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On Positive Solutions of Some Periodic Parabolic Eigenvalue Problem with a Weight Function.

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ABSTRACT - Let Ω be a bounded domain in R^n and let m be a T-periodic function such that its restriction to $\Omega \times (0, T)$ is in $L^r(\Omega \times (0, T))$ for some $r > n+2$. We find necessary and sufficient conditions, on m, for the existence, uniqueness and simplicity of the principal eigenvalue for the Dirichlet and Neumann periodic parabolic eigenvalue problem with weight m.

1. Introduction.

Let Ω be a bounded domain in $\Re^n$ with $C^{2+\theta}$ boundary $(0 < \theta < 1)$, and $\{a_{i,j}(x, t)\}_{1 \leq i, j \leq n}$ $\{a_j(x, t)\}_{1 \leq j \leq n}$ two families of $(\theta, \theta/2)$-Hölder continuous and T-periodic in t functions on $\overline{\Omega} \times \Re$. We also assume that $a_{i,j} = a_{j,i}$ and that $c \sum_i \xi_i^2 \leq \sum_{i,j} a_{i,j}(x, t)\, \xi_i \xi_j$ for some $c > 0$ and all $(x, t) \in \overline{\Omega} \times \Re$, $(\xi_1, \ldots, \xi_n) \in \Re^n$.

Let $m(x, t)$ be a T-periodic in t real function on $\Omega \times \Re$. Our aim is to consider, in a suitable weak sense, the following periodic parabolic

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eigenvalue problem on $\Omega \times \Re$

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum_{i,j} a_{i,j} \dfrac{\partial^2 u}{\partial x_i \partial x_j} - \sum_j a_j \dfrac{\partial u}{\partial x_j} = \lambda m u \,, \\ B(u) = 0 \,, \\ u(x, t) = u(x, t+T) \,, \end{cases}$$

where either $B(u) = u_{|\partial\Omega \times \Re}$ (Dirichlet condition) or $B(u) = (\partial u / \partial \nu)_{|\partial\Omega \times \Re}$ (Neumann condition). If $m \in C^{\theta, \theta/2}(\overline{\Omega} \times \Re)$ and $B(u) = u_{|\partial\Omega \times \Re}$, Beltramo and Hess, in [B-H], found necessary and sufficient conditions on m for the existence, uniqueness and simplicity of the positive principal eigenvalue of the above problem. Beltramo extended these results to more general boundary conditions (that include the Neumann condition) in [B]. The case m continuous and $B(u) = (\partial u / \partial \nu)_{|\partial\Omega \times \Re}$ is studied in [P] and the case $m \in L^\infty(\Omega \times \Re)$ is treated in [G-L-P] under the additional hypothesis $a_{i,j} \in C^1(\overline{\Omega} \times \Re)$, $1 \leq i, j \leq n$.. Our purpose is to obtain, under this additional assumtion, similar results if $\left(\int_{\Omega \times (0,T)} |m|^r \right)^{1/r} < \infty$.

2. Notation and preliminaries.

Let Ω, $a_{i,j}$, a_j; be as above with $a_{i,j} \in C^1(\overline{\Omega} \times \Re)$, $1 \leq i, j \leq n$. We fix, for the whole paper, p, q and r such that $n + 2 < p < q < r$. We consider, for $u \in C^{2,1}(\overline{\Omega} \times \Re)$, $L(u) = \partial u / \partial t + A(x, t, D) u$ where $A(x, t, D)\, u = - \sum a_{i,j}(x, t)\, D_{i,j} u - \sum a_j(x, t)\, D_j u$.

Let E be a vector space of functions on $\Omega \times \Re$, we set

$$E_B = \{ u \in E \cap D(B) \colon Bu = 0 \} \,,$$

where $D(B)$ is the domain of the boundary condition. For $1 \leq s \leq \infty$, let $L^s_T(\Omega \times \Re)$ denote the space of the measurable functions $f \colon \Omega \times \Re \to C$ such that $f(x, t) = f(x, t+T)$ a.e. $(x, t) \in \Omega \times \Re$ and $\|f\|_s < \infty$, where $\|f\|_s = \left(\int_{\Omega \times (0,T)} |f|^s \right)^{1/s}$ if $s < \infty$ and $\|f\|_\infty = \operatorname{ess\,sup}_{(x,t) \in \Omega \times (0,T)} |f(x,t)|$. If S is a bounded operator from $L^p_T(\Omega \times \Re)$ into $L^q_T(\Omega \times \Re)$, we write $\|S\|_{p,q}$ for the norm operator with respect to the above norms. If a is a real or complex valued function on $\Omega \times \Re$, we still denote by a the operator multiplication by a.

Let $X = L^p(\Omega)$. If $a(x, t)$ is a T periodic function in $C^{\theta, \theta/2}(\Omega \times \Re)$ and satisfies $a \geq 0$ if $B(u) = u_{|\partial\Omega \times \Re}$ and $a \geq 0$, $a \neq 0$ if $B(u) =$

$= (\partial u/\partial \nu)_{|\partial\Omega \times \mathfrak{R}}$, we consider $A_a(t)$: $W_B^{2,p}(\Omega) \subset L^p(\Omega) \to L^p(\Omega)$ defined by $A_a(t)u = A(\cdot, t, D)u + a(\cdot, t)u$. Once we fix $k \in \mathfrak{R}$, $k > 1 + \|a\|_\infty$, we put $A = A_{a+k}(0)$ and for $\alpha \in [0, 1]$-let A^α be defined as in [H,1]. Let X_α be the domain of A^α. Then X_α is a Banach space with the norm $\|x\|_\alpha = \|A^\alpha x\|_{L^p(\Omega)}$. We have $X_0 = L^p(\Omega)$, $X_1 = W_B^{2,p}(\Omega)$ and $X_\alpha \subset X_\beta$ whenever $0 \leq \beta < \alpha \leq 1$, the inclusion being compact. Moreover, for $1/2 + n/2p < \alpha < 1$, we have $X_\alpha \subset C_B^{1+\gamma}$ for some $\gamma = \gamma(\alpha) < 1$ and this inclusion is compact (see [H,1] or [A]).

For $\omega > 0$, $f \in C^\sigma([0, T+\omega], X)$, $\sigma \in (0, 1]$ the linear evolution equation

$$\frac{du}{dt} + A_{a+k}(t)\, u(t) = f(t), \qquad u(0) = u_0$$

has a unique solution $u \in C([0, T+\omega,], X) \cap C^1((0, T+\omega], X)$ if $u_0 \in X_0$ and $u \in C^1([0, T+\omega], X)$ if $u_0 \in X_1$. This solution is given by

$$u(t) = U_{a+k}(t, 0)u_0 + \int_0^t U_{a+k}(t, \tau)\, f(\tau)\, d\tau\,,$$

where $U_{a+k}(t, \tau)$ is the associated evolution operator. The change $u(t) = e^{-kt} v(t)$ reduces the problem

$$\frac{dv}{dt} + A_a(t)\, v(t) = f(t), \qquad v(0) = u_0$$

to the above problem and gives us

$$v(t) = U_a(t, 0)\, u_0 + \int_0^t U_a(t, \tau)\, f(\tau)\, d\tau\,,$$

where $U_a(t, \tau) = e^{k(t-\tau)}\, U_{a+k}(t, \tau)$.

