RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

YOSHIAKI FUKUMA

On k-very ampleness of tensor products of ample line bundles on abelian surfaces

Rendiconti del Seminario Matematico della Università di Padova, tome 101 (1999), p. 209-227

<http://www.numdam.org/item?id=RSMUP_1999__101__209_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 101 (1999)

On k-Very Ampleness of Tensor Products of Ample Line Bundles on Abelian Surfaces (*).

YOSHIAKI FUKUMA (**)

ABSTRACT - In this paper, we study abelian surfaces X and ample line bundles $L_1, \ldots, L_t$ such that $L := L_1 + \ldots + L_t$ is not k-very ample for $t = k, k+1$. And we also study polarized abelian surfaces (X, L) such that $(k-t)L$ is not k-very ample with $t \geq 1$ under some condition. As corollaries of the above results, we get the classification of (X, L) such that $(k-t)L$ is not k-very ample for $t = -1, 0, 1$ and 2.

0. Introduction.

Let X be an abelian variety over the complex number field $\mathbb{C}$ and let L be an ample line bundle on X. Then it is well-known that $2L$ is spanned and $3L$ is very ample. (See [LB].) In [BaSz1] and [BaSz2], Bauer and Szemberg studied a sufficient condition of k-very ampleness of $L_1 + \ldots + L_{k+1}$ for ample line bundles $L_1, \ldots, L_{k+1}$. In particular, in [BaSz2], as a corollary, they proved that $L_1 + \ldots + L_{k+2}$ is k-very ample for any ample line bundles $L_1, \ldots, L_{k+2}$.

Next the following question arises from the Bauer-Szemberg result;

(*) Research Fellow of the Japan Society for the Promotion of Science.
(**) Indirizzo dell'A.: E-mail: fukuma@math.titech.ac.jp. Current Address: Department of Mathematics, College of Education, Naruto University of Education, Takashima, Naruto-cho, Naruto-shi 772-8502, Japan; E-mail: fukuma@naruto-u.ac.jp

QUESTION. Let X be an abelian variety X and let $L_1, \dots, L_s$ be ample line bundles on X for $s \leq k+1$. Then classify $(X, L_1, \dots, L_s)$ such that $L_1 + \dots + L_s$ is not k-very ample.

Ohbuchi ([O]) studied polarized abelian varieties (X, L) such that $2L$ is not very ample.

In sect. 2, we study abelian surfaces X and ample line bundles $L_1, \dots, L_t$ such that $L := L_1 + \dots + L_t$ is not k-very ample for $t = k$, $k+1$.

In sect. 3, we study polarized abelian surfaces (X, L) such that $(k - - t) L$ is not k-very ample with $t \geq 1$ under some condition. In particular, we characterize (X, L) such that $(k-t)L$ is not k-very ample with $t = 1$ or 2. (See Theorem 3.4 and Theorem 3.5.)

We work over the complex number field and we use the customary notation in algebraic geometry.

Acknowledgment. The author would like to express his hearty thanks to Professor Tomasz Szemberg for giving him some useful comments and suggestions. The author also would like to thank the referee for giving him useful comments.

1. Preliminaries.

DEFINITION 1.1. (See [BeSo].) – Let (X, L) be a polarized surface. Then L is called k-very ample if for any 0-dimensional subscheme $(Z, \mathcal{O}_Z)$ with length $\mathcal{O}_Z = k+1$, the map

$$\Gamma(L) \to \Gamma(L \otimes \mathcal{O}_Z)$$

is surjective.

LEMMA 1.2. *Let X be an abelian surface. Assume that X contains a smooth elliptic curve D. Then there exists an elliptic fibration $f \colon X \to C$ such that C is a smooth elliptic curve, D is a fiber of f, and any fiber of f is isomorphic to D.*

PROOF. By a translation of D, we may assume that D contains the origin of X and D is an abelian subvariety of X. Then there exist the quotient X/D and the surjective homomorphism $f \colon X \to X/D$. Then X/D is a smooth elliptic curve and every fiber of f is isomorphic to the fiber over the origin of X/D which is D by construction. ■

LEMMA 1.3. *Let X be an abelian surface. Assume that there exist a smooth elliptic curve C and a surjective morphism $f: X \to C$ with connected fibers such that f has a section S. Then $X \cong C \times F$ and f is identified with the first projection via this isomorphism, and S is a fiber of the second projection, where F is a fiber of f.*

PROOF. We remark that f is an elliptic fibration such that any fiber of f is smooth since X is an abelian surface. Let S be a section of f. Then by Lemma 1.2, there exist a smooth elliptic curve C' and an elliptic fibration $h: X \to C'$ such that any fiber of h is a smooth elliptic curve and S is a fiber of h. Moreover any fiber of h (resp. f) is a section of f (resp. h). In particular $C' \cong F$. Then there exists a morphism $\pi: X \to C \times C'$ such that $f = p_1 \circ \pi$ and $h = p_2 \circ \pi$, where p_1 (resp. p_2) is the projection $C \times C' \to C$ (resp. $C \times C' \to C'$). We remark that π is bijective by construction. Let $F_f = f^*(x)$ and $F_h = h^*(y)$, where $x \in C$ and $y \in C'$. Then $F_f = \pi^* \circ p_1^*(x)$ and $F_h = \pi^* \circ p_2^*(y)$. Then

$$1 = F_f F_h = (\pi^* \circ p_1^*(x))(\pi^* \circ p_2^*(y)) = \deg(\pi)(p_1^*(x)\, p_2^*(y)) .$$

Hence π is birational. Therefore by Zariski Main Theorem, we obtain that π is an isomorphism. ■

REMARK 1.3.1. Let (X, L) be a polarized abelian surface, and let D be an effective divisor on X such that $LD = 1$. Then D is irreducible and reduced. By Hodge index Theorem, we get that $D^2 = 0$. Moreover D is a smooth elliptic curve since X is an abelian surface and $g(D) = 1$. By Lemma 1.2, there exists a fiber space $f: X \to C$ such that C is a smooth elliptic curve, D is a fiber of f, and any fiber of f is isomorphic to D. Since $h^0(L) > 0$ and $LD = 1$, there exists a section of f. So by Lemma 1.3, $X \cong C \times F$ for a fiber F of f, and f is identified with the first projection via this isomorphism.

THEOREM 1.4 (Terakawa). *Let (X, L) be a polarized abelian surface. Then L is k-very ample if and only if $L^2 \geqslant 4k + 6$ and there exists no effective divisor D satisfying the inequalities*

$$LD \leqslant 2g(D) + k - 1 \leqslant 2k + 1 .$$

PROOF. See Theorem 3.15 in [Te].

