RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

FRANCESCO DE GIOVANNI
ALESSIO RUSSO

Groups with dense subnormal subgroups

Rendiconti del Seminario Matematico della Università di Padova, tome 101 (1999), p. 19-27
<http://www.numdam.org/item?id=RSMUP_1999__101__19_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 101 (1999)

Groups with Dense Subnormal Subgroups.

FRANCESCO DE GIOVANNI - ALESSIO RUSSO (*)

1. Introduction.

Let χ be a property pertaining to subgroups of a group. We shall say that a group G has *dense χ-subgroups* if for each pair (H, K) of subgroups of G such that $H < K$ and H is not maximal in K, there exists a χ-subgroup X of G such that $H < X < K$. Obviously every group in which all subgroups have the property χ has dense χ-subgroups, and conversely in many cases it can be proved that, if a group G has dense χ-subgroups, then the set of all subgroups of G which do not have the property χ is small in some sense. Several authors have investigated the structure of groups with dense χ-subgroups for many different choices of the property χ (see for instance [3], [4], [7], [13]). In particular, Mann [7] considered groups with dense normal or subnormal subgroups, proving that any infinite group with dense subnormal subgroups is locally nilpotent. The consideration of the symmetric group of degree 3 shows that there exist finite non-nilpotent groups with dense normal subgroups.

The aim of this article is to obtain further information on the structure of infinite groups with dense subnormal subgroups. It will be proved in particular that, if G is an infinite group with dense subnormal subgroups, then all subgroups of G are subnormal. In the proof of this result, we will use the relevant theorem of Möhres [10] stating that every group, in which all subgroups are subnormal, is soluble. In the last part of the article groups for which the set of subnormal subgroups with defect bounded by a fixed positive integer is dense are studied.

Most of our notation is standard and can be found in [5] and [12].

(*) Indirizzo degli AA: Dipartimento di Matematica e Applicazioni, Università di Napoli «Federico II», Complesso Universitario Monte S. Angelo, Via Cintia, I 80126 Napoli (Italy).

We are grateful to the referee for the useful comments, improving in particular the statement of Theorem 2.4.

2. Density of subnormal subgroups.

Recall that a group G is called a *Baer group* if it is generated by its abelian subnormal subgroups, or equivalently if every finitely generated subgroup of G is subnormal.

LEMMA 2.1. *Let G be an infinite group with dense subnormal subgroups. Then G is a Baer group.*

PROOF. The group G is locally nilpotent by a result of Mann (see [7], Theorem 2), and clearly it can be assumed that G is not nilpotent. Let H be any finitely generated subgroup of G. Then G contains another finitely generated subgroup K such that $H < K$ and H is not maximal in K. Thus there exists a subnormal subgroup X of G such that $H < X < K$. Since X is nilpotent, it follows that H is subnormal in G, so that G is a Baer group. ∎

The proof of the next result is a slight modification of one given in [8]. We give it here in details for the convenience of the reader.

LEMMA 2.2. *Let G be a Baer group whose hyperabelian subgroups are soluble. Then G is soluble.*

PROOF. Let A be an abelian non-trivial subnormal subgroup of G with smallest defect c, and assume that $c > 1$. Consider a finite series

$$A = A_0 \lhd A_1 \lhd \ldots \lhd A_c = G .$$

By hypothesis the Fitting subgroup F of A_1 is soluble, with derived length n, say. Then $F^{(n-1)}$ is normal in A_2, and hence it is an abelian non-trivial subnormal subgroup of G with defect at most $c-1$. This contradiction shows that G contains an abelian non-trivial normal subgroup. On the other hand, if N is any normal subgroup of G having an ascending G-invariant series with abelian factors, the hypotheses are inherited by the factor group G/N. It follows that the group G is hyperabelian, and hence soluble. ∎

The following lemma shows that in any group with dense subnormal subgroups the «join property» of subnormal subgroups holds.