REMARK 2.1. For the rest of the paper we fix $\alpha \in (1/2 + n/2p, 1 - 1/p)$, let k be as above and set

$$K_a = U_a(T, 0)\,|_{X_\alpha}: X_\alpha \to X_\alpha.$$

Let $C_T^\gamma(\mathfrak{R}, X_\alpha)$, $C_T(\overline{\Omega} \times \mathfrak{R})$ and $C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})$ denote the subspace of T-periodic functions in $C^\gamma(\mathfrak{R}, X_\alpha)$, $C(\overline{\Omega} \times \mathfrak{R})$ and $C_B^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})$ respectively. We identify $L_T^p(\Omega \times \mathfrak{R})$ with $L_T^p(\mathfrak{R}, L^p(\Omega))$ in the obvious way. Then, if a is as above, (see e.g. [G-L-P], lemma 3.1), there exists

$\gamma \in (0, 1)$ such that the operator $S_a: L^p(\Omega \times [0, T+\omega]) \to C^\gamma([0, T+\omega], X_a)$ defined by

$$(S_a(g))(t) = U_a(t, 0)(I - K_a)^{-1} \int_0^T U_a(T, \tau)\, g(\tau)\, d\tau + \int_0^t U_a(t, \tau)\, g(\tau)\, d\tau$$

is injective, positive and bounded. Moreover $S_a(g)$ has a unique T-periodic extension to $\Omega \times \mathfrak{R}$, still denoted by $S_a(g)$ such that for $q \geqslant p$

$$S_a(g)_{|L_T^q(\Omega \times \mathfrak{R})}: L_T^q(\Omega \times \mathfrak{R}) \to L_T^q(\Omega \times \mathfrak{R})$$

is a compact operator. Moreover, for some $\gamma^* \in (0, 1)$, the same is true for

$$S_a(g)_{|C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})}: C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R}) \to C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R}).$$

This follows from the observation that the following inclusions are continuous and that the second is compact

$$C_T^\gamma(\mathfrak{R}, X_a) \subset C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R}) \subset C_T(\overline{\Omega} \times \mathfrak{R}) \subset L_T^q(\Omega \times \mathfrak{R}).$$

REMARK 2.2. We now take $\lambda > 0$ and define $W = S_\lambda(L_T^p(\Omega \times \mathfrak{R}))$ and $\tilde{L}: W \to L_T^p(\Omega \times \mathfrak{R})$ by $\tilde{L} = S_\lambda^{-1} - \lambda I$. Then W and $\tilde{L}$ do not depend on λ (see e.g. [G-L-P], remark 3.5), $\tilde{L}$ is an extension of L and $(\tilde{L} + \lambda)^{-1}: L_T^p(\Omega \times \mathfrak{R}) \to W$ is a positive operator (see e.g. [G-L-P] Lemma 3.7). Note that, if we consider on W the topology induced by $C_{T,B}^{1+\gamma^*, \gamma^*}(\overline{\Omega} \times \mathfrak{R})$, then $\tilde{L}$ is a closed operator.

LEMMA 2.3. $\lim_{\delta \to +\infty} \|(\tilde{L} + \delta)^{-1}\|_{r, \infty} = 0.$

PROOF. For $\delta > 0$, we have $U_a(t, \tau) = e^{\delta(t-\tau)} U_{a+\delta}(t, \tau)$. For $g \in L_T^q(\Omega \times \mathfrak{R})$, $0 \leqslant \tau \leqslant T$, we set $G_\delta(\tau) = e^{\delta\tau} g(\tau)$, Hölder inequality gives us

$$\|G_\delta\|_{L^p(\Omega \times [0, t])} \leqslant c\delta^{-(1/p - 1/q)} e^{\delta t} \|g\|_{L^q(\Omega \times [0, t])}.$$

For $0 \leqslant t \leqslant T$ we have $(\tilde{L} + \delta + 1)^{-1}(g)(t) = S^{(1)}(g)(t) + S^{(2)}(g)(t)$, where

$$S^{(1)}(g)(t) = e^{-\delta t} \int_0^t U_1(t, \tau)\, G_\delta(\tau)\, d\tau$$

and

$$S^{(2)}(g)(t) = e^{-\delta(t+T)} U_1(t, 0)(I - e^{-\delta T} K_1)^{-1} \int_0^t U_1(T, \tau)\, G_\delta(\tau)\, d\tau$$

reasoning as in lemma 3.1 in [G-L-P], we get

$$\|S^{(1)}(g)(t)\|_{L^\infty(\Omega)} \leqslant c\|S^{(1)}(g)(t)\|_\alpha \leqslant ce^{-\delta t}\|G_\delta\|_{L^p(\Omega\times(0,t))} \leqslant$$
$$\leqslant c\delta^{-(1/p-1/q)}\|g\|_{L^q(\Omega\times[0,T])}.$$

So

$$\|S^{(1)}(g)(t)\|_{L^\infty(\Omega\times(0,T))} \leqslant c\delta^{-(1/p-1/q)}\|g\|_{L^q(\Omega\times(0,T))}.$$

The maximum principle implies

$$\|U_1(t, 0)\|_{C(\overline{\Omega}),\, C(\overline{\Omega})} \leqslant 1\,, \qquad \|K_1^{-1}\|_{C(\overline{\Omega}),\, C(\overline{\Omega})} \leqslant 1\,,$$

then

$$\left\| e^{-\delta(t+T)} U_1(t, 0)(I - e^{-\delta T}K_1)^{-1} \int_0^t U_1(T, \tau)\, G_\delta(\tau)\, d\tau \right\|_{L^\infty(\Omega\times(0,T))} \leqslant$$

$$\leqslant ce^{-\delta T}\left\| \int_0^t U_1(T, \tau)\, G_\delta(\tau)\, d\tau \right\|_{L^\infty(\Omega\times(0,T))} \leqslant$$

$$\leqslant ce^{-\delta T}\left\| \int_0^T U_1(T, \tau) G_\delta(\tau)\, d\tau \right\|_\alpha \leqslant c\delta^{-(1/p-1/q)} e^{\delta t}\|g\|_{L^q(\Omega\times[0,T])}$$

and the lemma follows. ■

REMARK 2.4. By lemma 2.3 there exists a non increasing function $\delta_0 \colon \mathfrak{R}^{>0} \to \mathfrak{R}^{>0}$ such that $\|(\tilde{L} + \delta)^{-1}\|_{r,\infty} < \varepsilon$ if $\delta > \delta_0(\varepsilon)$. We set $W^q = S_\lambda(L_T^q(\Omega \times \mathfrak{R}))$. Note that W^q does not depend on the choice of λ. , moreover $W = W^p$. We have

LEMMA 2.5. *If $a \in L_T^r(\Omega \times \mathfrak{R})$ and $\delta \in \mathfrak{R}$, $\delta > \delta_0(1/\|a\|_r)$, then*

i) *$(\tilde{L} + a + \delta)_{|W^q}$ is a bijection between W^q and $L_T^q(\Omega \times \mathfrak{R})$.*

ii) *$((\tilde{L} + a + \delta)_{|W^q})^{-1} \colon L_T^q(\Omega \times \mathfrak{R}) \to L_T^q(\Omega \times \mathfrak{R})$ is a compact operator, moreover it is positive if $a \geqslant 0$.*

