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Homogenization of a linear transport equation with time depending coefficient

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Homogenization of a Linear Transport Equation with Time Depending Coefficient.

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ABSTRACT - We are concerned with the effective behaviour of the transport equation $\partial_t u_\varepsilon + a_\varepsilon(t, y)\, \partial_x u_\varepsilon = 0$ when $a_\varepsilon \rightharpoonup a$ in L^∞ weak* and the Cauchy problem related to the equation with memory satisfied by a weak* limit of the sequence of solutions. The memory term is represented by an averaging operator. The homogenized equation has a unique solution, established considering a kinetic formulation.

1. Introduction.

Let $T > 0$ be fixed, $\Omega \subset \boldsymbol{R}^N$ be an open set and let (a_ε) be a sequence in $L^\infty((0, T) \times \Omega)$ that satisfies

$$
\begin{aligned}
&0 \leq \alpha_- \leq a_\varepsilon(t, y) \leq \alpha_+ \ \text{ a.e. in } (0,T) \times \Omega, \\
&a_\varepsilon \rightharpoonup a \ \text{ in } L^\infty((0, T) \times \Omega) \text{ weak*}.
\end{aligned}
\tag{1.1}
$$

We are interested in the asymptotic behaviour of the solution u_ε of a transport equation having $a_\varepsilon(t, y)$ as a coefficient oscillating in a transverse direction (shear flow):

$$
\partial_t \ u_\varepsilon + a_\varepsilon(t, y)\, \partial_x u_\varepsilon = 0 \qquad \text{with } u_\varepsilon(0, x, y) = u_0(x, y), \tag{1.2}
$$

where $u_0(x, y) \in L^\infty(\boldsymbol{R} \times \Omega)$.

The problem (1.2) is a model for studying the global behaviour of concentration of fluids in porous media.

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Transport equations with oscillating coefficients arise also for the Liouville equation associated with an oscillating hamiltonian or when studying the oscillations in Euler equation (see Amirat-Hamdache-Ziani [3] for examples and an application to a multidimensional miscible flow in porous media).

The homogenization of (1.2) brings out a diffusion operator in the x variable with memory effect in the time variable t, besides the natural transport operator.

In the case of time independent coefficient $a_\varepsilon(y)$, treated in the papers [2], [4]-[6], [8], [10], [14], [16], the memory term is of kind convolution in time and in particular, in the nonlocal homogenization framework developed by L. Tartar [16] and Y. Amirat, K. Hamdache, A. Ziani [4]-[5], it is described by a parametrized measure by use of the integral representation of Nevanlinna-Pick's holomorphic functions.

The representation allows to define a kinetic system equivalent to the homogenized equation and prove existence and uniqueness of the limit solution by the semigroup theory (see [7]).

The same method has been applied to the model (1.2) in [3] and yields a memory term which depends from the effective solution by an equation of division of distributions and thus not explicitly.

By Fourier transform in x the problem (1.2) writes:

$$(1.3)\quad \begin{cases} \partial_t \widehat{u}_\varepsilon(t, \xi, y) + 2\pi i \xi\, a_\varepsilon(t, y)\, \widehat{u}_\varepsilon(t, \xi, y) = 0 & \text{in } (0, T) \times \boldsymbol{R} \times \Omega\,, \\ \widehat{u}_\varepsilon(0, \xi, y) = \widehat{u}_0(\xi, y) & \text{in } \boldsymbol{R} \times \Omega \end{cases}$$

and considering the frequence ξ as a parameter it can be treated with the method developed by Tartar in [17] for an ordinary equation. Thus, up to a subsequence, the weak limit of $\widehat{u}_\varepsilon$ satisfies:

$$(1.4)\quad \partial_t \widehat{u}(t, \xi, y) + 2\pi i \xi a(t, y)\, \widehat{u}(t, \xi, y) - \\ - (2\pi i \xi)^2 \int_0^t e^{-2\pi i \xi \int_s^t a(\sigma, y)\, d\sigma} K(t, s, y; \xi)\, \widehat{u}(t, \xi, y)\, ds = 0$$

where, for any $\xi \in \boldsymbol{R}$, the kernel $K(t, s, y; \xi)$ is the solution of a resolvent Volterra equation.

Under the assumption of equicontinuity of the coefficients $a_\varepsilon(t, y)$, by an estimate derived from the resolvent equation, R. Alexandre in [1] has characterized the kernel $K(t, s, y; \xi)$ as the symbol of a pseudo-differential operator in x and thus has showed a class of homogenized equations.