LEMMA 2.3. *Let G be a group with dense subnormal subgroups, and let H and K be subnormal subgroups of G. Then also the subgroup $\langle H, K\rangle$ is subnormal in G.*

PROOF. It can obviously be assumed that both H and K are properly contained in $J = \langle H, K\rangle$, so that also H^J and K^J are proper subgroups of J. Then $H^J \cap K^J$ is not maximal in J, and so there exists a subnormal subgroup X of G such that

$$H^J \cap K^J < X < J .$$

It follows that $H^J \cap K^J$ is subnormal in G. Since $[H, K] \leqslant H^J \cap K^J$, we obtain that also $[H, K]$ is subnormal in G. Therefore J is a subnormal subgroup of G (see [5], Theorem 1.2.3). ■

We are now in a position to prove the main result of the article. Recall that a group is called *hypoabelian* if it has a descending series with abelian factors, i.e. if it does not contain perfect non-trivial subgroups.

THEOREM 2.4. *Let G be an infinite group with dense subnormal subgroups. Then every subgroup of G is subnormal.*

PROOF. Assume first that the result is false for infinite soluble groups, and among the counterexamples with minimal derived length choose a group G containing a non-subnormal subgroup K with smallest derived length. Then the commutator subgroup K' of K is subnormal in G and we can also choose the counterexample in such a way that the defect c of K' is minimal. Since G is a Baer group by Lemma 2.1, the subgroup K must be infinite. Moreover, every subgroup of finite index of G is subnormal. Assume that K is not abelian, and consider the normal closure $N = (K')^G$ of K'. Then $H = KN$ is subnormal in G, and hence K is not subnormal in H. Clearly K' is not normal in G and $(K')^H = (K')^N$, so that K' has defect $c-1$ in H, and K is subnormal in H by the minimal choice of G. This contradiction shows that K is abelian. If K contains a subgroup L such that K/L is a finite non-simple group, then there exists a subnormal subgroup X of G such that $L < X < K$, and $K = \langle X, E\rangle$ where E is finitely generated. Since G is a Baer group, it follows from

Lemma 2.3 that K is subnormal in G, a contradiction. Therefore the finite residual J of K has finite index in K, so that J is not subnormal in G, and replacing K by J it can also be assumed that K is divisible. Assume that K contains a subgroup M such that

$$K/M = (K_1/M) \times (K_2/M),$$

where both K_1/M and K_2/M are non-trivial. Then there exist subnormal subgroups X_1 and X_2 of G such that $K_1 < X_1 < K$ and $K_2 < X_2 < K$, and hence $K = \langle X_1, X_2 \rangle$ is a subnormal subgroup of G. This contradiction proves that K is a group of type p^∞ for some prime p. As G is locally nilpotent, it contains a unique Sylow p-subgroup P. Obviously K is not subnormal in P, and hence without loss of generality it can be assumed that G is a p-group. Let A be the smallest non-trivial term of the derived series of G. Then KA is a subnormal subgroup of G, so that K is not subnormal in KA, and replacing G by KA we may also suppose that $G = KA$, where A is an abelian normal subgroup of G. Let X be any subnormal subgroup of G containing K. Then $U = X \cap A$ is a normal subgroup of $G = XA$, and

$$KU = K(X \cap A) = KA \cap X = X,$$

so that $X/U \simeq K/K \cap A$ is a group of type p^∞. It follows that the subnormal subgroup X/U of G/U has defect at most 2 (see [12] Part 1, p. 136). Therefore X has defect at most 2 in G, and so also the intersection K_0 of all subnormal subgroups of G containing K is subnormal. On the other hand, either $K = K_0$ or K is a maximal subgroup of K_0, so that in all cases K is normal in K_0, and hence it is a subnormal subgroup of G. This contradiction proves that the theorem holds when G is a soluble group.