PROOF. To see that $\tilde{L}+a+\delta$ is injective we note that $(\tilde{L}+a+\delta)\cdot\cdot w=0$, $w\in W^q$, implies $(I+(\tilde{L}+\delta)^{-1}a)w=0$ (since $\tilde{L}+\delta$ is injective). Now Lemma 2.3 give us the injectivity. Note also that, for $w\in W^q$, $u\in L_T^q(\Omega\times\Re)$, $(\tilde{L}+a+\delta)w=u$ is equivalent to

$$w+(\tilde{L}+\delta)^{-1}(aw)=(\tilde{L}+\delta)^{-1}u$$

then Lemma 2.3 implies that for $u\in L_T^q(\Omega\times\Re)$ this equation has an unique solution w in $L_T^\infty(\Omega\times\Re)$. Moreover, the solution is given by

$$w=(\tilde{L}+\delta)^{-1}u-(\tilde{L}+\delta)^{-1}(aw)$$

then $w\in W^q$ and so $(\tilde{L}+a+\delta)_{|W^q}$ is bijective. On the other hand, Hölder inequality gives us $\|(\tilde{L}+\delta)^{-1}a\|_{\infty,\infty}<1$. Therefore

$$I+((\tilde{L}+\delta)^{-1}a)_{|C_T(\Omega\times\Re)}\colon C_T(\Omega\times\Re)\to C_T(\Omega\times\Re)$$

has a bounded inverse. Since $(\tilde{L}+\delta)^{-1}_{|L_T^q(\Omega\times\Re)}$ is a compact operator on $L_T^q(\Omega\times\Re)$, the first statement of (ii) follows from the identity

$$((\tilde{L}+a+\delta)_{|W^q})^{-1}=((I+(\tilde{L}+\delta)^{-1}a)_{|C_T(\Omega\times\Re)})^{-1}(\tilde{L}+\delta)^{-1}.$$

Now we take a sequence $\{a_j\}_{j\in N}$ of nonnegative and T periodic Hölder continuous functions with support contained in $\Omega\times\Re$ that converges to a in $L_T^r(\Omega\times\Re)$, then the sequence $((\tilde{L}+a_j+\delta)_{|W^q})^{-1}$ converges to $((\tilde{L}+a+\delta)_{|W^q})^{-1}$ in the norm topology on $B(L_T^q(\Omega\times\Re))$. Indeed

$$\|((\tilde{L}+a+\delta)_{|W^q})^{-1}-((\tilde{L}+a_j+\delta)_{|W^q})^{-1}\|_{q,q}\leq$$

$$\leq\|((\tilde{L}+a+\delta)_{|W^q})^{-1}\|_{r,q}\|a_j-a\|_{\infty,r}\|((\tilde{L}+a_j+\delta)_{|W^q})^{-1}\|_{q,\infty}\leq$$

$$\leq 2\|((\tilde{L}+a+\delta)_{|W^q})^{-1}\|_{r,q}\|a_j-a\|_r\|((\tilde{L}+\delta)_{|W^q})^{-1}\|_{q,\infty}.$$

Since each $((\tilde{L}+a_j+\delta)_{|W^q})^{-1}$ is a positive operator, the lemma follows. ■

LEMMA 2.6. *Let (λ_1,λ_2) be an open interval with $\lambda_1>0$. Suppose $m\in L_T^r(\Omega\times\Re)$ and let $s_0=s_0(\lambda_1,\lambda_2,m)=(1/\lambda_1)\,\delta_0(1/(4\lambda_2\|m\|_{\infty,r}))$, where δ_0 is defined as in remark 2.4. Then for $\lambda\in(\lambda_1,\lambda_2)$ and $s>s_0$ we have*

(i) *$(\tilde{L}+\lambda(s-m))_{|W^q}\colon W^q\to L_T^q(\Omega\times\Re)$ is a bijection.*

(ii) $\|((\tilde{L}+\lambda(s-m))_{|W^q})^{-1}\|_{\infty,\infty}\leq 4\|((\tilde{L}+\lambda s)_{|W^q})^{-1}\|_{\infty,\infty}$.

(iii) $((\tilde{L}+\lambda(s-m))|_{W^q})^{-1}: L^q_T(\Omega\times\mathfrak{R})\to L^q_T(\Omega\times\mathfrak{R})$ *is a compact and positive operator.*

PROOF. We write, as usual, $m=m^+-m^-$ where $m^+=\max\{m,0\}$ and $m^-=-\min\{m,0\}$. Suppose that $(\tilde{L}+\lambda(s-m))w=0$ for some $w\in W$, then $(I-\lambda(\tilde{L}+\lambda s)^{-1}m)w=0$ and so $w=0$, since

$$\lambda\|(\tilde{L}+\lambda s)^{-1}m\|_{\infty,\infty}\leqslant\lambda\|(\tilde{L}+\lambda s)^{-1}\|_{r,\infty}\|m\|_{\infty,r}<1.$$

By Lemma 2.5 $((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}$ is a compact and positive operator on $L^q_T(\Omega\times\mathfrak{R})$. We have that

$$\|((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}|m|\|_{\infty,\infty}\leqslant 2\|((\tilde{L}+\lambda s)_{|W^q})^{-1}|m|\|_{\infty,\infty}<\frac{1}{2}. \tag{2.7}$$

Then $I-((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}|m|$, as operator on $L^\infty_T(\Omega\times\mathfrak{R})$, has a bounded inverse. Since

$$\left(I-\left((\tilde{L}+\lambda(s+2m^-))_{|W^q}\right)^{-1}|m|\right)^{-1}= \tag{2.8}$$
$$=\sum_{j\geqslant 0}\left(\left((\tilde{L}+\lambda(s+2m^-))_{|W^q}\right)^{-1}|m|\right)^j$$

this inverse is positive.

If $u\in L^q_T(\Omega\times\mathfrak{R})$, let

$$w=\left(I-\left((\tilde{L}+\lambda(s+2m^-))_{|W^q}\right)^{-1}|m|\right)^{-1}((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}u. \tag{2.9}$$

Then

$$w-((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}(|m|w)=((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}u.$$

Thus $w\in W$ and $(\tilde{L}+\lambda(s-m))w=u$. Therefore (i) holds. From (2.8), (2.9) and (2.7), we obtain (ii). The compactness stated in (iii) follows from (2.8), since (by Lemma 2.5) $((\tilde{L}+\lambda(s+2m^-))_{|W^q})^{-1}$ is a compact operator on $L^q_T(\Omega\times\mathfrak{R})$. ∎

REMARK 2.10. If m is a T-periodic function in $C^\infty(\Omega\times\mathfrak{R})$ then for s large enough, $(L+\lambda(s-m))^{-1}_{|C_T(\Omega\times\mathfrak{R})}$ is a bounded operator on $C_T(\Omega\times\mathfrak{R})$. Let ϱ denote its spectral radius and let $\mu_m:\mathfrak{R}\to\mathfrak{R}$ be defined as in [H,1], p. 38, then for $\lambda\in\mathfrak{R}$, $\mu_m(\lambda)$ is the unique real number such that there exists $u_\lambda>0$, $u_\lambda\in C^{2,1}(\Omega\times\mathfrak{R})$ satisfying $Lu_\lambda=\lambda m u_\lambda+$

$+\mu_m(\lambda)\, u_\lambda$, therefore

$$\frac{1}{\lambda s + \mu_m(\lambda)} = \varrho$$

DEFINITION 2.11. *Let S be a bounded operator on $L^q(\Omega \times [0, T])$ and let E be a measurable subset of $\Omega \times [0, T]$. As in* [S], *we say that E is invariant relative to S if $Sf = 0$ a.e. in E whenever $f = 0$ a.e. on E and that S is irreducible if there are no nontrivial invariant subsets relative to S.*

In the following we identify $L^q_T(\Omega \times \mathfrak{R})$ with $L^q(\Omega \times [0, T])$.