The main purpose of this paper is to give to $K(t, s, y; \xi)$ a representation by an averaging operator as in the time independent case and to establish existence and uniqueness of solution for the limiting problem.

We will work under the assumption (1.1) on the coefficients and exploit some results of [15] concerning the analysis of the ordinary equation which follows from the ideas contained in [17] and [7].

By an asymptotic analysis in the frequence ξ and application of Phragmén-Lindelof principle joint to the Paley-Wiener theorem, we identify a bounded parametrized measure $k_{t,s,y}(d\mu_1, d\mu_2, d\mu)$ by which the memory kernel is represented as the Fourier transform $K(t, s, y; \xi) = \langle dk_{t,s,y}\mu_1\mu_2\delta_{x=\mu}dx, e^{-2\pi ix\xi}\rangle$.

The moments of $dk_{t,s,y}$ are all determined through the quantities $a_l(t, s, y)$, $l \geqslant 2$ defined, up to subsequences, by:

$$a_\varepsilon(t, y)\, a_\varepsilon(s, y) \int_s^t a_\varepsilon(\tau_1, y) \dots a_\varepsilon(\tau_{l-2}, y)\, d\tau_1 \dots d\tau_{l-2} \overset{*}{\rightharpoonup} a_l(t, s, y)$$

and one has $\displaystyle\int_{\Lambda\times\Lambda} dk_{t,s,y}\mu_1\mu_2 := k_0(t, s, y) = a_2(t, s, y) - a(t, y)a(s, y)$.
We will suppose that $k_0(t, y) := k_0(t, t, y) > 0$ a.e. in $(0, T) \times \Omega$.

The representation ensures that the operator

$$\text{(1.5)} \qquad Ku = \int_0^t \int \langle k_{t,s,y}, \mu_1\mu_2 e^{2\pi i\xi\left(x - \int_s^t a(\tau, y)\, d\tau - \mu\right)} \rangle\, \hat{u}(s, \xi, y)\, d\xi\, ds$$

is bounded on $L^\infty(0, T; L^2(\boldsymbol{R} \times \Omega))$, it is as a pseudo-differential operator in x whose symbol belongs to the class $S^0_{0,0}(\boldsymbol{R})$ for almost any t, s, y and the homogenized equation writes:

$$\text{(1.6)} \qquad \partial_t u + a(t, y)\, \partial_x u - \partial^2_{xx} Ku = 0\,.$$

Moreover, through the resolvent equation we get that the measure $dk_{t,s,y}$ is linked with the Young measure $\omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda)$ associated to $\left(b_\varepsilon(t, y), b_\varepsilon(s, y), \int_s^t b_\varepsilon(\sigma, y)\, d\sigma\right)$, where $b_\varepsilon := a_\varepsilon - a$, by the relation:

$$(1.7) \quad \langle k_{t,s,y}, \mu_1\mu_2 e^{-2\pi i\xi\mu}\rangle = \langle \omega_{t,s,y}, \lambda_1\lambda_2 e^{-2\pi i\xi\lambda}\rangle + \\ + 2\pi i\xi \int_s^t \langle \omega_{t,\tau,y}, \lambda_1 e^{-2\pi i\xi\lambda}\rangle\langle k_{\tau,s,y}, \mu_1\mu_2 e^{-2\pi i\xi\mu}\rangle\, d\tau$$

a.e. for t, s, y, for any $\xi \in \boldsymbol{R}$.

The link above enables us to give to the homogenized equation (1.6) a kinetic formulation as a system and then prove that the whole sequence u_ε converges to the weak limit u.

Actually, introducing the auxiliary function $z = \partial_x Ku$ and the operators C and H related to $d\omega_{t,s,y}\lambda_1\delta_{x=\int_s^t a(\sigma,y)\,d\sigma + \lambda}\,dx$ and $d\omega_{t,s,y}\lambda_1\lambda_2\delta_{x=\int_s^t a(\sigma,y)\,d\sigma+\lambda}\,dx$ as in (1.4), (1.5) is equivalent to the system:

$$(1.8) \quad \begin{cases} \partial_t u(t,x,y) + a(t,y)\,\partial_x u(t,x,y) - \partial_x z(t,x,y) = 0 \\ z(t,x,y) - \partial_x(Cz)(t,x,y) - \partial_x(Hu)(t,x,y) = 0 \end{cases}$$

for which existence and uniqueness of solution hold in view of a generalization of the fixed point theorem.