Suppose now that G is hyperabelian, and let

$$1 = G_0 < G_1 < \ldots < G_\tau = G$$

be an ascending normal series with abelian factors of G. Assume that G contains a non-subnormal subgroup K. Clearly there exists a smallest non-limit ordinal $\alpha \leqslant \tau$ such that K is a proper non-maximal subgroup of KG_α. Then for each non-limit ordinal $\beta < \alpha$ we have that either $K = KG_\beta$ or K is maximal in KG_β, so that G_β' is contained in K. It follows that the commutator subgroup $G_{\alpha-1}'$ of $G_{\alpha-1}$ is contained in K, and replacing G by $G/G_{\alpha-1}'$ it can be assumed that the normal subgroup $V = G_\alpha$ of G is metabelian. By hypothesis there exists a subnormal subgroup W of G with $K < W < KV$. Let X be any subnormal subgroup of G such that

$K \leqslant X \leqslant W$. Then $W = X(V \cap W)$, and by induction on the defect of X in W it can be easily proved that X contains some term of the derived series of W with finite ordinal type. Therefore $W^{(\omega)}$ is contained in the intersection K_0 of all subnormal subgroups of W containing K. On the other hand, either $K = K_0$ or K is a maximal subgroup of K_0, so that K is normal in K_0, and $W^{(\omega+1)}$ lies in K. As K is not subnormal in W, the group $W/W^{(\omega+1)}$ must be infinite, and replacing G by $W/W^{(\omega+1)}$ we may suppose without loss of generality that G is also hypoabelian. If K is not abelian, the subgroup K'' is not maximal in K, so that there exists a subnormal subgroup X of G such that $K'' < X < K$, and hence K'' is subnormal in G. Therefore, replacing eventually K by K', we may suppose that K' is subnormal G, and the counterexample G and its subgroup K can be chosen in such a way that the defect of K' in G is minimal. Consider the normal closure $N = (K')^G$ of K'. As in the first part of the proof we have that K is subnormal in KN, so that KN is not subnormal in G, and hence it can be assumed that K is abelian. The same argument used in the case of soluble groups allows us to suppose that K is a group of type p^∞ for some prime p. Then K is not ascendant in G, and so there exists an ordinal $\alpha < \tau$ such that KG_α is not subnormal in $KG_{\alpha+1}$. On the other hand, the factor group $KG_{\alpha+1}/G_\alpha$ is soluble, and hence all its subgroups are subnormal by the first part of the proof. This contradiction proves that the theorem also holds when G is hyperabelian.

In the general case, we have that, if H is any hyperabelian subgroup of G, either H is finite or all subgroups of H are subnormal. Therefore H is soluble by a result of Möhres ([10], Satz 7, see also [1]). It follows from Lemma 2.2 that also the group G is soluble, and the first part of the proof applies. ■

We give now a series of consequences of Theorem 2.4, depending on Möhres' results concerning groups in which all subgroups are subnormal.

COROLLARY 2.5. *Let G be a group with dense subnormal subgroups. Then G is soluble.*

PROOF. The statement is obvious if G is finite. Suppose that G is infinite. Then all subgroups of G are subnormal by Theorem 2.4, and it follows from the above quoted result of Möhres that G is soluble. ■

COROLLARY 2.6. *Let G be an infinite group with dense subnormal subgroups. If G has finite exponent, then it is nilpotent.*

PROOF. By Theorem 2.4 all subgroups of G are subnormal, and hence the group G is nilpotent (see [9], Satz 12). ■

COROLLARY 2.7. *Let G be a periodic hypercentral group with dense subnormal subgroups. Then G is nilpotent.*

PROOF. Clearly it can be assumed that G is infinite. Then all subgroups of G are subnormal by Theorem 2.4, and it follows from a result of Möhres that G is nilpotent (see [11], Satz 2.7). ■

COROLLARY 2.8. *Let G be a torsion-free group with dense subnormal subgroups. Then G is hypercentral.*

PROOF. By Lemma 2.4 every subgroup of G is subnormal, and hence G is hypercentral (see [8], Satz 11). ■

3. Density of subnormal subgroups with bounded defect.

In this section we consider groups for which the set of subnormal subgroups with defect bounded by a fixed positive integer is dense. If H is a subgroup of a group G, the *series of normal closures* of H in G is defined by the positions $H^{G,0} = G$ and $H^{G,n+1} = H^{H^{G,n}}$ for each non-negative integer n. It follows immediately from the definition that, if H and K are subgroups of a group G such that $H \leq K$, then $H^{G,n} \leq K^{G,n}$ for all n.