LEMMA 2.12. (i) *Suppose $a \in L^r(\Omega \times [0, T])$ and let δ be a positive real number as in lemma 2.5, then $((\tilde{L} + a + \delta)_{|W^q})^{-1}$ is irreducible on $L^q(\Omega \times [0, T])$*

(ii) *Let (λ_1, λ_2) be a finite open interval with $\lambda_1 > 0$. Suppose $m \in L^r_T(\Omega \times \mathfrak{R})$ and let $s_0 = s_0(\lambda_1, \lambda_2, m)$ be defined as in lemma 2.6. Then for $\lambda \in (\lambda_1, \lambda_2)$ and $s > s_0$*

$$((\tilde{L} + \lambda(s - m))|_{W^q})^{-1}\colon L^q_T(\Omega \times \mathfrak{R}) \to L^q_T(\Omega \times \mathfrak{R})$$

is irreducible on $L^q(\Omega \times [0, T])$.

PROOF. To prove (i) we take $b \in C^\infty_T(\Omega \times \mathfrak{R})$ such that

$$\|b - a - \delta\|_r < \|((\tilde{L} + a + \delta)_{|W^q})^{-1}\|^{-1}_{r,\infty}$$

and note that

$$((\tilde{L} + b)_{|W^q})^{-1} =$$

$$= \left[\sum_{j=0}^{\infty} (-1)^j (((\tilde{L} + a + \delta)_{|W^q})^{-1}(b - a - \delta))^j\right] ((\tilde{L} + a + \delta)_{|W^q})^{-1}.$$

Then if $E \subset \Omega \times [0, T]$ is invariant relative to $((\tilde{L} + a + \delta)_{|W^q})^{-1}$it is also invariant relative to $((\tilde{L} + b)_{|W^q})^{-1}$. But this last operator on $L^q_T(\Omega \times \mathfrak{R})$ is irreducible, indeed, pick $c \in \mathfrak{R}$ such that $b \leq c$. If $f \in L^q_T(\Omega \times \mathfrak{R})$ and $f > 0$ then $(\tilde{L} + b)^{-1} f \geq (\tilde{L} + c)^{-1} f$ and the right hand side of this inequality is positive a.e.

To prove (ii) we take $b \in L^\infty_T(\Omega \times \mathfrak{R})$ such that $\|b - m\|_r < 1$. Then, for

$\lambda\varepsilon[\lambda_1, \lambda_2]$, we have $\|\lambda((\tilde{L}+\lambda(s-m))_{|W^q})^{-1}\|_{\infty,\infty} \leq 1/2$. Thus

$$((\tilde{L}+\lambda(s-b))_{|W^q})^{-1} =$$

$$= \sum_{j=0}^{\infty}(-1)^j\lambda^j\Big(\big((\tilde{L}+\lambda(s-m))_{|W^q}\big)^{-1}(m-b)\Big)^j\big((\tilde{L}+\lambda(s-m))_{|W^q}\big)^{-1}.$$

then (ii) follows from (i) and the identity

$$((\tilde{L}+\lambda s)_{|W^q})^{-1} =$$

$$= \sum(-1)^j\lambda^j\Big(\big((\tilde{L}+\lambda(s-b))_{|W^q}\big)^{-1}b\Big)^j((\tilde{L}+\lambda(s-b))_{|W^q})^{-1}. \quad \blacksquare$$

COROLLARY 2.13. *Let $s_0(m, \lambda_1, \lambda_2)$ be as in Lemma 2.6, suppose $s > s_0(m, \lambda_1, \lambda_2)$, and consider the operator $S = ((\tilde{L}+\lambda(s-m))_{|W^q})^{-1}$ on $L^q(\Omega \times (0, T))$, then its spectral radius ϱ is an algebraically simple positive eigenvalue which has an a.e. positive eigenfunction, and no other eigenvalue has a positive eigenfunction. Moreover it is also an eigenvalue with an a.e. positive eigenfunction for the adjoint operator.*

PROOF. S is compact, irreducible and positive and so is its adjoint $S^*: L^{q'}(\Omega \times (0, T)) \to L^{q'}(\Omega \times (0, T))$. Then the spectral radius of S and S^* are positive (See [Z], p. 410) and the theorem follows from theorem 8 and lemma 16 in [S]. ■

DEFINITION 2.14. *Given $m \in L_T^r(\Omega \times \mathfrak{R})$ and $\lambda > 0$. Pick λ_1, λ_2 such that $0 < \lambda_1 < \lambda < \lambda_2$. Let $s_0(m, \lambda_1, \lambda_2)$ be as in Lemma 2.6 and pick $s > s_0(m, \lambda_1, \lambda_2)$. We define $\mu_m(\lambda)$ by $1/(\lambda s + \mu(\lambda)) = \varrho$, where ϱ is the spectral radius of*

$$((\tilde{L}+\lambda(s-m))_{|W^q})^{-1} \in B(L_T^q(\Omega \times \mathfrak{R})) .$$

It is easy to check that $\mu_m(\lambda)$ does not depend on the particular λ_1, λ_2 and s chosen. Additionally we define $\mu_m(0) = 0$ for the Neumann boundary condition and $\mu_m(0)$ equal to the principal eigenvalue of L^{-1} for the Dirichlet boundary condition.

REMARK 2.15. Suppose $B(u) = u_{|\partial\Omega \times \mathfrak{R}}$. Let $\lambda_0 > 0$ be the principal eigenvalue of L and let $u_0 > 0$ be such that $Lu_0 = \lambda_0 u_0$. Consider $\tilde{L}^{-1}$ as operator on $L_T^q(\Omega \times \mathfrak{R})$. Then (see [S], Theorem 8), λ_0^{-1} is a simple eigenvalue of $\tilde{L}^{-1*}: L_T^{q'}(\Omega \times \mathfrak{R}) \to L_T^{q'}(\Omega \times \mathfrak{R})$ with a unique positive eigenfunction Ψ_D satisfying $\int_{\Omega \times [0,T]} \Psi_D = 1$. Similarly, for the Neumann condition, let Ψ_N be the positive eigenfunction of $(\tilde{L}+1)^{-1*}: L_T^{q'}(\Omega \times \mathfrak{R}) \to$ $\to L_T^{q'}(\Omega \times \mathfrak{R})$ with eigenvalue 1 satisfying $\int_{\Omega \times [0,T]} \Psi_N = 1$.

Lemma 2.16. *Suppose $0 < \lambda_1 < \lambda_2$ and let $\{m_j\}_{j=1}^{\infty}$ be a sequence in $C_T^{\infty}(\overline{\Omega} \times \mathfrak{R})$ such that m_j converges to m in $L_T^r(\Omega \times \mathfrak{R})$. Then $\{\mu_{m_j}\}_{j=1}^{\infty}$ converges to μ_m uniformly on $[\lambda_1, \lambda_2]$.*

Proof. We choose $s > s_0(\|m\|_r, \lambda_1, 2\lambda_2)$. For each $j \in N$ and $\lambda \in$ $\in [\lambda_1, 2\lambda_2]$ there exists $u_{j,\lambda} \in C^{2,1}(\overline{\Omega} \times \mathfrak{R})$, $u_{j,\lambda}$ real analytic in λ, $u_{j,\lambda} > 0$ in $\Omega \times \mathfrak{R}$ such that

$$\begin{cases} (L + \lambda(s - m_j))\, u_{j,\lambda} = (\lambda s + \mu_{m_j}(\lambda))\, u_{j,\lambda}, \\ Bu_j(\lambda) = 0\,, \qquad \|u_{j,\lambda}\|_{\infty} = 1 \end{cases}$$