When the coefficients $a_\varepsilon(t, y)$ are absolutely continuous in t and verify $\partial_t a_\varepsilon(t,y) \overset{*}{\rightharpoonup} \partial_t a(t,y)$ in $L^\infty((0,T)\times\Omega)$, there exist an operator $\partial_t K\colon H^1(\boldsymbol{R}) \to L^2(\boldsymbol{R})$ bounded for a.e. t, s, y represented by a bounded measure $k_{t,s,y}(dm_1, d\mu_1, d\mu_2, d\mu)$ such that (1.6) has a kinetic formulation with the time derivative of z.

As an example we will consider a sequence with small amplitude oscillations, i.e. $a_\varepsilon(t, y; \gamma) = a(t, y) + \gamma b_\varepsilon(t, y) + \gamma^2 c_\varepsilon(t, y) + o(\gamma^2)$ a.e. for $(t, y) \in (0, T)\times\Omega$ and γ small, absolutely continuous in t, with $a_\varepsilon(t, y; \gamma) \overset{*}{\rightharpoonup} a_\gamma(t, y)$, b_ε, $\partial_t b_\varepsilon \overset{*}{\rightharpoonup} b$, $\partial_t b$, $c_\varepsilon \overset{*}{\rightharpoonup} c$ in L^∞ weak*.

Moreover, let $(b_\varepsilon(t,y) - b(t,y))(b_\varepsilon(s,y) - b(s,y)) \overset{*}{\rightharpoonup} k_0(t,s,y)$ in $L^\infty((0,T)\times(0,T)\times\Omega)$ weak* with $k_0(t,t,y) > 0$ a.e. t, y.

In this case the related measure admits an asymptotic expansion

$$\langle k_{t,s,y;\gamma}, \sigma_1\sigma_2\delta_{z=\int_s^t a_\gamma(\tau,y)\,d\tau+\sigma}\,dz\rangle = \gamma^2 k_0(t,s,y)\,\delta_{z=\int_s^t a(\tau,y)\,d\tau}\,dz + o(\gamma^2)$$

and u satisfies up to order γ^2 the equation

$$\partial^2_{tt} u + (a_\gamma + a)\, \partial^2_{tx} u + (a a_\gamma - k_0(t, y))\, \partial^2_{xx} u + \partial_t a_\gamma(t, y)\, \partial_x u -$$

$$- \gamma^2 \int_0^t \partial_t k_0(t, s, y)\, \partial^2_{xx} u \left(s, x - \int_s^t a(\tau, y)\, d\tau, y \right) ds = 0 .$$

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2. The Cauchy problem when $a_\varepsilon \rightharpoonup a$.

Let (a_ε) be a bounded sequence in $L^\infty((0, T) \times \Omega)$ that satisfies (1.1). In the following we shall denote $b_\varepsilon := a_\varepsilon - a$, $\alpha := \alpha_+ - \alpha_-$, $\Lambda = [-\alpha, \alpha]$, $\Lambda_T = [-T\alpha, T\alpha]$.

Clearly $|b_\varepsilon(t, y)| \leq \alpha$ a.e. in $(0, T) \times \Omega$ and $b_\varepsilon \overset{*}{\rightharpoonup} 0$ in L^∞ weak*.

We consider the transport equation:

$$\text{(2.1)} \qquad \partial_t u_\varepsilon + a_\varepsilon(t, y)\, \partial_x u_\varepsilon = 0 \ \text{ in } (0, T) \times \boldsymbol{R} \times \Omega,$$

$$u_\varepsilon\,|_{t=0} = u_0(x, y) \ \text{ in } \boldsymbol{R} \times \Omega$$

in which $u_0(x, y) \in L^\infty(\boldsymbol{R} \times \Omega)$.

The homogenization of (2.1) is studied dealing in Fourier transform in x with the ordinary equation:

$$\text{(2.2)} \qquad \partial_t \widehat{u}_\varepsilon(t, \xi, y) + 2\pi i \xi\, a_\varepsilon(t, y)\, \widehat{u}_\varepsilon(t, \xi, y) = 0 \ \text{ in } (0, T) \times \boldsymbol{R} \times \Omega$$

with $\widehat{u}_\varepsilon(0, \xi, y) = \widehat{u}_0(\xi, y)$ in $\boldsymbol{R} \times \Omega$.