THEOREM 1.5. *Let (X, L) be a polarized abelian surface. If $g(L) = 2$, then (X, L) is one of the following*:

(I) $X \cong J(B)$, *and L is the class of a translation of the theta divisor, where B is a smooth projective curve of genus two, $J(B)$ is the jacobian variety of B.*

(II) $X \cong E_1 \times E_2$, and $L = p_1^* \mathcal{D}_1 \otimes p_2^* \mathcal{D}_2$, where *$E_1$ and E_2 are smooth elliptic curves, $p_i: E_1 \times E_2 \to E_i$ is the i-th projection, and $\mathcal{D}_i \in \mathrm{Pic}(E_i)$ with* $\deg \mathcal{D}_1 = \deg \mathcal{D}_2 = 1$.

PROOF. See [OoU], [BeLP], or [Fj]. ■

THEOREM 1.6. *Let (X, L) be a polarized abelian surface with $L^2 \geq 2a$, where $a \in \mathbb{N}$. Assume that there exists an irreducible reduced curve D on X such that $LD = 2$ and $D^2 = 0$. Then (X, L) satisfies one of the following*:

(1) $X \cong E_1 \times E_2$ *and* $L = p_1^* \mathcal{B}_1 \otimes p_2^* \mathcal{B}_2$ *with* $\deg \mathcal{B}_1 \geq 1$ *and* $\deg \mathcal{B}_2 = 2$, *or* $\deg \mathcal{B}_1 = 2$ *and* $\deg \mathcal{B}_2 \geq 1$.

(2) *There exists a surjective morphism $f: X \to C$ with connected fibers such that C is a smooth elliptic curve and any fiber of f is a smooth elliptic curve, and (X, L) is one of the following*:

(2-1) *There exists an ample spanned vector bundle $\mathcal{E}$ of rank two on C such that $\mathcal{E} = \mathcal{E}' \otimes \mathcal{M}_1$ with* $\deg \mathcal{M}_1 \geq \lceil a/2 \rceil$, *$X$ is a double covering of $\mathbb{P}(\mathcal{E})$ whose branch locus is smooth and linearly equivalent to $-2K_{\mathbb{P}(\mathcal{E})}$, $f = p \circ \pi$, and $L = \mathcal{O}(T) \otimes f^* \mathcal{M}_1$*,

(2-2) $X \cong J(B)$ *and $L = \mathcal{O}_X(A) \otimes f^* \mathcal{M}_2$ such that A is a translation of the theta divisor, $AF = 2$ for a fiber F of f, and* $\deg \mathcal{M}_2 \geq \lceil (a-1)/2 \rceil$,

(2-3) $X \cong E_1 \times E_2$ *and $L = p_1^* \mathcal{D}_1 \otimes p_2^* \mathcal{D}_2 \otimes f^* \mathcal{M}_3$ such that any fiber of p_i is a section of f for $i = 1, 2$,* $\deg \mathcal{D}_1 = \deg \mathcal{D}_2 = 1$, *and* $\deg \mathcal{M}_3 \geq \lceil (a-1)/2 \rceil$,

where in (1) *and* (2-3) *E_1 and E_2 are smooth elliptic curves, $p_i: E_1 \times E_2 \to E_i$ is the i-th projection, $\mathcal{B}_i$, $\mathcal{D}_i \in \mathrm{Pic}(E_i)$ for $i = 1, 2$, $\mathcal{M}_3 \in \mathrm{Pic}(C)$, in* (2-1) *$\mathcal{E}' = \mathcal{O}_C \oplus \mathcal{L}_1$ for $\mathcal{L}_1 \in \mathrm{Pic}(C)$ with $\mathcal{L}_1 \not\cong \mathcal{O}_C$ and $2\mathcal{L}_1 \cong \mathcal{O}_C$, $\mathcal{M}_1 \in \mathrm{Pic}(C)$, $p: \mathbb{P}(\mathcal{E}) \to C$ is the natural projection, $\pi: X \to \mathbb{P}(\mathcal{E})$ is the double covering, and T is a smooth elliptic curve with $TF = 2$ for a fiber F of f, and in* (2-2) *B is a smooth projective curve of genus* *two, $J(B)$ is the jacobian variety, and $\mathcal{M}_2 \in \mathrm{Pic}(C)$. (For $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the smallest integer which is greater than or equal to x.)*

REMARK 1.6.1. In the case (2-2) in Theorem 1.6, B cannot be general. Actually if B is general, then the Néron-Severi group $\mathrm{NS}(X)$ is generated by the class of the theta divisor. In particular, X does not contain any elliptic curve.

PROOF. This can be proved by the same argument as in the proof of Theorem 2.2 in [Fk]. By assumption, D is a smooth elliptic curve. Hence by Lemma 1.2, there exists a surjective morphism $f: X \to C$ such that C is a smooth elliptic curve, D is a fiber of f, and any fiber of f is smooth. Since $LF = LD = 2$ for any fiber F of f, we get that $f^* \circ f_*(L) \to L$ is surjective. Let $\mathcal{E} := f_* L$. Then $\mathcal{E}$ is a locally free sheaf of rank two on C, and there exists a morphism $\pi: X \to \mathbb{P}(\mathcal{E})$ such that $f = p \circ \pi$, where $p: \mathbb{P}(\mathcal{E}) \to C$ is the bundle map. By construction, π is a double covering. Since X is an abelian surface, the branch locus B is smooth and linearly equivalent to $-2K_{\mathbb{P}(\mathcal{E})}$. Furthermore $L = \pi^*(H(\mathcal{E}))$ and $\mathcal{E}$ is ample with $\deg \mathcal{E} \geq a$, where $H(\mathcal{E})$ is the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Since $|-2K_{\mathbb{P}(\mathcal{E})}|$ has a smooth member, by the same argument as in the proof of Proposition 2.3 in [Fk1], we can prove that there exists a vector bundle $\mathcal{E}'$ on C and a line bundle $\mathcal{M}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{M}$, and $\mathcal{E}'$ and $\mathcal{M}$ satisfy one of the following three types;

(*A*) $\mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{O}_C$ and $\deg \mathcal{M} \geq \lceil a/2 \rceil$,

(*B*) $\mathcal{E}' \cong \mathcal{O}_C \oplus \mathcal{L}_1$ and $\deg \mathcal{M} \geq \lceil a/2 \rceil$, where $\mathcal{L}_1 \in \mathrm{Pic}(C)$ with $\mathcal{L}_1 \not\cong \mathcal{O}_C$ and $2\mathcal{L}_1 \cong \mathcal{O}_C$,

(*C*) there exists a nontrivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E}' \to \mathcal{L}_2 \to 0$$

and $\deg \mathcal{M} \geq \lceil (a-1)/2 \rceil$, where $\mathcal{L}_2 \in \mathrm{Pic}(C)$ with $\deg \mathcal{L}_2 = 1$.

Let $\iota: \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}')$ be the isomorphism such that $p = p' \circ \iota$. Let $\pi' = \iota \circ \pi$.

(1) The case in which $\mathcal{E}'$ is the type (A) or (B).