(see [H,1], Lemma 15.1). Since $((L + \lambda(s - m_j))_{|W^q})^{-1}$ is positive, we have $\mu_{m_j}(\lambda) \geqslant -\lambda s$. Suppose the Dirichlet condition an let Ψ_D be as in remark 2.15. We derive with respect to λ at $\lambda = 0$ the identity

$$\langle u_{j,\lambda}, \Psi_D \rangle = \langle L^{-1}(\lambda m_j + \mu_{m_j}(\lambda))\, u_{j,\lambda}, \Psi_D \rangle$$

to obtain $d\mu_{m_j}/d\lambda\,|_{\lambda=0} = -\langle m_j u_0, \Psi_D \rangle / \langle u_0, \Psi_D \rangle$ where u_0 is a positive eigenfunction of L with eigenvalue λ_0. Analogously we get $d\mu_{m_j}/d\lambda\,|_{\lambda=0} = -\langle m_j, \Psi_N \rangle / \langle 1, \Psi_N \rangle$ for the Neumann case. In either case the sequence $\{d\mu_{m_j}/d\lambda_{\,|\lambda=0}\}_{j=1}^{\infty}$ is bounded from above. Also, since μ_{m_j} is concave and satisfies $\mu_{m_j}(\lambda) \geqslant -\lambda s$ on $[\lambda_1, 2\lambda_2]$, it is easy to see that $\{d\mu_{m_j}/d\lambda\,|_{\lambda=\lambda_2}\}_{j=1}^{\infty}$ is bounded from below. Then $\{(d\mu_{m_j}/d\lambda)(\lambda)\}_{j \in N,\, \lambda \in [\lambda_1, \lambda_2]}$ is bounded, so, by the Ascoli-Arzelà theorem there exists subsequence $\{\mu_{m_{j_k}}\}_{k \in N}$ that converges uniformly on $[\lambda_1, \lambda_2]$. Let $\sigma(\lambda) = \lim_{k \to \infty} \mu_{m_{j_m}}(\lambda)$, then $\mu_m(\lambda) = \sigma(\lambda)$ for $\lambda \in [\lambda_1, \lambda_2]$. Indeed, let $v_{j,\lambda} = u_{j,\lambda}/\|u_{j,\lambda}\|_{\infty}$. From

$$v_{j,\lambda} = L^{-1}(\mu_{m_j}(\lambda) + \lambda m_j)\, v_{j,\lambda} \tag{2.17}$$

and since $\tilde{L}^{-1}: L_T^p(\Omega \times \mathfrak{R}) \to C_T(\Omega \times \mathfrak{R})$ is compact, there exists an $C_T(\Omega \times \mathfrak{R})$ convergent subsequence $\{v_{j_k,\lambda}\}_{k \in N}$. Let $v_\lambda = \lim_{k \to \infty} v_{j_k,\lambda}$, we note that $v_\lambda \neq 0$. Taking limits in (2.17) we get $v_\lambda = \tilde{L}^{-1}(\sigma(\lambda) + \lambda m)\, v_\lambda$, and so, Corollary 2.13 implies $\mu_m(\lambda) = \sigma(\lambda)$. Finally we observe that the above argument shows that any subsequence of $\{\mu_{m_j}(\lambda)\}$ has a subsequence that converges to $\mu_m(\lambda)$ and so the sequence itself converges to $\mu_m(\lambda)$. ■

LEMMA 2.18. *If* $m \in L_T^r(\Omega \times (0, T))$ *then* μ_m *is continuous on* $[0, \infty)$ *and analytic on* $(0, \infty)$.

PROOF. Let $\{m_j\}_{j=1}^{\infty}$ be a sequence in $C_T^{\infty}(\overline{\Omega} \times \mathfrak{R})$ that converges to m in $L_T^r(\Omega \times \mathfrak{R})$, we pick $s > s_0(m, \lambda_1, \lambda_2)$. By lemma 2.14 μ_m is continuous on $(0, \infty)$. The continuity of μ_m at 0 follows from the facts that μ_m is the pointwise limit of a sequence of concave functions and that $\{d\mu_{m_j}/d\lambda\,|_{\lambda=0}\}_{j=1}^{\infty}$ has an upper bound.

It remains to see the analyticity. Let J be the inclusion from W^q into $L_T^q(\Omega \times \mathfrak{R})$. We consider W^q as a Banach space, with the topology inherited from the graph norm, then $\tilde{L} + \lambda(s - m)$: $W^q \to L_T^q(\Omega \times \mathfrak{R})$ is a Banach space isomorphism and $\tilde{L} + \lambda(s - m) - (\lambda s + \mu_m(\lambda))J$ is a compact perturbation of it, therefore it is a Fredholm operator of index 0, then

$$\dim \operatorname{Ker}(\tilde{L} + \lambda(s - m) - (\lambda s + \mu_m(\lambda))) =$$

$$= \operatorname{co}\dim\left(\operatorname{Im}\left(\tilde{L} + \lambda(s - m) - (\lambda s + \mu_m(\lambda))\right)\right).$$

Corollary 2.13 implies that $\lambda s + \mu_m(\lambda)$ is a J simple eigenvalue of $\tilde{L} + \lambda(s - m)$. It follows from Lemma 1.3 in [C-R] that there exists $\varepsilon > 0$ such that if $U \in B(W^q, L_T^q(\Omega \times \mathfrak{R}))$ and $\|\tilde{L} + \lambda(s - m) - U\| < \varepsilon$ then U has an unique eigenvalue $\varrho(U)$ satisfying $|\varrho(U) - (\lambda s + \mu_m(\lambda))| < \varepsilon$. Moreover, $\varrho(U)$ is a J simple eigenvalue and the application $U \to \varrho(U)$ is analytic. For λ' close enough to λ, $\lambda' s + \mu_m(\lambda')$ is a J simple eigenvalue of the operator $\tilde{L} + \lambda'(s - m)$. Since μ_m is continuous on $(0, \infty)$ we must have $\varrho(\tilde{L} + \lambda'(s - m)) = \mu_m(\lambda')$, so the analyticity follows. ■

REMARK 2.19. For $m \in L_T^r(\Omega \times \mathfrak{R})$ and $t \in \mathfrak{R}$, let $\tilde{m}(t) = \operatorname*{ess\,sup}_{x \in \Omega} m(x, t)$, $m^*(t) = \operatorname*{ess\,inf}_{x \in \Omega} m(x, t)$, and $P(m) = \int_0^T \tilde{m}(t)\,dt$. If m is independent of x, then $\mu_m(\lambda) = \mu_m(0) - \lambda(P(m)/T)$. Indeed, this is true if in addition $m \in C_T^{\infty}(\Omega \times \mathfrak{R})$ (see [H,1], Lemma 15.3) and so, by Lemma 2.16, for an arbitrary $m \in L_T^r(\Omega \times \mathfrak{R})$.

3. – The Main results.