By considering the frequence ξ as a parameter we can follow the analysis done in [15] for the case of a linear equation and subsequently make an asymptotic analysis in ξ.

We first reformulate some statements of [15], referring to the corresponding results for proofs.

We denote by $\omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda)$ the parametrized family of Young measures generated by the vector-valued sequence

$$\left(b_\varepsilon(t, y), b_\varepsilon(s, y), \int_s^t b_\varepsilon(\sigma, y)\, d\sigma \right),$$

with support in $\Lambda \times \Lambda \times \Lambda_T$ for any t, s, y and by $\omega_{t,s,y}(d\lambda_1, d\lambda)$,

$\omega_{t,s,y}(d\lambda_2, d\lambda)$, $\omega_{t,s,y}(d\lambda)$ its projections (see Lemma 2.0.1 in [15]):

$$\omega_{t,s,y}(d\lambda_1, d\lambda) = \mathrm{proj}\,|_{\lambda_2 \in \Lambda}\, \omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda),$$

$$\omega_{t,s,y}(d\lambda_2, d\lambda) = \mathrm{proj}\,|_{\lambda_1 \in \Lambda}\, \omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda),$$

$$\omega_{t,s,y}(d\lambda) = \mathrm{proj}\,|_{\lambda_1 \in \Lambda,\, \lambda_2 \in \Lambda}\, \omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda).$$

The projection $\omega_{t,s,y}(d\lambda)$ coincides with the Young measure associated to the sequence $\left(\int_s^t b_\varepsilon(\sigma, y)\, d\sigma\right)$, has compact support in Λ_T; its Fourier transform $\widehat{\omega}_{t,s,y}(\xi) = \int_{\Lambda_T} \omega_{t,s,y}(d\lambda)\, e^{-2\pi i \xi \lambda}$ verifies $\widehat{\omega}_{s,s,y}(\xi) = 1$ and it is absolutely continuous in t and s, with second mixt derivative in t, s. The functions

$$(2.3) \qquad \begin{cases} F(t, s, y; \xi) = \widehat{\omega}_{t,s,y}(\xi) - 1\,, \\ C(t, s, y; \xi) = \langle \omega_{t,s,y}(d\lambda_1, d\lambda), \lambda_1 e^{-2\pi i \xi \lambda} \rangle, \\ G(t, s, y; \xi) = \langle \omega_{t,s,y}(d\lambda_2, d\lambda), \lambda_2 e^{-2\pi i \xi \lambda} \rangle, \\ H(t, s, y; \xi) = \langle \omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda), \lambda_1 \lambda_2 e^{-2\pi i \xi \lambda} \rangle \end{cases}$$

are related as follows:

$$(2.4) \qquad \begin{cases} \partial_t F(t, s, y; \xi) = -2\pi i \xi\, C(t, s, y; \xi), \\ \partial_s F(t, s, y; \xi) = 2\pi i \xi\, G(t, s, y; \xi), \\ \partial_t \partial_s F(t, s, y; \xi) = -(2\pi i \xi)^2 H(t, s, y; \xi); \end{cases}$$

moreover F, C, G all vanish at $t = s$, whereas

$$H(s, s, y; \xi) = \langle \nu_{s,y}(d\lambda_1), (\lambda_1)^2 \rangle \geq 0\,,$$

where $\nu_{t,y}(d\lambda_1)$ denotes the Young measure associated to $(b_\varepsilon(t, y))$.

For a subsequence, $\widehat{u}_\varepsilon \overset{*}{\rightharpoonup} \widehat{u} = e^{-2\pi i \xi \int_0^t a(\sigma, y) d\sigma}\, \widehat{\omega}_{t,0,y}(\xi)\, \widehat{u}_0(\xi, y)$ in $W^{1,\infty}(0, T; H^1(\boldsymbol{R} \times \Omega))$ weak*, with

$$\partial_t \widehat{u} + 2\pi i \xi\, a(t, y)\, \widehat{u} = -2\pi i\, \xi C(t, 0, y; \xi)\, \widehat{u}_0(\xi, y)$$

and we have:

PROPOSITION 2.1. *Under hypothesis* (1.1), *after extraction of a subsequence, there is a kernel* $K(t, s, y; \xi)$ *defined on* $(0, T) \times (0, T) \times \Omega$ *for any* $\xi \in \boldsymbol{R}$, *such that the subsequence* $\widehat{u}_\varepsilon$ *of solutions of* (2.2) *con-*

verges weak to the solution* $\widehat{u}$ *of*:

$$\partial_t \widehat{u}(t, \xi, y) + 2\pi i \xi a(t, y)\, \widehat{u}(t, \xi, y) - \tag{2.5}$$

$$- (2\pi i \xi)^2 \int_0^t e^{-2\pi i \xi \int_s^t a(\sigma, y) d\sigma} K(t, s, y; \xi)\, \widehat{u}(s, \xi, y)\, ds = 0$$

in $(0, T) \times \boldsymbol{R} \times \Omega$ *with* $\widehat{u}(0, \xi, y) = \widehat{u}_0(\xi, y)$ *in* $\boldsymbol{R} \times \Omega$.

The kernel $K(t, s, y; \xi)$ *is given by*

$$2\pi i \xi K(t, s, y; \xi) = \partial_s D(t, s, y; \xi) \tag{2.6}$$

where $D(t, s, y; \xi)$ *is the solution of the Volterra equation*:

$$C(t, s, y; \xi) = D(t, s, y; \xi) - 2\pi i \xi \int_s^t D(t, \tau, y; \xi)\, C(\tau, s, y; \xi)\, d\tau\,. \tag{2.7}$$

For the proof we refer to Theorem 2.1 in [15].

In the next Lemma we summarize the properties of D and K:

LEMMA 2.1. *The solution* $D(t, s, y; \xi)$ *to* (2.7) *is measurable in t, s, y, analytic in* ξ. *The kernel* $K(t, s, y; \xi)$ *solves for any* $\xi \in \boldsymbol{R}$ *the family of Volterra equations*:

$$K(t, s, y; \xi) - 2\pi i \xi \int_s^t C(t, \tau, y; \xi)\, K(\tau, s, y; \xi)\, d\tau = H(t, s, y; \xi) \tag{2.8}$$

on $(0, T) \times (0, T) \times \Omega$ *and it is given by*

$$K(t, s, y; \xi) = H(t, s, y; \xi) + 2\pi i \xi \int_s^t D(t, \tau, y; \xi)\, H(\tau, s, y; \xi)\, d\tau\,. \tag{2.9}$$

PROOF. We refer to Lemmas 2.1.1 and 2.1.2 in [15] recalling that in view of (2.7)

$$D(t, s, y; \xi) = \sum_{k \geq 0} (2\pi i \xi)^k D_k(t, s, y)$$

with

$$|D_k(t, s, y)| \leq c_k (\alpha_+ - \alpha_-)^{k+1} \frac{(t-s)^k}{k!}, \qquad c_k \leq 2^{k-1} \quad \text{if } k \geq 1$$

and $c_0 = 0$, $c_1 = 1$, whereas K has an expansion obtained by H and C

through (2.9),

$$K(t, s, y; \xi) = \sum_{j \geq 0} (2\pi i \xi)^j k_j(t, s, y)$$

in which the coefficients k_j satisfy the bound

$$|k_j(t, s, y)| \leq c_j(\alpha_+ - \alpha_-)^{j+2} \frac{(t-s)^j}{j!}, \qquad c_j \leq 2^j. \qquad \blacksquare$$

In particular, (2.9) yields:

$$K(s, s, y; \xi) := k_0(s, y) = \langle \nu_{s,y}(d\lambda_1), (\lambda_1)^2 \rangle. \tag{2.10}$$

We recall also the following equivalence between (2.5) and the kinetic system formulated through relation (2.8) by introducing the function

(2.11) $\hat{z}(t,\xi,y) =$

$$= 2\pi i \xi \int_0^t e^{-2\pi i \xi \int_s^t a(\sigma, y) d\sigma} K(t, s, y; \xi)\, \hat{u}(s, \xi, y)\, ds := 2\pi i \xi \widetilde{Ku}$$

and denoting by $\widetilde{Cz}$, $\widetilde{Hu}$ the functions defined through $C(t, s, y; \xi)$, $H(t, s, y; \xi)$ as above:

PROPOSITION 2.2. *For any $\xi \in \boldsymbol{R}$ the Cauchy problem related to equation* (2.5) *is equivalent in* $(0, T) \times \Omega$ *to the following system:*

$$\begin{cases} \partial_t \hat{u} + 2\pi i \xi\, a(t, y)\, \hat{u} - 2\pi i \xi\, \hat{z} = 0 \\ \hat{z} - 2\pi i \xi\, \widetilde{Cz} - 2\pi i \xi\, \widetilde{Hu} = 0 \end{cases} \tag{2.12}$$

with $(\hat{u}, \hat{z})(0, y; \xi) = (\hat{u}_0(y; \xi), 0)$ *in* Ω.