Let $C_0 \in |H(\mathcal{E}')|$. Then C_0 is an irreducible reduced curve by Proposition 2.8 in [Ha, Ch. V]. Let B be the branch locus of π'. We remark that $-2K_{\mathbb{P}(\mathcal{E}')} = 4H(\mathcal{E}')$.

Since $-2K_{\mathbb{P}(\mathcal{E}')} C_0 = 0$, we get $C_0 \subset B$ or $C_0 \cap B = \emptyset$.

(1-1) The case in which $C_0 \subset B$.

Then $(\pi')^*(C_0) = 2B_1$. Since $LF = 2$ for a fiber F of f and B_1 is not contained in a fiber of f, B_1 is a section of f. Hence by Lemma 1.3, $X \cong C \times F$ and f is identified with the first projection, and B_1 is a fiber of the second projection $h: C \times F \to F$. So we get

$$L = \pi^* H(\mathcal{E}) \cong \pi^* \circ p^*(\mathcal{M}) \otimes (\pi')^* H(\mathcal{E}') \cong$$

$$\cong f^* \mathcal{M} \otimes (\pi')^* \mathcal{O}_{\mathbf{P}(\mathcal{E}')}(C_0) \cong f^* \mathcal{M} \otimes h^* \mathcal{P},$$

where $\mathcal{P} \in \operatorname{Pic}(F)$ with $\deg \mathcal{P} = 2$. Therefore we get the type (1) in Theorem 1.6.

(1-2) The case in which $C_0 \cap B = \emptyset$. Then we obtain one of the following:

(1-2-1) $(\pi')^*(C_0) = B_2$,
(1-2-2) $(\pi')^*(C_0) = B_3 + B_4$,

where B_i is an irreducible reduced curve for $i = 2, 3, 4$ with $B_3 \neq B_4$.
First we consider the case (1-2-1).

CLAIM 1.7. *If $(\pi')^*(C_0) = B_2$, then $\mathcal{E}'$ is the type (B).*

PROOF. We remark that B_2 is a smooth elliptic curve. Hence by Lemma 1.2, there exists a surjective morphism $f_1: X \to C_1$ such that C_1 is a smooth elliptic curve and B_2 is a fiber of f_1. Hence $h^0((\pi')^*(C_0)) = h^0(B_2) = 1$. On the other hand, since π' is a double covering, we have $h^0((\pi')^*(C_0)) = h^0(H(\mathcal{E}'))$. Hence $h^0(H(\mathcal{E}')) = 1$. Therefore $\mathcal{E}'$ is the type (B). This completes the proof of Claim 1.7. ■

Since $L = (\pi')^*(C_0) \otimes f^* \mathcal{M}$, we obtain the type (2-1) in Theorem 1.6. We remark that $B_2 F = 2$ for a fiber F of f.

Next we consider the case (1-2-2).

If $(\pi')^*(C_0) = B_3 + B_4$, then we get $B_3^2 = B_4^2 = 0$ and $B_3 B_4 = 0$. Since B_3 and B_4 are not contained in a fiber of f, we get that $B_3 F = B_4 F = 1$ for a fiber F of f because $(\pi')^*(C_0)F = 2$. Hence B_3 and B_4 are sections of f. Hence by Lemma 1.3, we get that $X \cong C \times F$, f is identified with the first projection, and B_i is a fiber of the second projection $h: X \to F$ for $i = 3, 4$. Hence by the same argument as the case (1-1), we obtain $L = f^* \mathcal{M} \otimes h^* \mathcal{P}'$, where $\mathcal{P}' \in \operatorname{Pic}(F)$ with $\deg \mathcal{P}' = 2$.

Therefore we get the type (1) in Theorem 1.6.

(2) The case in which $\mathcal{E}'$ is the type (C).

Then $H(\mathcal{E}')$ is ample with $H(\mathcal{E}')^2 = 1$. We put $\mathcal{O}_X(A) = (\pi')^*(H(\mathcal{E}'))$. Then $AF = 2$ for a fiber F of f. Since $A^2 = 2$ and X is an abelian surface, we obtain $g(A) = 2$ and we get that (X, A) is one of the type (I) or (II) in Theorem 1.5. We remark that $L = (\pi')^*(H(\mathcal{E}')) \otimes f^* \mathcal{M} = \mathcal{O}_X(A) \otimes f^* \mathcal{M}$. If (X, A) is the type (II) in Theorem 1.5, then any fiber of f is not contained in a fiber of p_i and any fiber of p_i is a section of f for $i = 1, 2$ since $AF = 2$. Therefore if (X, A) is the type (I) (resp. (II)) in Theorem 1.5, then we get the type (2-2) (resp. (2-3)) in Theorem 1.6. This completes the proof of Theorem 1.6. ∎

DEFINITION 1.8. Let X be an abelian surface and let $L, L_1, \dots, L_n$ be ample line bundles on X.

(*A*) If (X, L) is the type (2-1) (resp. (2-2), (2-3)) in Theorem 1.6 with $L^2 \geqslant 2a$, then we say that (X, L) is the type (I; a) (resp. (II; a), (III; a)).

(*B*) If $X \cong J(B)$ and $L_i \equiv B$ for a smooth curve B of genus two and $i = 1, \dots, n$, then we call $(X, L_1, \dots, L_n)$ the type (J), where $\equiv$ denotes numerical equivalence.

(*C*) If $X \cong E_1 \times E_2$ and $L = p_1^*(\mathcal{D}_1) \otimes p_2^*(\mathcal{D}_2)$ with $(\deg \mathcal{D}_1, \deg \mathcal{D}_2) = (a, b)$ or $(\deg \mathcal{D}_1, \deg \mathcal{D}_2) = (b, a)$, then we call (X, L) the type (P; $\{a, b\}$), where E_1 and E_2 are smooth elliptic curves, p_i is the i-th projection, and $\mathcal{D}_i \in \mathrm{Pic}(E_i)$ for $i = 1, 2$.

(*D*) If $X \cong E_1 \times E_2$ and $L_i = p_t^*(\mathcal{D}_i) \otimes \mathcal{O}(S_i)$ with $\deg \mathcal{D}_i = a_i$, then we call (X, L) the type (PS; t; $a_1, \dots, a_n$), where E_1 and E_2 are smooth elliptic curves, p_t is the t-th projection, S_i is a section of p_t, and $\mathcal{D}_i \in \mathrm{Pic}(E_t)$ for $i = 1, \dots, n$.

2. The case in which $L_1 + \dots + L_t$ is not k-very ample for $t = k+1$ or k.

THEOREM 2.1. *Let X be an abelian surface, and let L_1 and L_2 be ample line bundles on X. We put $L := L_1 + L_2$. Then L is not very ample if and only if (X, L_1, L_2) satisfies one of the following*:

(1) (X, L_1, L_2) *is the type* (J),

(2) (X, L_1, L_2) *is the type* (PS; t; a_1, a_2) *with* $a_1 > 0$, $a_2 > 0$, *and* $t = 1, 2$.