LEMMA 3.1. *Let $m_1 > m_2$ be functions in $L_T^r(\Omega \times \mathfrak{R})$. Then $\mu_{m_1}(\lambda) < \mu_{m_2}(\lambda)$, for all $\lambda > 0$.*

PROOF. If $\lambda > 0$ we pick $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 < \lambda < \lambda_2/2$. We also pick $s > \max\{s_0(m_1, \lambda_1, \lambda_2), s_0(m_2, \lambda_1, \lambda_2)\}$. We set T_1, T_2: $L_T^q(\Omega \times \mathfrak{R}) \to L_T^r(\Omega \times \mathfrak{R})$ defined by $T_i = ((\tilde{L} + \lambda(s - m_i))_{|W^q})^{-1}$. Then T_1 and T_2 are positive operators and, by (iii) of Lemma 2.6

$$T_1 - T_2 = \lambda((\tilde{L} + \lambda(s - m_1))_{|W^q})^{-1}(m_1 - m_2)((\tilde{L} + \lambda(s - m_2))_{|W^q})^{-1} > 0 .$$

Thus $\varrho(T_1) \geq \varrho(T_2)$. Suppose for contradiction that $\varrho(T_1) = \varrho(T_2)$ and let ϱ denote this value. Let $u \in W^q$ $u > 0$ such that $T_2 u = \varrho u$, then $u(x, t) > 0$ $a.\ e.(x, t) \in \Omega \times \mathfrak{R}$ (see [S], Lemma 16) and then

$$\frac{1}{\varrho} u = (\tilde{L} + \lambda(s - m_2))\, u =$$

$$= (\tilde{L} + \lambda(s - m_1))\, u + \lambda(m_1 - m_2)\, u > (\tilde{L} + \lambda(s - m_1))\, u .$$

Thus $T_1 u > \varrho u$. Contradiction. ∎

Let $\Gamma \in C^2(\mathfrak{R}, \Omega)$ be a T-periodic curve in Ω and Ω_0 a domain in $\mathfrak{R}^n$ with C^∞ boundary such that $\Gamma(t) + \Omega_0 \subset \Omega$ for every $t \in \mathfrak{R}$. We define

$$B_{\Gamma, \Omega_0} = \{(\Gamma(t) + w, t) \colon w \in \Omega_0, t \in [0, T]\}$$

and

$$P_{\Gamma, \Omega_0}(m) = \int_{B_{\Gamma, \Omega_0}} m .$$

THEOREM 3.2. *Let $\Gamma \in C^2(\mathfrak{R}, \Omega)$ be a T-periodic curve in Ω and Ω_0 a domain in $\mathfrak{R}^n$ with C^∞ boundary such that $\Gamma(t) + \Omega_0 \subset \Omega$ for every $t \in \mathfrak{R}$. Suppose $m \in L_T^r(\Omega \times \mathfrak{R})$. Assume in addition that $a_{i,j}$ has continuous spacial derivates $\partial a_{i,j}/\partial x_i$, $1 \leq i, j \leq n$. Then we have*

(i) *If $P_{\Gamma, \Omega_0}(m) > 0$, then there exists $\lambda^D > 0$ and $u^D > 0$, solution of the periodic eigenvalue Dirichlet problem $\tilde{L}u = \lambda m u$ in $\Omega \times \mathfrak{R}$, $u_{|\partial\Omega \times \mathfrak{R}} = 0$.*

(ii) *Suppose the Neumann boundary condition and $\tilde{m} \neq m^*$. Let*

Ψ_N be defined as in remark 2.15. If $P_{\Gamma,\Omega_0}(m) > 0$ and $\langle \Psi^N, m\rangle < 0$ then there exists $\lambda^N > 0$ and $u^N > 0$ that solve the problem $\tilde{L}u = \lambda m u$ in $\Omega \times \mathfrak{R}$, $\partial u/\partial \nu_{|\partial\Omega\times\mathfrak{R}} = 0$.

PROOF. We pick $c \in \mathfrak{R}$ such that $P_{\Gamma,\Omega_0}(m) > c > 0$. Let $\{m_j\}_{j\in N}$ be a sequence of functions in $C_T^\infty(\mathfrak{R}^n \times \mathfrak{R})$ such that supp $m_j \subset \Omega \times \mathfrak{R}$, $\lim_{j\to\infty} m_j = m$ in $L_T^r(\Omega \times \mathfrak{R})$. Without lost of generality we can assume that $P_{\Gamma,\Omega_0}(m_j) > c$ for all $j \in N$. In the Neumann case we can also assume that $\langle \Psi, m_j\rangle < 0$ and $\tilde{m}_j \not\equiv m_j^*$. Let λ_j^D, u_j^D (λ_j^N, u_j^N)be the principal eigenvalue and the corresponding positive eigenvector for the Dirichlet (Neumann) boundary condition (see [H,1], Theorems 16.1 and 16.3) corresponding to the weight m_j such that $\|u_j^D\|_\infty = 1$, ($\|u_j^N\|_\infty = 1$).

We first consider the Dirichlet case. We introduce the change of coordinates given by Φ: $\Omega \times \mathfrak{R} \to \mathfrak{R}^n \times \mathfrak{R}$ where $\Phi(w, t) = (w - \Gamma(t), t)$. In the new coordinates the equation $Lu_j = \lambda_j m_j u_j$ on $\Omega \times \mathfrak{R}$ becomes $L^\Phi u_j^\Phi = \lambda_j m_j^\Phi u_j^\Phi$ on $\Phi(\Omega \times \mathfrak{R})$, where $m_j^\Phi = m_j \circ \Phi^{-1}$ and $u_j^\Phi = u_j \circ \Phi^{-1}$.

Take $\sigma_j > 0$ and $v_j > 0$ satisfying $L^\Phi v_j = \sigma_j m_j v_j$ on $\Omega_0 \times \mathfrak{R}$, v_j T periodic and $v_{j|\partial\Omega_0\times\mathfrak{R}} = 0$ and $\|v_j\|_\infty = 1$. Since $\partial a_{i,j}/\partial x_i \in C(\overline{\Omega} \times \mathfrak{R})$ and $\int_{\Omega_0\times\mathfrak{R}} m_j^\Phi > c$ we can apply proposition 3.1 in [H,2] to obtain that the sequence $\{\sigma_j\}$ is bounded. Reasoning as in [H,1] Lemma 15.4, we see that $\lambda_j < \sigma_j$ and so $\lambda_j \leqslant c$ for some $c > 0$ and all $j \in N$. Then we can find a subsequence (which we still denote $\{\lambda_j\}$) that converges to some $\lambda^D \geqslant 0$. Use that $\{\lambda_j^D m_j u_j^D\}_{j\in N}$ is bounded in $L_T^r(\Omega \times \mathfrak{R})$, $u_j^D = L^{-1}(\lambda_j m_j u_j^D)$ and that $\tilde{L}^{-1}$ is a compact operator on $L_T^r(\Omega \times \mathfrak{R}) \to C_T(\Omega \times \mathfrak{R})$ to conclude that there exists a subsequence $u_{j_k}^D$ that converges to some u^D in $L_T^\infty(\Omega \times \mathfrak{R})$. Then $u^D = \tilde{L}^{-1}(\lambda^D m u^D)$. Moreover, $\lambda^D > 0$, otherwise $(\tilde{L}+1)^{-1}$ would have 2 positive eigenvalues with positive eigenfunctions.