(For the proof see Theorem 2.2 in [15]).

We will check the boundedness of the memory kernel $K(t, s, y; \xi)$ by an asymptotic analysis in the frequence.

To this end we introduce the solution M to

$$G(t, s, y; \xi) = M(t, s, y; \xi) - 2\pi i \xi \int_s^t G(t, \tau, y; \xi)\, M(\tau, s, y; \xi)\, d\tau \tag{2.13}$$

on $(0, T) \times (0, T) \times \Omega$, $\xi \in \boldsymbol{R}$ and state:

PROPOSITION 2.3. *There exists a bounded measure $b_{t,s,y}(d\mu)$ parametrized in t, s, y with support in Λ_T, absolutely continuous in t and s, with second mixt derivative in t, s, such that $\hat{b}_{t,s,y}(\xi)$ verifies*:

$$\partial_t \hat{b}_{t,s,y}(\xi) = -2\pi i\xi D(t, s, y; \xi), \quad \partial_s \hat{b}_{t,s,y}(\xi) = 2\pi i\xi M(t, s, y; \xi) \tag{2.14}$$

a.e. in t, s, y, for any $\xi \in \boldsymbol{R}$. The moments of $db_{t,s,y}$ are all determined by the measure $d\omega_{t,s,y}$ and in particular

$$\langle b_{t,s,y}(d\mu), 1\rangle = \langle \omega_{t,s,y}(d\lambda), \lambda^2\rangle \geq 0 . \tag{2.15}$$

PROOF. For F, G as in (2.3), the solution $B(t, s, y; \xi)$ to:

$$B(t, s, y; \xi) - 2\pi i\xi \int_s^t G(t, \tau, y; \xi)\, B(\tau, s, y; \xi)\, d\tau = F(t, s, y; \xi) \tag{2.16}$$

with $B(s, s, y; \xi) = 0$ is given by:

$$B(t, s, y; \xi) = F(t, s, y; \xi) + 2\pi i\xi \int_s^t M(t, \tau, y; \xi)\, F(\tau, s, y; \xi)\, d\tau \tag{2.17}$$

and it is such that

$$B(t, s, y; \xi) = \sum_{k \geq 0} B_k(t, s, y)(2\pi i\xi)^k$$

with

$$B_0(t, s, y) = B_1(t, s, y) = 0, \quad B_2(t, s, y) = F_2(t, s, y) = \langle d\omega_{t,s,y}, (\lambda)^2\rangle$$

and

$$|B_k(t, s, y)| \leq c_k(\alpha_+ - \alpha_-)^k \frac{(t-s)^k}{k!}, \qquad c_k \leq 2^{k-2} \quad \text{if } k \geq 2 .$$

The bound on the coefficients $B_k(t, s, y)$ yields:

$$|B(t, s, y; z)| \leq e^{2(\alpha_+ - \alpha_-)(t-s)|2\pi i z|}, \qquad z \in \boldsymbol{C} ; \tag{2.18}$$

moreover from equation (2.16) we have:

$$|B(t, s, y; \xi)| \leq \text{const} \quad \text{when } \xi \in \boldsymbol{R} .$$

We can therefore apply the Phragmén-Lindelof result and get

$$|B(t, s, y; z)| \leq e^{2(\alpha_+ - \alpha_-)(t-s)2\pi|\mathrm{Im}\, z|}, \qquad z \in \boldsymbol{C}. \tag{2.19}$$

In view of the Paley-Wiener theorem (see Hörmander [11], Gel'fand-Shilov [9]), inequality (2.18) yields that

$$B(t, s, y; \xi) = \widehat{b}_{t,s,y}(\xi) = \int_\Gamma b_{t,s,y}(\mu)\, e^{-2\pi i \xi\mu}\, d\mu$$

for a distribution $b_{t,s,y}(\mu)$ of order 0 with support contained in Λ_T. From equation (2.17) we see that $B(t, s, y; \xi)$ is absolutely continuous in t, s and (2.14) holds. ∎

It follows that $B(t, s, y; \xi)$ is a symbol in the class $S^0_{0,0}(\boldsymbol{R})$ for a.e. t, s, y (see Hörmander [11]) and the same holds for its derivatives:

Proposition 2.4. *The functions $D(t, s, y; \xi)$, $M(t, s, y; \xi)$, $K(t, s, y; \xi)$ are symbols in $S^0_{0,0}(\boldsymbol{R})$ for almost every t, s, y. Moreover, there exists a measure $k_{t,s,y}(d\mu_1, d\mu_2, d\mu)$ parametrized in t, s, y, with support in $\Lambda \times \Lambda \times \Lambda_T$, such that*

$$K(t, s, y; \xi) = \langle k_{t,s,y}(d\mu_1, d\mu_2, d\mu), \mu_1\mu_2 e^{-2\pi i \xi\mu}\rangle$$

with

$$\langle k_{t,s,y}(d\mu_1, d\mu_2, d\mu), \mu_1\mu_2\rangle = \langle \omega_{t,s,y}(d\lambda_1, d\lambda_2, d\lambda), \lambda_1\lambda_2\rangle.$$

Proof. By still using the Phragmém-Lindelof principle joint to the Paley-Wiener theorem. ∎

Thus the homogenized equation writes:

$$\partial_t u + a(t, y)\, \partial_x u - \partial^2_{xx} K u = 0, \tag{2.20}$$

where the operator K defined by

$$Ku(t, x, y) = \int_0^t \int e^{2\pi i \xi\left(x - \int_s^t a(\sigma, y)\, d\sigma\right)} K(t, s, y; \xi)\, \widehat{u}(s, \xi, y)\, d\xi\, ds \tag{2.21}$$

is bounded on $L^\infty(0, T; L^2(\boldsymbol{R} \times \Omega))$ with

$$\|Ku\|_{L^\infty(0, T; L^2(\boldsymbol{R} \times \Omega))} \leqslant \alpha^2 T \|u\|_{L^\infty(0, T; L^2(\boldsymbol{R} \times \Omega))}. \tag{2.22}$$

The proof of existence and uniqueness of solution for equation (2.20) is based on a kinetic formulation as a system.

Let introduce the spaces $Y_0 = L^2(\boldsymbol{R} \times \Omega)$, $X_0 = H^1(\boldsymbol{R}; L^2(\Omega))$, $X = W^{1,\infty}(0, T; X_0)$, $Y = L^\infty(0, T; Y_0)$ and denote by C, H the operators associated to symbols C, H as in (2.21).

If z is the function in (2.10) we get:

THEOREM 2.1. *The homogenized equation* (2.20) *is equivalent in* $(0, T) \times \boldsymbol{R} \times \Omega$ *to the system*

$$\begin{cases} \partial_t u + a(t, y)\, \partial_x u - \partial_x z = f(t, x, y), \\ z(t, x, y) - \partial_x(Cz) - \partial_x(Hu) = \int_0^t g(s, x, y)\, ds \end{cases} \tag{2.23}$$

with $f = 0 = g$ *and* $(u, z)(0, x, y) = (u_0, 0)$ *in* $\boldsymbol{R} \times \Omega$.

Assuming that

$$k_0(t, y) > 0 \quad a.e. \text{ in } \quad (0, T) \times \Omega\,, \tag{2.24}$$

for every $(u_0, z_0) \in X_0 \times X_0$ *and* $(f, g) \in L^1(0, T; Y_0) \times L^1(0, T; Y_0)$ *the Cauchy problem related to* (2.23) *admits a unique solution* $(u, z) \in X \times Y$.

PROOF. In view of Proposition 2.2, we have the equivalence between (2.5) and (2.12). By integrating in time the first equation and denoting $F = (\hat{f}, \hat{g})$, we can write (2.12) under the general form:

$$\widehat{U}(t, \xi, y) - 2\pi i \xi \int_0^t R(t, s, y; \xi)\, \widehat{U}(s, \xi, y)\, ds = \tag{2.25}$$

$$= \widehat{U}_0(\xi, y) + \int_0^t \widehat{F}(s, \xi, y)\, ds$$

in which $R(t, s, y; \xi)$ is the kernel of an operator bounded in $Y \times Y$:

$$R(t, s, y; \xi) = \begin{pmatrix} -a(s, y) & 1 \\ e^{-2\pi i \xi \int_s^t a(\sigma, y)\, d\sigma} H(t, s, y; \xi) & e^{-2\pi i \xi \int_s^t a(\sigma, y)\, d\sigma} C(t, s, y; \xi) \end{pmatrix}$$

whose related operator is bounded in $Y \times Y$.