THEOREM 3.1. *Let G be an infinite group for which the set of subnormal subgroups with defect at most n is dense for some positive integer n. Then G is nilpotent and every subgroup of G has defect at most $n+1$.*

PROOF. By Lemma 2.1 we have that G is a Baer group. Let H be any finitely generated subgroup of G such that $H^{G,n} \neq H$, and assume that H is not maximal in $H^{G,n}$. By hypothesis there exists a subnormal subgroup X of G with defect at most n such that $H < X < H^{G,n}$. Then

$$H^{G,n} \leq X^{G,n} = X,$$

and this contradiction shows that H is a maximal subgroup of $H^{G,n}$ so that in particular H is normal in $H^{G,n}$. Therefore every finitely generated subgroup of G is subnormal with defect at most $n+1$, and it is well-known that in this case every subgroup of G has the same property (see [12] Part 2, Lemma 7.41). Finally, the group G is nilpotent by a result of Roseblade (see [12] Part 2, Theorem 7.42). ■

The above theorem has the following obvious consequence, which was already proved by Mann (see [7], Theorem 4).

COROLLARY 3.2. *Let G be an infinite group with dense normal subgroups. Then G is nilpotent with class at most* 3.

PROOF. It follows from Theorem 3.1 that G is a nilpotent group in which every subgroup has defect at most 2. Then G has class at most 3 by a result of Mahdavianary (see [6], Theorem 1). ■

We note now the following property of infinite periodic groups with dense subnormal subgroups of bounded defect.

THEOREM 3.3. *Let G be an infinite periodic group for which the set of subnormal subgroups with defect at most n is dense for some positive integer n. Then either every subgroup of G is subnormal with defect at most n, or G is a central extension of a group of type p^∞ (p prime) by a finite group in which all subgroups are subnormal with defect at most n.*

PROOF. The group G is nilpotent by Theorem 3.1. Let H be any infinite subgroup of G, and consider an arbitrary element x of the subgroup $H^{G,n}$. Clearly there exists a finite subgroup E of H such that x belongs to $E^{G,n}$. Moreover, G contains a subgroup X with defect at most n such that $E < X < H$. In particular,

$$E^{G,n} \leqslant X^{G,n} = X < H\,,$$

so that $x \in H$ and $H^{G,n} = H$. Therefore every infinite subgroup of G has defect at most n. Application of Theorem B of [2] yields that either every subgroup of G has defect at most n, or G is an extension of a subgroup P of type p^∞ (p prime) by a finite group whose subgroups have defect at most n. In the latter case, since G is nilpotent we have also that P is contained in $Z(G)$. ■

In the last part of this section we consider groups for which the set of subnormal subgroups with defect at most 2 is dense.

COROLLARY 3.4. *Let G be an infinite periodic group for which the set of subnormal subgroups with defect at most* 2 *is dense. Then G is nilpotent with class at most* 4.

PROOF. It follows from Theorem 3.3 that every subgroup of $G/Z(G)$ is subnormal with defect at most 2, so that G is nilpotent with class at most 4 by the already mentioned result of Mahdavianary [6]. ■

LEMMA 3.5. *Let G be a torsion-free group for which the set of subnormal subgroups with defect at most* 2 *is dense. Then G is nilpotent with class at most* 2.

