{v}) + \chi_{ijk}(\boldsymbol{v})\, \chi_{ijk}(\boldsymbol{v}) + \gamma_{,i}\gamma_{,i} + \gamma^2]\, dV,$$

for any $\boldsymbol{v} = (v_i, \psi_i, \gamma) \in W$.

We consider the operators $N_k \boldsymbol{v}$, $k = 1, 2, \ldots, 49$, mapping W into $L_2(B)$

(20)
$$\begin{cases} N_i \boldsymbol{v} = \varepsilon_{1i}(\boldsymbol{v}), & N_{3+i} \boldsymbol{v} = \varepsilon_{2i}(\boldsymbol{v}), & N_{6+i} \boldsymbol{v} = \varepsilon_{3i}(\boldsymbol{v}), \\ N_{9+i} \boldsymbol{v} = \gamma_{1i}(\boldsymbol{v}), & N_{12+i} \boldsymbol{v} = \gamma_{2i}(\boldsymbol{v}), & N_{15+i} \boldsymbol{v} = \gamma_{3i}(v), \\ N_{18+i} \boldsymbol{v} = \chi_{11i}(v), & N_{21+i} \boldsymbol{v} = \chi_{12i}(v), & N_{24+i} \boldsymbol{v} = \chi_{13i}(v), \\ N_{27+i} \boldsymbol{v} = \chi_{21i}(v), & N_{30+i} \boldsymbol{v} = \chi_{22i}(v), & N_{33+i} \boldsymbol{v} = \chi_{23i}(v), \\ N_{36+i} \boldsymbol{v} = \chi_{31i}(v), & N_{39+i} \boldsymbol{v} = \chi_{32i}(v), & N_{42+i} \boldsymbol{v} = \chi_{33i}(v), \\ N_{45+i} \boldsymbol{v} = \sigma_{,i}(v), & N_{49} \boldsymbol{v} = \sigma(v) & (i = 1, 2, 3). \end{cases}$$

It easy to see that, in fact, the $N_k \boldsymbol{v}$ operators defined above have the general form

(21)
$$N_k \boldsymbol{v} = \sum_{r=1}^{n} \sum_{p \leqslant k_r} n_{kr\alpha} D^{\alpha} v_r, \qquad p = |\alpha|,$$

where $n_{kr\alpha}$ are bounded and measurable on B and D^{α} is

$$D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2}\, \partial x_3^{\alpha_3}}.$$

The $N_k \boldsymbol{v}$ operators form a coercive system on W if, for each $\boldsymbol{v} \in W$, we have

(22) $$\sum_{k=1}^{22} |N_k \boldsymbol{v}|^2_{L_2(B)} + \sum_{r=1}^{13} |v_r|^2_{L_2(B)} \geqslant c_1 |\boldsymbol{v}|^2_W , \qquad c_1 > 0 ,$$

where c_1 does not depend on $\boldsymbol{v}$. Also, $|\cdot|_{L_2}$, $|\cdot|_W$ denote the usual norms in $L_2(B)$ and W, respectively. We have the following theorem, [2],

THEOREM 1. *Let $n_{ps\alpha}$ be constants for $|\alpha| = k_s$. Then the $N_k \boldsymbol{v}$ system is coercive on W if and only if the rank of the matrix*

(23) $$(N_{ps}\,\xi) = \left(\sum_{|\alpha| = k_s} n_{ps\alpha}\,\xi_\alpha \right),$$

is equal to m for each $\xi \in C_3$, $\xi \neq 0$, where C_3 denotes the complex three-dimensional space and $\xi_\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3}$.

We assume furthermore that for each $\boldsymbol{v} \in W$

(24) $$A(\boldsymbol{v}, \boldsymbol{v}) \geqslant c_2 \sum_{k=1}^{49} |N_k \boldsymbol{v}|^2_{L_2(B)} , \qquad c_2 > 0 ,$$

where c_2 does not depend on $\boldsymbol{v}$. Let

(25) $$\mathcal{P} = \left\{ \boldsymbol{v} \in V, \sum_{k=1}^{49} |N_k \boldsymbol{v}|^2_{L_2(B)} = 0 \right\}.$$

We denote by $V/\mathcal{P}$ the factor space of classes $\tilde{\boldsymbol{v}} = \{\boldsymbol{v} + p, \boldsymbol{v} \in V, p \in \mathcal{P}\}$ with the norm

$$|\tilde{\boldsymbol{v}}|_{V/\mathcal{P}} = \inf_{p \in \mathcal{P}} |\boldsymbol{v} + p|_W .$$

From [2] we deduce the following theorem

THEOREM 2. *Assume that $A(\boldsymbol{v}, \boldsymbol{u}) = [\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{u}}]$ defines a bilinear form for each $\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{u}}$ from $W/\mathcal{P}$, $\boldsymbol{u} \in \tilde{\boldsymbol{v}}$, $\boldsymbol{v} \in \tilde{\boldsymbol{v}}$. Further we suppose that* (22) *and* (24) *hold. Then a necessary and sufficient condition for the existence of a weak solution of the boundary-value problem is*

(26) $$p \in \mathcal{P} \Rightarrow f(p) + g(p) = 0 .$$

Further,

$$A(\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{v}}) \geqslant c_3 \, |\tilde{\boldsymbol{v}}|_{W/\mathcal{P}}, \qquad c_3 > 0, \tag{27}$$

for every $\tilde{\boldsymbol{v}} \in W/\mathcal{P}$.

Now, we shall apply the above results to prove the existence and uniqueness of a weak solution of our boundary-value problem. Clearly, from (19) and (20), we obtain (24). The matrix (23) has the rank 13 for each $\xi \in C_3$, $\xi \neq 0$. Thus by Theorem 1 we conclude that the system of N_s operators, defined in (20), is coercive on W. According to the definition (25) of $\mathcal{P}$, for each $\boldsymbol{v} \in \mathcal{P}$, we have $\varepsilon_{ij}(\boldsymbol{v}) = 0$, $\gamma_{ij}(\boldsymbol{v}) = 0$, $\gamma = 0$, such that

$$\mathcal{P} = \{\boldsymbol{v} = (v_i, \psi_{jk}, \gamma) \in V,\ v_i = a_i + \varepsilon_{ijk} b_j x_k,\ \psi_{jk} = \varepsilon_{jks} b_s,\ \gamma = c\}, \tag{28}$$

where a_i, b_i and c are arbitrary constants and ε_{ijk} is the alternating symbol.

First, we assume that S_u is a non-empty set. Then we deduce that

$$\mathcal{P} = \{0\},$$

and (26) is satisfied. In view of Theorem 2 it follows that

THEOREM 3. *Let* $\mathcal{P} = \{0\}$. *There exists one and only one weak solution* $\boldsymbol{u} \in W$.

Let us consider the case when S_u is empty. Then $\mathcal{P}$ is given by (28), where a_i and b_i are arbitrary constants. We are led to the following theorem

THEOREM 4. *The necessary and sufficient conditions for the existence of a weak solution* $\boldsymbol{u} \in W$ *of the traction problem are given by*

$$\int_B \varrho F_i \, dV + \int_{\partial B} \tilde{t}_i \, dA = 0,$$

$$\int_B \varrho \varepsilon_{ijk} (x_j F_k + G_{jk}) \, dV + \int_{\partial B} \varepsilon_{ijk} (x_j \tilde{t}_k + \tilde{\mu}_{jk}) \, dA = 0.$$

where ε_{ijk} *is the alternating symbol.*

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