PROOF. First we prove the «if» part.

If (X, L_1, L_2) is the type (1), then $L^2 = (L_1 + L_2)^2 = 8$ and $h^0(L) = L^2/2 = 4$. If L is very ample, then X is a hypersurface of degree 8 in $\mathbb{P}^3$. But this is impossible because X is an abelian surface. Hence L is not very ample.

If (X, L_1, L_2) is the type (2), then $LF_1 = 2$, where F_1 is a fiber of p_1. If L is very ample, then $F_1 \cong \mathbb{P}^1$. But this is impossible because X is an abelian surface.

Next we prove the «only if» part. Assume that L is not very ample. Then $L^2 = (L_1 + L_2)^2 \geqslant 8$ because $L_1^2 \geqslant 2$, $L_2^2 \geqslant 2$, and $L_1 L_2 \geqslant 2$.

(A) The case in which $L^2 = 8$.

Then $L_1 L_2 = 2$ and $L_1^2 = L_2^2 = 2$. By Hodge index Theorem, we obtain that $L_1 \equiv L_2$. Since $g(L_1) = 2$, we get that (X, L_1) is the type (I) or (II) in Theorem 1.5.

If (X, L_1) is the type (I), then $L_2 \equiv B$.

If (X, L_1) is the type (II), then we can easily prove $L_2 = p_1^* \mathcal{D}_1' \otimes p_2^* \mathcal{D}_2'$, where $\mathcal{D}_i' \in \mathrm{Pic}(E_i)$ with $\deg \mathcal{D}_1' = \deg \mathcal{D}_2' = 1$.

(B) The case in which $L^2 \geqslant 10$.

By Reider's Theorem, there exists an effective divisor D on X such that $(LD, D^2) = (2, 0)$ or $(1, 0)$ because the value of D^2 is even. Since $L = L_1 + L_2$ and L_i is ample for $i = 1, 2$, we get $(LD, D^2) = (2, 0)$. So we have $L_i D = 1$ for each i. By Remark 1.3.1, there exists a fiber space $f: X \to C$ such that C is a smooth elliptic curve, $X \cong C \times F$, f is identified with the first projection via this isomorphism, and D is a fiber of f, where F is a fiber of f. Since $L_i F = 1$ for each i, we obtain that $L_i = f^* \mathcal{D}_i \otimes \mathcal{O}_X(S_i)$ for $i = 1, 2$, where $\mathcal{D}_i \in \mathrm{Pic}(C)$ and S_i is a section of f. Since L_i is ample, we get $L_i S_i > 0$. Hence $\deg \mathcal{D}_i > 0$ because $S_i^2 = 0$. This completes the proof of Theorem 2.1. ■

COROLLARY 2.1.1. *Let X be an abelian surface and let L be an ample line bundle on X. Then $2L$ is not very ample if and only if (X, L) satisfies one of the following*:

(1) *(X, L) is the type* (J),
(2) *(X, L) is the type* (P; $\{a, b\}$) *with $a > 0$ and $b = 1$.*

PROOF. We put $L_1 = L$ and $L_2 = L$. Then we get the above result by Theorem 2.1. We remark that if $X \cong E_1 \times E_2$, then we use Lemma 1.3. ∎

THEOREM 2.2. *Let X be an abelian surface and let $L_1, \dots, L_{k+1}$ be ample line bundles on X. Assume that $k \geq 2$. We put $L := L_1 + \dots + L_{k+1}$. Then L is not k-very ample if and only if $(X, L_1, \dots, L_{k+1})$ is the type* (PS; t; $a_1, \dots, a_{k+1}$) *with $a_i > 0$ for any i, and $t = 1, 2$.*

PROOF. First we prove the «if» part. Let F_1 be a fiber of p_1. Then $LF_1 = k + 1$. But by Theorem 1.4, L is not k-very ample.

Next we prove the «only if» part. Then

$$L^2 = \sum_{i=1}^{k+1} L_i^2 + 2 \sum_{i>j} L_i L_j \geq 2(k+1) + 2 \times \frac{k(k+1)}{2} \times 2 =$$

$$= 2k^2 + 4k + 2 \geq 6(k+1) = 4k + 2k + 6 \geq 4k + 10\,.$$

Assume that L is not k-very ample. Then by Theorem 1.4, there exists an effective divisor D such that

$$LD \leq 2g(D) + k - 1 \leq 2k + 1\,.$$

Because $LD = (L_1 + \dots + L_{k+1})D$, there exists an ample line bundle L_i such that $L_i D = 1$. We may assume that $i = 1$. By Remark 1.3.1, there exists a fiber space $f\colon X \to C$ such that C is a smooth elliptic curve, $X \cong C \times F$, f is identified with the first projection via this isomorphism, and D is a fiber of f, where F is a fiber of f. On the other hand, by Theorem 1.4, we get $LD \leq k + 1$ since $g(D) = 1$. Hence $L_i D = 1$ for any i. Since D is a fiber of f and $h^0(L_i) > 0$ for $i = 1, \dots, k+1$, we get that $L_i \cong f^* \mathcal{D}_i \otimes \mathcal{O}_X(S_i)$ for $i = 1, \dots, k+1$, where $\mathcal{D}_i \in \mathrm{Pic}(C)$, S_i is a section of f. Since L_i is ample, we get $L_i S_i > 0$. Hence $\deg \mathcal{D}_i > 0$ because $S_i^2 = 0$ for any i. This completes the proof of Theorem 2.2. ∎

By Theorem 2.2, we can prove the following Corollary.

COROLLARY 2.2.1. *Let X be an abelian surface and let L be an ample line bundle on X. Then for $k \geq 2$, $(k+1)L$ is not k-very ample if and only if (X, L) is the type* (P; $\{a, b\}$) *with $a > 0$ and $b = 1$.* ∎

THEOREM 2.3. *Let X be an abelian surface and let $L_1, \dots, L_k$ be ample line bundles on X. Assume that $k \geq 3$. We put $L := L_1 + \dots + L_k$.*

Then L is not k-very ample if and only if $(X, L_1, \dots, L_k)$ is one of the following:

(1) $k=3$ *and* (X, L_1, L_2, L_3) *is the type* (J),

(2) $(X, L_1, \dots, L_{k-1})$ *is the type* $(\mathrm{PS}; t; a_1, \dots, a_{k-1})$, *and* $L_k = p_t^*(\mathcal{D}_k) \otimes \mathcal{O}_X(T)$, *where* $\mathcal{D}_k \in \mathrm{Pic}(E_t)$ *with* $\deg \mathcal{D}_k \geq 0$, p_t *is the t-th projection, T is a divisor on X with $TF_t = 2$ for a fiber F_t of p_t, a_i is a positive integer for $i = 1, \dots, k-1$, and $t = 1, 2$,*

(3) $(X, L_1, \dots, L_k)$ *is the type* $(\mathrm{PS}; t; a_1, \dots, a_k)$ *with* $a_i > 0$ *for* $i = 1, \dots, k$, *and* $t = 1, 2$.