Let's consider the Neumann case. It follows from $\lambda_j^N < \lambda_j^D$that there exists a convergent subsequence $\lambda_{j_k}^N$. Let $\lambda^N = \lim \lambda_{j_k}^N$. Moreover, we can assume that $u_{j_k}^N$ converges (in $L_T^\infty(\Omega \times \mathfrak{R})$) to some u^N. Then we get, as above, $u^N \in W$ and $\tilde{L}u^N = \lambda^N m u^N$. To see that $\lambda^N > 0$ assume for contradiction that $\tilde{L}u^N = 0$. Then $u_N \equiv 1$, and we have

$$\langle u_j^N, \Psi\rangle = \langle (\tilde{L}+1)^{-1}(\lambda_j^N m_j + 1)\, u_j^N, \Psi\rangle = \langle \lambda_j^N m_j u_j^N, \Psi\rangle + \langle u_j^N, \Psi\rangle.$$

Thus $\langle m_j u_j^N, \Psi^N\rangle = 0$ and then $\langle m, \Psi^N\rangle = 0$. Contradiction. ∎

THEOREM 3.3. *Let Γ and Ω_0 be as above and $m \in L_T^r(\Omega \times \Re)$. We have*

(i) *Assume the Dirichlet Boundary condition and $P_{\Gamma, \Omega_0}(m) > 0$. Then there exists at most one $\lambda^D > 0$ such that $\mu_m(\lambda^D) = 0$.*

(ii) *Suppose the Neumann boundary condition, $P_{\Gamma, \Omega_0}(m) > 0$, $\langle \Psi^N, m \rangle < 0$ and $\widetilde{m} \neq m^*$ then there exists at most one $\lambda^N > 0$ such that $\mu_m(\lambda^N) = 0$.*

PROOF. (i) follows from the facts that μ_m is concave on $[0, \infty)$ and $\mu_m(0) > 0$. To see (ii) suppose that there exist $\lambda_1 > 0$, $\lambda_2 > 0$ such that $\mu_m(\lambda_1) = \mu_m(\lambda_2) = 0$. Since μ_m is concave and analytic we must have $\mu \equiv 0$ on $[0, \infty)$. Since $m^* \neq \widetilde{m}$, $m < \widetilde{m}$ then given $\varepsilon > 0$ there exists $h \in L_T^r(\Omega \times \Re)$, $h > 0$ such that $m + h < \widetilde{m}$ and $\|h\|_r < \varepsilon$. Moreover we can choose h such that $m + h$ it is not function of t alone. For ε small enough we must have $P(m + h) > 0$ and $\langle \Psi, m + h \rangle < 0$ so by theorem 3.2 there exists $\lambda > 0$ such that $\mu_{m+h}(\lambda) = 0$, but by Lemma 3.1 $\mu_{m+h}(\lambda) < \mu_m(\lambda) = 0$. Contradiction. ■

REMARK 3.4. Let m, $a_{i,j}$ $1 \leqslant i, j \leqslant n$ be as in Theorem 3.2 and let $\{m_j\}_{j=1}^{\infty}$ be a sequence in $C_T^{\infty}(\Omega \times \Re)$, $\sup p\, m_j \subset K_j \times \Re$ for some compact subset $K_j \subset \Omega_j$. Then the sequence $\{\lambda_j\}$ of principal eigenvalues associated to the weights m_j converges to the principal eigenvalue λ corresponding to the weight m. Indeed, for every subsequence $\{\lambda_{j_k}\}$ we can prove, as in theorem 3.2, that there exists a subsequence $\{\lambda_{j_{k_s}}\}$ convergent to some λ satisfying $\mu_m(\lambda) = 0$. So the assertion follows from lemma 3.8. A diagonal process gives us the following

COROLLARY 3.5. *Let m, $a_{i,j}$ $1 \leqslant i, j \leqslant n$ be as in Theorem 3.2 and let $\{m_j\}_{j=1}^{\infty}$ be a sequence in $L_T^r(\Omega \times \Re)$ such that m_j converges to m in $L_T^r(\Omega \times \Re)$. Then the sequence $\{\lambda_j\}$ of principal eigenvalues associated to the weights m_j converges to the principal eigenvalue λ corresponding to the weight m.*

We set $\pi: \Re^n \times \Re \to \Re$ defined by $\pi(x, t) = t$. If $B \subset \Re^n \times \Re$ and $t \in \Re$ we put $B_t = \{x \in \Re^n : (x, t) \in B\}$. If $\Omega \subset \Re^n$ we define for $\delta > 0$, $\Omega_\delta = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \delta\}$.

LEMMA 3.6. *Suppose that $m \in L_T^r(\Omega \times \Re)$ has an upper bound and that $\int_a^b \widetilde{m}(t)\, dt > c$. Suppose also that $\delta > 0$ is such that $\Omega_\delta \neq \emptyset$. Then* *there exists a finite family* $\{Q_r\}_{1\leqslant r\leqslant N}$ *of pairwise disjoints congruent open cubes with edges of length* l *and parallel to the coordinate axis such that*

(1) $l \leqslant (\delta/2(n+1))$ *and* $Q_r \subseteq \Omega_{\delta/2} \times [a, b]$ *for* $1 \leqslant r \leqslant N$.

(2) *The family* $\{\pi(Q_r)\}_{r=1}^{N}$ *is pairwise disjoint.*

(3) $\sum_{r=1}^{N} |\pi(Q_r)| = b - a$.

(4) $\int_{\bigcup_{1\leqslant r\leqslant N} Q_r} m > cl^n$.

PROOF. Without lost of generality we can assume that $m \leqslant 1$. Let $\widetilde{m}_j$ defined as in remark 2.19. It is easy to see that $\widetilde{m}_j$ is a measurable function on $[a, b]$. Also $\widetilde{m}_j(t) \leqslant \widetilde{m}_{j+1}(t)$ and $\lim_{j\to\infty} \widetilde{m}_j(t) = \widetilde{m}(t)$. Then we can fix k large such that $\int_a^b \widetilde{m}_j(t)\,dt > c$, $|\Omega_k| \geqslant |\Omega|/2$ and $k \geqslant 1/\delta$.

For $0 < \theta < \delta < \eta$ we define

$$E(\eta, \theta) = \{(x, t) \in \Omega_{1/k} \times [a, b]\colon m(x, t) \geqslant \widetilde{m}_k(t) - \eta + \theta\}$$

then $m - (\widetilde{m} - \eta) \geqslant \theta$ on $E(\eta, \theta)$. Let $E^d(\eta, \theta)$ be the set of the points $(x, t) \in E(\eta, \theta)$ such that (x, t) is a Lebesgue point for $m(x, t) - (\widetilde{m}_k(t) - \eta)$ and, for $r > 0$, let $E^{(r)}(\eta, \theta)$ be the set of the points (x, t) in $E^d(\eta, \theta)$ such that

$$\frac{1}{|Q|}\int_Q (m - (\widetilde{m} - \eta)) \geqslant \frac{\theta}{2}$$

holds for every open cube Q with edges parallel to the coordinate axis with diameter less that $1/r$ containing (x, t). $E^{(r)}(\eta, \theta)$ is a measurable set. Note that $E^{(r)}(\eta, \theta) \subseteq E^{(s)}(\eta, \theta)$ if $r < s$. Also $E^d(\eta, \theta) \subseteq \bigcup_{r\in N} E^{(r)}(\eta, \theta)$. Moreover, from $|E^{(r)}(\eta, \theta)_t| \neq 0$ a.e. $t \in [a, b]$ it follows that $|\pi(E^d(\eta, \theta))| = b - a$. Then

$$\lim_{r\to\infty} |\pi(E^{(r)}(\eta, \theta))| \geqslant |\pi(E^d(\eta, \theta))| = b - a\,.$$