PROOF. The group G is nilpotent by Theorem 3.1. Assume that G contains a cyclic subgroup $\langle x\rangle$ with defect $n > 2$, and let H be any subgroup of G with defect 2 such that $\langle x\rangle < H \leqslant \langle x\rangle^{G,2}$. Then

$$\langle x\rangle^{G,2} \leqslant H^{G,2} = H\,,$$

and hence $H = \langle x\rangle^{G,2}$. It follows that $\langle x\rangle$ is a maximal subgroup of $\langle x\rangle^{G,2}$, so that all non-trivial subgroups of $\langle x\rangle^{G,2}$ have finite index, and $\langle x\rangle^{G,2}$ is infinite cyclic (see [12] Part 1, Theorem 4.33). Thus $\langle x\rangle$ is normal in its normal closure $\langle x\rangle^{G}$, and this contradiction shows that every cyclic subgroup of G has defect at most 2. Therefore the group G is nilpotent with class at most 2 (see [6], Theorem A). ■

THEOREM 3.6. *Let G be an infinite group for which the set of subnormal subgroups with defect at most* 2 *is dense. Then G is soluble with derived length at most* 4.

PROOF. The group G is nilpotent by Theorem 3.1. If G is periodic the statement is an obvious consequence of Corollary 3.4. Thus it can be assumed without loss of generality that G contains an element of infinite order x. Let T be the subgroup of G consisting of all elements of finite order, and let y be any element of T. It is well-known that the finitely generated nilpotent group $\langle x, y\rangle$ contains a central element z of infinite order (see [12] Part 1, Theorem 2.24). Since $\langle y\rangle$ is not a maximal subgroup of $\langle y, z\rangle$, there exists a subgroup X of G with defect at most 2 such that $\langle y\rangle < X < \langle y, z\rangle$. Clearly $\langle y\rangle$ is the subgroup of all elements of finite order of X, so that it is characteristic in X and hence has defect at most 2 in G. Therefore every cyclic subgroup of T has defect at most 2, and so T has nilpotency class at most 3 (see [6], Theorem 1). On the other hand, the factor group G/T has nilpotent class at most 2 by Lemma 3.5, and it follows that G has derived length at most 4. ■

REFERENCES

[1] C. CASOLO, *Groups in which all subgroups are subnormal*, Rend. Accad. Naz. Sci. XL, **104** (1986), pp. 247-249.

[2] S. FRANCIOSI - F. DE GIOVANNI, *Groups with infinite subnormal subgroups of bounded defect*, Boll. Un. Mat. Ital., (6) 4D (1985), pp. 49-56.

[3] S. FRANCIOSI - F. DE GIOVANNI, *Groups with dense normal subgroups*, Ricerche Mat., **42** (1993), pp. 283-305.

[4] L. A. KURDACHENKO - N. F. KUZENNYĬ - N. N. SEMKO, *Groups with a dense system of infinite almost normal subgroups*, Ukrain. Math. J., **43** (1991), pp. 904-908.

[5] J. C. LENNOX - S. E. STONEHEWER, *Subnormal Subgroups of Groups*, Clarendon Press, Oxford (1987).

[6] S. K. MAHDAVIANARY, *A special class of three-Engel groups*, Arch. Math. (Basel), **40** (1983), pp. 193-199.

[7] A. MANN, *Groups with dense normal subgroups*, Israel J. Math., **6** (1968), pp. 13-25.

[8] W. MÖHRES, *Torsionfreie Gruppen, deren Untergruppen alle subnormal sind*, Math. Ann., **284** (1989), pp. 245-249.

[9] W. MÖHRES, *Auflösbare Gruppen mit endlichem Exponenten, deren Untergruppen alle subnormal sind II*, Rend. Sem. Mat. Univ. Padova, **81** (1989), pp. 269-287.

[10] W. MÖHRES, *Auflösbarkeit von Gruppen, deren Untergruppen alle subnormal sind*, Arch. Math. (Basel), **54** (1990), pp. 232-235.

[11] W. MÖHRES, *Hyperzentrale Torsiongruppen, deren Untergruppen alle subnormal sind*, Illinois J. Math., **35** (1991), pp. 147-157.

[12] D. J. S. ROBINSON, *Finiteness Conditions and Generalized Soluble Groups*, Springer, Berlin (1972).

[13] G. VINCENZI, *Groups with dense pronormal subgroups*, Ricerche Mat., **40** (1991), pp. 75-79.

Manoscritto pervenuto in redazione il 22 luglio 1996.