PROOF. First we prove the «if» part.

If $(X, L_1, \dots, L_k)$ is the type (1) in Theorem 2.3, then $k=3$, $LB = (L_1 + L_2 + L_3)B = 6$, and $g(B) = 2$. Then by Theorem 1.4, L is not 3-very ample.

If $(X, L_1, \dots, L_k)$ is the type (2) or (3) in Theorem 2.3, then $LF_1 \leq k+1$ for a fiber F_1 of p_1. Then by Theorem 1.4, L is not k-very ample.

Next we prove the «only if» part.

We calculate L^2:

$$L^2 = (L_1 + \dots + L_k)^2 = \sum_{i=1}^{k} L_i^2 + 2\sum_{i>j} L_i L_j \geq 2k + 2 \times \frac{k(k-1)}{2} \times 2 = 2k^2.$$

Since $k \geq 3$, we get $2k^2 \geq 4k+6$. So we have $L^2 \geq 4k+6$. Assume that L is not k-very ample. Then by Theorem 1.4, there exists an effective divisor D on X such that

$$LD \leq 2g(D) + k - 1 \leq 2k + 1 .$$

We may assume that

$$L_1 D \leq L_2 D \leq \dots \leq L_k D .$$

Then we obtain that $L_1 D \leq 2$. By Hodge index Theorem, we get $D^2 \leq 2$.

(2.3.1) The case in which $D^2 = 2$.

Then $L_1 D = 2$ and $L_1 \equiv D$ by Hodge index Theorem. Since $g(L_1) = 2$, we obtain that (X, L_1) is the type (I) or (II) in Theorem 1.5. On the other hand, by $g(D) = 2$, $L_1 D = 2$, and Theorem 1.4, we get

$$2k \leq (L_1 + \dots + L_k)D = LD \leq 2g(D) + k - 1 = k + 3 .$$

Hence $k=3$ and $L_1 D = L_2 D = L_3 D = 2$. So by Hodge index Theorem, we get $L_1 \equiv L_2 \equiv L_3 \equiv D$. Therefore we get the type (1) or (3) in Theorem 2.3.

(2.3.2) The case in which $D^2 = 0$.
(2.3.2.1) The case in which $L_1 D = 2$.

Then by assumption, we have $LD \geqslant 2k$. Hence by Theorem 1.4, we get $2k \leqslant LD \leqslant 2g(D) + k - 1 = k + 1$. Therefore $k \leqslant 1$ and this is a contradiction.

(2.3.2.2) The case in which $L_1 D = 1$.

By Remark 1.3.1, there exists a fiber space $f\colon X \to C$ such that C is a smooth elliptic curve, $X \cong C \times F$, f is identified with the first projection via this isomorphism, and D is a fiber of f, where F is a fiber of f. Since $k \leqslant (L_1 + \ldots + L_k) D \leqslant 2g(D) + k - 1 = k + 1$, we get

$$(L_1 D, \ldots, L_{k-1} D, L_k D) = (1, \ldots, 1, 2) \text{ or } (1, \ldots, 1, 1).$$

If $(L_1 D, \ldots, L_{k-1} D, L_k D) = (1, \ldots, 1, 2)$, then $(X, L_1, \ldots, L_k)$ is the type (2) in Theorem 2.3, and if $(L_1 D, \ldots, L_{k-1} D, L_k D) = (1, \ldots, 1, 1)$, then $(X, L_1, \ldots, L_k)$ is the type (3) in Theorem 2.3 by using the same argument as in the proof of the above Theorems. This completes the proof of Theorem 2.3. ■

THEOREM 2.4. *Let X be an abelian surface and let L be an ample line bundle on X. Then for $k \geqslant 2$, kL is not k-very ample if and only if (X, L) is one of the following:*

(1) *$k = 2$ or 3, and (X, L) is the type* (J),
(2) *(X, L) is the type* (P; $\{a, b\}$) *with $a > 0$ and $b = 1$.*

PROOF. For $k \geqslant 3$, this is a corollary of Theorem 2.3. Assume that $k = 2$. By the same argument as in the proof of Theorem 2.3, we can prove the «if» part. So we prove the «only if» part. Assume that L is not 2-very ample.

(2.4.1) The case in which $L^2 \geqslant 4$.

Then $(2L)^2 \geqslant 16$, and by Theorem 1.4, there exists an effective divisor D on X such that $(2L) D \leqslant 2g(D) + 1 \leqslant 5$. If $D^2 > 0$, then by Hodge index Theorem, we get $LD \geqslant 3$ and this is impossible. Hence $D^2 = 0$ and $g(D) = 1$. Therefore $2LD \leqslant 3$, that is, we obtain $LD = 1$. So D is a smooth elliptic curve. By Lemma 1.2 and Lemma 1.3, we can prove that (X, L) is the type (P; $\{a, b\}$) with $a > 0$ and $b = 1$.

(2.4.2) The case in which $L^2 = 2$.

Then (X, L) is one of the type in Theorem 1.5.
This completes the proof of Theorem 2.4. ∎

3. The case in which $(n-t)L$ is not k-very ample for $t=1$ and 2.

THEOREM 3.1. *Let (X, L) be a polarized abelian surface. Let (k, t) be a pair of integer which satisfies the following inequalities;*

(A) $k \geqslant 3t + 1$,
(B) $t \geqslant 1$.

Then $(k-t)L$ is not k-very ample if and only if one of the following types holds;

(1) $(k, t) = (4, 1), (5, 1)$, *or* $(7, 2)$, *and* (X, L) *is the the type* (J),
(2) (X, L) *is the type* (P; $\{a, b\}$) *with* $a > 0$ *and* $b = 1$.

PROOF. First we remark that $(k-t)^2 L^2 \geqslant 4k + 6$ unless $(L^2, k, t) = (2, 4, 1)$.

We prove the «if» part of Theorem 3.1. If (X, L) is the type (1) in Theorem 3.1, then $(k-t)LB \leqslant 2g(B) + k - 1 \leqslant 2k + 1$. Hence by Theorem 1.4, L is not k-very ample. If (X, L) is the type (2) in Theorem 3.1, then $(k-t)LF_1 \leqslant 2g(F_1) + k - 1 \leqslant 2k + 1$ for any fiber F_1 of p_1. Hence L is not k-very ample by Theorem 1.4.

Next we prove the «only if» part.

(3.1.1) The case in which $L^2 \geqslant 4$.