Given $\varepsilon > 0$ we fix $r > 2k$ such that $|\pi(E^{(r)}(\eta, \theta))| \geqslant b - a - \varepsilon$, also we choose $0 < l < 1/r(n+1)$ such that $Nl = b - a$ for some natural number N. Let $\{t_i\}_{0\leqslant i\leqslant N}$ be the partition of $[a, b]$ given by $t_i = a + il$, $i = 0, \dots, N$. Let I be the set of the indices i, $0 \leqslant i \leqslant N$ such that the strip $\mathfrak{R}^n \times (t_{i-1}, t_i)$ intersects $E^{(r)}(\eta, \theta)$ and let I^c be its complement. For $i \in I$ we choose a cube Q_i such that $Q_i \cap E^{(r)}(\eta, \theta) \neq \emptyset$ and $\pi(Q_i) = (t_{i-1}, t_i)$. Since $E^{(r)}(\eta, \theta) \subseteq \Omega_{1/k}$ and $\operatorname{diam}(Q_i) < < 1/2k\sqrt{n+1}$ we have that $Q_i \subseteq \Omega_{1/(2k)} \times (t_{i-1}, t_i)$. Since $|\pi E^{(r)}(\eta, \theta)| \geq b - a - \varepsilon$, I^c satisfies $\sum_{i \in I^c} (t_i - t_{i-1}) < \varepsilon$. Let $F = = \bigcup_{i \in I^c} (t_{i-1}, t_i)$, then $|\Omega \times F| \leq \varepsilon |\Omega|$. Now

$$\int_{\Omega \times F} |m| < \|m\|_r |\Omega \times F|^{1/r'} \leq \varepsilon^{1/r'} \|m\|_r |\Omega|^{1/r'}$$

with r' defined by $1/r + 1/r' = 1$. To cover $\Re^n$ we use cubes with vertices on the points of the lattice lZ^n. Let $Q_1^*, \ldots, Q_M^*$ be the cubes in the mesh meeting Ω_k, so $Q_j^* \subseteq \Omega$, $1 \leq j \leq M$. Since $|\Omega| \leq 2|\Omega_k| \leq 2Ml^n$, then $M \geq |\Omega|/(2l^n)$. Since

$$\|m\|_r \varepsilon^{1/r'} |\Omega|^{1/r'} \geq \int_{\Omega \times F} |m| \geq \sum_{s=1}^{M} \int_{Q_s^* \times F} |m|$$

we have, for some s, $1 \leq s \leq M$ that

$$\int_{Q_s^* \times F} |m| \leq \|m\|_r \varepsilon^{1/r'} |\Omega|^{1/r'} M^{-1} \leq 2l^n \|m\|_r \varepsilon^{1/r'} |\Omega|^{-1/r'}.$$

We define, for $i \in I^c$, $Q_i = Q_s^*$. Then, for $i \in I$

$$\int_{Q_i} m \geq \int_{Qi} \widetilde{m} - \eta |Q_i| + \theta |Q_i|/2 = l^n \left(\int_{t_{i-1}}^{t_i} \widetilde{m}_k(t)\, dt - \eta l + \theta l/2 \right).$$

Then

$$\sum_{i \in I} \int_{Q_i} m \geq l^n \left(\sum_{i \in I} \int_{t_{i-1}}^{t_i} \widetilde{m}_k(t)\, dt - \eta l \operatorname{Card}(I) - \frac{\theta l \operatorname{Card}(I)}{2} \right) \geq$$

$$\geq l^n \sum_{i \in I} \int_{t_{i-1}}^{t_i} \widetilde{m}_k(t)\, dt - l^n \left(\eta(b-a) + \theta \frac{b-a}{2} \right).$$

Since $\sum_{i \in I^c} \left| \int_{Q_i} m \right| \leq 2l^n \|m\|_r \varepsilon^{1.r'} |\Omega|^{-1/r}$ we get

$$\sum_{i=1}^{N} \int_{Q_i} m \geq l^n \left(\sum_{i \in I} \int_{t_{i-1}}^{t_i} \widetilde{m}_k(t)\, dt - \eta(b-a) - \theta \frac{b-a}{2} - 2\|m\|_r \varepsilon^{1/r'} |\Omega|^{-1/r'} \right)$$

and so $\sum_{i=1}^{N} \int_{Q_i} m \geqslant cl^n$ for η, θ, and ε small enough. ■

Now, reasoning as in remarks 4.2, 4.3 and lemma 4.4 of [G-L-P], we obtain

REMARK 3.7. Suppose that $m \in L_T^r(\Omega \times \mathfrak{R})$ is bounded from above and that $\int_0^T \widetilde{m}(t)\, dt > 0$. Then there exists a T-periodic curve $\gamma \in C^2(\mathfrak{R}, \Omega)$ and a domain Ω_0 with smooth boundary such that

(i) $\gamma(t) + \Omega_0 \subset \Omega$ for every $t \in \mathfrak{R}$.

(ii) $P_{\Gamma, \Omega_0}(m) > 0$.

THEOREM 3.8. *Let m, $a_{i,j}$ $1 \leqslant i, j \leqslant n$ be as in Theorem 3.2. Suppose that there exists $\lambda > 0$, $u \in D(\tilde{L})$ $u > 0$ solution of the periodic eigenvalue problem $\tilde{L}u = \lambda mu$, $B(u) = 0$, where either $B(u) = u_{|\partial\Omega \times \mathfrak{R}}$ or $B(u) = \partial u/\partial \nu_{|\partial\Omega \times \mathfrak{R}}$. If $B(u) = \partial u/\partial \nu_{|\partial\Omega \times \mathfrak{R}}$ we also assume that $m \neq \widetilde{m}$, if $B(u) = \partial u/\partial \nu_{|\partial\Omega \times \mathfrak{R}}$. Then there exists a T-periodic curve $\gamma \in C^2(\mathfrak{R}, \Omega)$ and a domain Ω_0 with smooth boundary satisfying $\gamma(t) + \Omega_0 \subset \Omega$, $t \in \mathfrak{R}$ and such that $P_{\Gamma, \Omega_0}(m) > 0$. Moreover, if $B(u) = \partial u/\partial \nu_{|\partial\Omega \times \mathfrak{R}}$, we also have $\langle \Psi^N, m \rangle < 0$.*

PROOF. Taking into account lemma 3.1 and remark 2.17 and reasoning as in the regular case (see [H,1], Lemma 15.6) we obtain, in both cases, $P(m) > 0$. Then lemma 3.6 gives us the first assertion of the theorem. To see that $\langle \Psi^N, m \rangle < 0$ we choose $m_j \in C_{c,T}^{\infty}(\Omega \times \mathfrak{R})$, $j \in N$, such that m_j converges to m in $L_T^r(\Omega \times \mathfrak{R})$. We pick $\lambda_1 < \lambda$. Then $\mu_m(\lambda_1) > 0$, therefore, by lemma 2.16, $\mu_{m_j}(\lambda_1) > \mu_m(\lambda_1)/2$ for all large enough j. Therefore

$$-\langle \Psi^N, m_j \rangle = \mu'_{m_j}(0) \geqslant \frac{\mu_{m_j}(\lambda_1)}{\lambda_1} > \frac{\mu_m(\lambda_1)}{2\lambda_1}$$

and then $\langle \Psi^N, m \rangle \leqslant -(\mu_m(\lambda_1)/2\lambda_1) < 0$. ■

REMARK 3.9. Let m, $a_{i,j}$ $1 \leqslant i, j \leqslant n$ be as in Theorem 3.2 and let M be the operator multiplication by m. Suppose either the Dirichlet condition or the Neumann condition. Taking into account corollary 2.13 and lemma 2.18 we have, with the same proof as in the regular case, (see

[H,1], Lemma 16.9) that the positive principal eigenvalue is an M simple eigenvalue of $\tilde{L}$.

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