Since $(k-t)^2 L^2 \geqslant 4k + 6$, by Theorem 1.4, there exists an effective divisor D on X such that $(k-t)LD \leqslant 2g(D) + k - 1 \leqslant 2k + 1$.

Assume that $D^2 > 0$. Then, by Hodge index Theorem, we get $LD \geqslant 3$. Hence $3(k-t) \leqslant (k-t)LD \leqslant 2k + 1$, and we obtain $k \leqslant 3t + 1$. By hypothesis we have $k = 3t + 1$ and $LD = 3$. So we get $D^2 = 2$ and $g(D) = 2$. By Theorem 1.4, we have $3(2t+1) = (k-t)LD \leqslant k + 3 = 3t + 4$, and so we have $t \leqslant 1/3$. But this is a contradiction.

Hence $D^2 = 0$ and $g(D) = 1$. By Theorem 1.4, we obtain $(k-t)LD \leqslant k + 1$.

Assume that $LD \geqslant 2$. Then we get $k-1 \leqslant 2t$. Since $k \geqslant 3t+1$, we get $3t \leqslant 2t$. This is impossible because $t \geqslant 1$. Hence $LD = 1$.

Therefore D is a smooth elliptic curve and by the same method as in the proof of the above theorems, we get that (X, L) is the type (P; $\{a, b\}$) with $a > 0$ and $b = 1$.

(3.1.2) The case in which $L^2 = 2$.

Then by Theorem 1.5, we get that (X, L) is one of the types in Theorem 1.5.

In order to prove Theorem 3.1, it is sufficient to study the case in which $X \cong J(B)$ and $L \equiv B$, where B is a smooth projective curve of genus 2 and $J(B)$ is its jacobian variety. If $(k, t) = (4, 1)$, then $(k-t)^2 L^2 < 4k+6$ and $(k-t)L$ is not k-very ample by Theorem 1.4. So we assume that $(k, t) \neq (4, 1)$. Then $(k-t)^2 L^2 \geqslant 4k+6$. Then by Theorem 1.4, there exists an effective divisor D on X such that $(k-t)\, LD \leqslant 2g(D) + k - 1 \leqslant 2k+1$. Here we remark that we can prove $LD \geqslant 2$ by Remark 1.3.1 since $L \equiv B$.

(3.1.2.1) The case in which $LD \geqslant 3$.

Then $3k - 3t \leqslant (k-t)\, LD \leqslant 2k+1$. So we have $k \leqslant 3t+1$. By hypothesis, we get $k = 3t+1$ and $LD = 3$. Since $L^2 = 2$, we obtain $D^2 \leqslant 4$. Hence $2k - 5 \leqslant 3t$. Since $k = 3t+1$, we get $2k-5 \leqslant k-1$ and $k \leqslant 4$. Because $k = 3t+1$, we obtain $(k, t) = (4, 1)$ and this is a contradiction.

(3.1.2.2) The case in which $LD = 2$.

If $D^2 = 0$, then $g(D) = 1$. So by Theorem 1.4, we get $2(k-t) = (k-t)\, LD \leqslant k+1$. Hence $k-1 \leqslant 2t$. Since $k \geqslant 3t+1$, this is impossible because $t \geqslant 1$.

If $D^2 > 0$, then $L \equiv D$ since $LD = 2$ and $L^2 = 2$. By Theorem 1.4, we get $2(k-t) = (k-t)\, LD \leqslant k+3$. Hence $k-3 \leqslant 2t$. Since $k \geqslant 3t+1$, we get $3t-2 \leqslant 2t$. Therefore $t \leqslant 2$.

If $t = 1$, then $2(k-1) = (k-t)\, LD \leqslant k+3$. Hence $k \leqslant 5$. Because $4 = 3t+1 \leqslant k$, we get $k = 5$ by the assumption that $(k, t) \neq (4, 1)$.

If $t = 2$, then $2(k-2) = (k-t)\, LD \leqslant k+3$. Hence $k \leqslant 7$. Because $7 = 3t+1 \leqslant k$, we get $k = 7$.

This completes the proof of Theorem 3.1. ■

THEOREM 3.2. *Let (X, L) be a polarized abelian surface. Let (k, t) be a pair of integers with $3t \geqslant k \geqslant 2t+1$, $t \geqslant 1$, and $(k, t) \neq (3, 1)$. Then*

$(k-t)L$ is not k-very ample if and only if one of the following holds:

(1) $(k, t) = (2t_1+1, t_1)$, $(2t_2+2, t_2)$, *or* $(2t_3+3, t_3)$ *for* $t_1 \geqslant 2$, $t_2 \geqslant 2$ *and* $t_3 \geqslant 3$, *and* (X, L) *is the the type* (J),

(2) (X, L) *is the type* (P; $\{a, b\}$) *with* $a \geqslant 1$ *and* $b = 1$,

(3) $k = 2t+1$ *and* (X, L) *is the type* (P; $\{a, b\}$) *with* $a \geqslant 1$ *and* $b = 2$,

(4) $k = 2t+1$ and (X, L) is one of the type (I; 2), (II; 2), or (III; 2).

PROOF. First we prove the «if» part. If (X, L) is the type (1) in Theorem 3.2, then $(k-t)LB \leqslant 2g(B)+k-1 \leqslant 2k$. Hence L is not k-very ample by Theorem 1.4. If (X, L) is the type (2) or (3) in Theorem 3.2, then $(k-t)LF_1 \leqslant 2g(F_1)+k-1 \leqslant 2k$ for a fiber F_1 of p_1. Hence L is not k-very ample by Theorem 1.4. If (X, L) is the type (4) in Theorem 3.2, then $(k-t)LF \leqslant 2g(F)+k-1 \leqslant 2k$ for a fiber F of f. Hence L is not k-very ample by Theorem 1.4.

Next we prove the «only if» part.

(A) The case in which $L^2 \geqslant 4$.

First we remark that $(k-t)^2L^2 \geqslant 4k+6$ by assumption. Hence by Theorem 1.4 there exists an effective divisor D on X such that $(k-t)LD \leqslant 2g(D)+k-1 \leqslant 2k+1$.

If $LD \geqslant 4$, then $4(k-t) \leqslant 2k+1$, that is, $2k \leqslant 4t+1$. Since $k \geqslant 2t+1$, we get that $4t+1 \geqslant 2k \geqslant 4t+2$. But this is impossible. Hence $LD \leqslant 3$.

If $LD = 3$, then $D^2 \leqslant 2$ since $L^2 \geqslant 4$. Therefore $3(k-t) = (k-t)LD \leqslant k+3$ because $g(D) \leqslant 2$. So we have $2k \leqslant 3t+3$. Since $k \geqslant 2t+1$, we obtain that $3t+3 \geqslant 4t+2$. Since $t \geqslant 1$, we have $t = 1$ and $k = 3$. But by assumption this is a contradiction. Hence $LD \leqslant 2$ and $D^2 = 0$.

If $LD = 1$, or $LD = 2$ such that D is not an irreducible reduced curve, then there exists an irreducible reduced curve B such that $LB = 1$. Then by Lemma 1.2 and Lemma 1.3, we get the type (2) in Theorem 3.2.

If $LD = 2$ such that D is an irreducible curve, then D is a smooth elliptic curve. Then $2(k-t) = (k-t)LD \leqslant 2g(D)+k-1 = k+1$. Hence $k \leqslant 2t+1$. By assumption $k = 2t+1$ in this case. Then by Theorem 1.6, we get the type (3) or (4) in Theorem 3.2.

(B) The case in which $L^2 = 2$.

Then (X, L) is the type (I) or (II) in Theorem 1.5. If (X, L) is the type (II) in Theorem 1.5, then (X, L) is the type (2) in Theorem 3.2. So it is sufficient to study the case in which (X, L) is the type (I) in Theorem 1.5.

Assume that $X \cong J(B)$ and L is the class of a translation of the theta divisor, where B is a smooth projective curve of genus 2 and $J(B)$ is its jacobian variety. Then we remark that $(k-t)^2 L^2 \geqslant 4k+6$ (resp. $(k-t)^2 L^2 < 4k+6$) if $t \geqslant 4$ (resp. $t \leqslant 3$). If $t \leqslant 3$, then we get that $(k-t)L$ is not k-very ample by Theorem 1.4. Since $3t \geqslant k \geqslant 2t+1$ and $(k, t) \neq (3, 1)$, we get that $(k, t) = (9, 3), (8, 3), (7, 3), (6, 2)$ or $(5, 2)$.

Assume that $t \geqslant 4$. Then by Theorem 1.4, there exists an effective divisor D on X such that $(k-t)LD \leqslant 2g(D) + k - 1 \leqslant 2k+1$.

If $LD \geqslant 4$, then this is impossible by the same argument as in the case (A). Hence $LD \leqslant 3$. We remark that $LD \geqslant 2$ since L is the class of a translation of the theta divisor. Since $L^2 = 2$, we get that $D^2 \leqslant 4$ and $g(D) \leqslant 3$.

If $LD = 3$, then $3(k-t) \leqslant k+5$. Hence $2k \leqslant 3t+5$. Since $k \geqslant 2t+1$, we get $4t+2 \leqslant 2k \leqslant 3t+5$, that is, $t \leqslant 3$. But this is a contradiction.

If $LD = 2$, then $D^2 \leqslant 2$. If $D^2 = 2$, then $g(D) = 2$ and $2(k-t) \leqslant k+3$. Thus we have $k \leqslant 2t+3$. Hence $k = 2t+1, 2t+2$, and $2t+3$.

If $D^2 = 0$, then $g(D) = 1$ and $2(k-t) \leqslant k+1$. Hence $k \leqslant 2t+1$. Since $k \geqslant 2t+1$, we get $k = 2t+1$.

This completes the proof of Theorem 3.2. ∎

COROLLARY 3.3. *Let (X, L) be a polarized abelian surface. Assume that $k \geqslant 2t+4$ for $(k, t) \in \mathbb{N}^{\oplus 2}$. Then $(k-t)L$ is not k-very ample if and only if (X, L) is the type* (P; $\{a, b\}$) *with $a > 0$ and $b = 1$.*

PROOF. This is obtained by Theorem 3.1 and Theorem 3.2. ∎

THEOREM 3.4. *Let (X, L) be a polarized abelian surface. Assume that $k \in \mathbb{N}$ with $k \geqslant 3$. Then $(k-1)L$ is not k-very ample if and only if one of the following holds*:

(1) *$k = 3, 4,$ or 5, and (X, L) is the type* (J),

(2) *(X, L) is the type* (P; $\{a_1, a_2\}$) *with $a_1 > 0$ and $a_2 = 1$,*

(3) *$k = 3$ and (X, L) is the type* (P; $\{b_1, b_2\}$) *with $b_1 \geqslant 1$ and $b_2 = 2$,*

(4) *$k = 3$ and (X, L) is one of the type* (I; 2), (II; 2), or (III; 2),

(5) *$k = 3$ and $L^2 = 4$.*

PROOF. We can easily prove the «if» part by Theorem 1.4. So we prove the «only if» part. If $k \geqslant 4$, then this is a corollary of Theorem 3.1, and (X, L) is one of the type (1) or (2) in Theorem 3.4. So we may assume that $k = 3$.

(A) The case in which $L^2 \geqslant 6$.

Then $4L^2 = (k-1)^2 L^2 \geqslant 18 = 4k + 6$. Hence by Theorem 1.4, there exists an effective divisor D on X such that $2LD \leqslant 2g(D) + 2 \leqslant 7$. Hence $LD \leqslant 3$. Since $L^2 \geqslant 6$ we get that $D^2 = 0$ and $g(D) = 1$. Therefore $2LD \leqslant \leqslant 4$, that is, $LD \leqslant 2$.

If $LD = 1$, or $LD = 2$ and D is not irreducible and reduced, then there exists an irreducible and reduced curve B on X such that $LB = 1$. Hence by Lemma 1.2 and Lemma 1.3, we get that (X, L) is the type (2) in Theorem 3.4.

If $LD = 2$ and D is irreducible and reduced, then by Theorem 1.6 we get that (X, L) is one of the type (3) or (4) in Theorem 3.4.

(B) The case in which $L^2 = 2$.

Then by Theorem 1.5, we get that (X, L) is one of the type (1) or (2) in Theorem 3.4.

(C) The case in which $L^2 = 4$.

Then (X, L) is the type (5) in Theorem 3.4.

This completes the proof of Theorem 3.4. ■

THEOREM 3.5. *Let (X, L) be a polarized abelian surface. Assume that $k \in \mathbb{N}$ with $k \geqslant 4$. Then $(k-2)L$ is not k-very ample if and only if one of the following holds*:

(1) $k = 4, 5, 6$, *or* 7, *and* (X, L) *is the type* (J),

(2) (X, L) *is the type* (P; $\{a_1, a_2\}$) *with* $a_1 > 0$ *and* $a_2 = 1$,

(3) $k = 4$ *or* 5, *and* (X, L) *is the type* (P; $\{b_1, b_2\}$) *with* $b_1 \geqslant 1$ *and* $b_2 = 2$,

(4) $k = 4$ *or* 5, *and* (X, L) *is one of the type* (I; 2), (II; 2), *or* (III; 2),

(5) $k = 4$ *and* $L^2 = 4$.

PROOF. We can easily prove the «if» part by Theorem 1.4. So we prove the «only if» part. If $k \geqslant 7$, then this is a corollary of Theorem 3.1, and (X, L) is one of the type (1) or (2) in Theorem 3.5.

If $6 \geqslant k \geqslant 5$, then this is a corollary of Theorem 3.2, and (X, L) is one of the type (1), (2), (3) or (4) in Theorem 3.5. So we assume $k = 4$.

(A) The case in which $L^2 \geqslant 6$.

Then $4L^2 = (k-2)^2 L^2 \geqslant 22 = 4k+6$. Hence by Theorem 1.4, there exists an effective divisor D on X such that $2LD \leqslant 2g(D) + 3 \leqslant 9$. Hence $LD \leqslant 4$. Since $L^2 \geqslant 6$ we get that $D^2 \leqslant 2$ and $g(D) \leqslant 2$. Therefore $2LD \leqslant \leqslant 7$, that is, $LD \leqslant 3$.

If $LD = 3$, then $D^2 = 0$ since $L^2 \geqslant 6$. Hence $2LD \leqslant 5$, that is, $LD \leqslant 2$. This is a contradiction. Hence $LD \leqslant 2$ and $D^2 = 0$.

If $LD = 1$, or $LD = 2$ and D is not irreducible and reduced, then there exists an irreducible and reduced curve B on X such that $LB = 1$. Hence by Lemma 1.2 and Lemma 1.3, we get that (X, L) is the type (2) in Theorem 3.5.

If $LD = 2$ and D is irreducible and reduced, then by Theorem 1.6 we get that (X, L) is one of the type (3) or (4) in Theorem 3.5.

(B) The case in which $L^2 = 2$.

Then by Theorem 1.5, we get that (X, L) is one of the type (1) or (2) in Theorem 3.5.

(C) The case in which $L^2 = 4$.

Then (X, L) is the type (5) in Theorem 3.5.

This completes the proof of Theorem 3.5. ■

In general, we can prove the following Theorem.

THEOREM 3.6. *Let (X, L) be a polarized abelian surface. Let $(k, t, u) \in \mathbb{N}^{\oplus 3}$ with $u \geqslant 2$ and $k \geqslant \{(u+3)t+1\}/(u+1)$. Assume that $(k-t)^2 L^2 \geqslant 4k+6$. If $(k-t)L$ is not k-very ample, then L is not u-very ample.*

PROOF. If $L^2 < 4u+6$, then by Theorem 1.4, L is not u-very ample. Hence we may assume that $L^2 \geqslant 4u+6$.

By assumption and Theorem 1.4, there exists an effective divisor D on X such that $(k-t)LD \leqslant 2g(D) + k - 1 \leqslant 2k+1$. We remark that $g(D) \geqslant 1$.

If $g(D)=1$, then $(k-t)LD \leqslant k+1$. Since

$$k \geqslant \frac{(u+3)t+1}{u+1} \geqslant \frac{(u+1)t+1}{u},$$

we get $LD \leqslant (k+1)/(k-t) \leqslant u+1$. Hence $LD \leqslant u+1 = 2g(D)+u-1 \leqslant 2u+1$. Therefore L is not u-very ample.

Assume that $g(D) \geqslant 2$. Then

$$LD \leqslant \frac{2g(D)+k-1}{k-t} = 2g(D) + \frac{2g(D)+k-1}{k-t} - 2g(D) \leqslant$$

$$\leqslant 2g(D) + \frac{2g(D)+k-1}{k-t} - 4 \leqslant 2g(D) + \frac{2k+1}{k-t} - 4\,.$$

On the other hand,

$$u-1-\left(\frac{2k+1}{k-t}-4\right) = u+3-\frac{2k+1}{k-t} =$$

$$= \frac{1}{k-t}\{(u+1)k-(u+3)t-1\} \geqslant 0\,.$$

Therefore $LD \leqslant 2g(D)+u-1$.

Hence it is sufficient to prove that $2g(D)+u-1 \leqslant 2u+1$.

If $D^2 \leqslant u$, then $2g(D)+u-1 \leqslant 2u+1$. Hence we may assume that $D^2 > u$. Since $L^2 \geqslant 4u+6$, we get

$$(LD)^2 \geqslant L^2D^2 \geqslant (4u+6)(u+1) = 4u^2+10u+6 =$$

$$= 4(u+1)^2+2u+2 > 4(u+1)^2.$$

Hence $LD \geqslant 2(u+1)+1$.

On the other hand,

$$2(u+1) - \frac{2k+1}{k-t} = \frac{1}{k-t}(2uk-2(u+1)t-1) =$$

$$= \frac{2u}{(k-t)}\left(k-\frac{(u+1)}{u}t-\frac{1}{2u}\right) \geqslant \frac{2u}{(k-t)}\left(k-\frac{(u+3)}{u+1}t-\frac{1}{u+1}\right) \geqslant 0\,.$$

Hence

$$LD \leqslant \frac{2k+1}{k-t} \leqslant 2(u+1).$$

But this is impossible. Therefore $LD \leqslant 2g(D) + u - 1 \leqslant 2u + 1$. By Theorem 1.4, we get that L is not u-very ample. ■

REFERENCES

[BaSz1] TH. BAUER - T. SZEMBERG, *On tensor products of ample line bundles on abelian varieties*, Math. Z., **223** (1996), pp. 79-85.

[BaSz2] TH. BAUER - T. SZEMBERG, *Higher order embeddings of abelian varieties*, Math. Z., **224** (1997), pp. 449-455.

[BeLP] M. C. BELTRAMETTI - A. LANTERI - M. PALLESCHI, *Algebraic surfaces containing an ample divisor of arithmetic genus two*, Ark. Mat., **25** (1987) pp. 189-210.

[BeSo] M. C. BELTRAMETTI - A. J. SOMMESE, *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Math., **16**, Walter de Gruyter, Berlin, New York (1995).

[Fj] T. FUJITA, *Classification of polarized manifolds of sectional genus two*, the *Proceedings of «Algebraic Geometry and Commutative Algebra» in Honor of Masayoshi Nagata* (1987), pp. 73-98.

[Fk] Y. FUKUMA, *On d-very ample line bundles on abelian surfaces*, preprint.

[Ha] R. HARTSHORNE, *Algebraic Geometry*, Graduate Texts in Math, vol. **52**, Springer-Verlag (1977).

[LB] H. LANGE - CH. BIRKENHAKE, *Complex Abelian Varieties*, Grundlehren der mathematischen Wissenschaften, **302**, Springer-Verlag (1992).

[O] A. OHBUCHI, *Some remarks on ample line bundles on abelian varieties*, Manuscripta Math., **57** (1987) pp. 225-238.

[OoU] F. OORT - K. UENO, *Principally polarized abelian varieties of dimension two or three are Jacobian varieties*, J. Fac. Sci. Univ. Tokyo, **20** (1973), pp. 377-381.

[Te] H. TERAKAWA, *The k-very ampleness and k-spannedness on polarized abelian surfaces*, to appear in Math. Nachr.

Manoscritto pervenuto in redazione il 21 luglio 1997
e in forma revisionata, il 5 gennaio 1998.