The resolvent equation:

$$S(t, s, y; \xi) - 2\pi i \xi \int_s^t S(t, \tau, y; \xi)\, R(\tau, s, y; \xi)\, d\tau = R(t, s, y; \xi) \tag{2.26}$$

has a unique solution in $(L^\infty((0, T) \times (0, T) \times \Omega; S^0_{0,0}(\boldsymbol{R})))^4$.

Actually, for any fixed ξ, (2.26) is an inhomogeneous integral equation in $(L^\infty((0, T)^2 \times \Omega))^4$ depending on ξ as a parameter.

By a generalization of the fixed point theorem, under assumption (2.24) it has a unique solution for any $\xi \in \boldsymbol{R}$, being a suitable power of the operator R a contraction in that Banach space.

One can directly calculate each term in $S(t, s, y; \xi)$ and check its boundedness in ξ, finding that

$$S(t, s, y; \xi) = e^{-2\pi i \xi \int_s^t a(\sigma, y)\, d\sigma} \begin{pmatrix} -\langle \omega_{t, s, y}, (\lambda_2 + a(s, y))\, e^{-2\pi i \xi \lambda} \rangle & 1 \\ \langle \omega_{t, s, y}, \lambda_1 (\lambda_2 + a(s, y))\, e^{-2\pi i \xi \lambda} \rangle & 0 \end{pmatrix}.$$

The solution to (2.25) is thus given by

$$\widehat{U}(t, \xi, y) = \widehat{U}_0(y, \xi) + \int_0^t \widehat{F}(s, \xi, y)\, ds + 2\pi i \xi \int_0^t S(t, s, y; \xi)\, ds \cdot \tag{2.27}$$

$$\cdot\, \widehat{U}_0(\xi, y) + \int_0^t \left(2\pi i \xi \int_s^t S(t, \sigma, \xi, y)\, d\sigma \right) \widehat{F}(s, \xi, y)\, ds\,.$$

It follows that $U \in X \times Y$ is indeed a solution of (2.23) and hypothesis (2.24) ensures its unicity. ■

We point out that (2.27) has a meaning also for $u_0 \in L^\infty(\boldsymbol{R} \times \Omega)$ and $f(t, x, y) \in L^1(0, T; L^\infty(\boldsymbol{R} \times \Omega))$ and it defines a generalized solution of the Cauchy problem (2.1).

3. The Cauchy problem when $\partial_t a_\varepsilon \rightharpoonup \partial_t a$.

In this Section we consider the model problem (2.1) assuming that $a_\varepsilon(t, y)$ is absolutely continuous in t and satisfies

$$\begin{aligned}(3.1)\qquad & 0 \leqslant \alpha_- \leqslant a_\varepsilon(t, y) \leqslant \alpha_+, \qquad -\beta \leqslant \partial_t a_\varepsilon(t, y) \leqslant \beta \text{ a.e. in} \\ & (0, T) \times \Omega \quad \text{with } a_\varepsilon \times \partial_t a_\varepsilon \to a_\varepsilon,\ \partial_t a_\varepsilon \overset{*}{\rightharpoonup} a,\ \partial_t a \text{ in } L^\infty \text{ weak*}\end{aligned}$$

$\beta > 0$ and give the corresponding kinetic formulation of the effective equation (2.20).

Let denote $\Lambda' = [-\beta, \beta]$ and consider the Young measure $\omega_{t,s,y}(dl_1, d\lambda_1, d\lambda_2, d\lambda)$ associated to the sequence

$$\left(\partial_t b_\varepsilon(t, y),\ b_\varepsilon(t, y),\ b_\varepsilon(s, y),\ \int_s^t b_\varepsilon(\sigma, y)\, d\sigma\right),$$

with support in $\Lambda' \times \Lambda \times \Lambda \times \Lambda_T$.

The functions $C(t, s, y; \xi)$, $H(t, s, y; \xi)$ and $K(t, s, y; \xi)$ defined in (2.3), (2.9) are naturally absolutely continuous in t and this ensures to equation (2.5) a different kinetic formulation, for any fixed $\xi \in \boldsymbol{R}$. We state those properties referring to [15]:

LEMMA 3.1. *The functions $C(t, s, y; \xi)$, $H(t, s, y; \xi)$ and $K(t, s, y; \xi)$ are absolutely continuous in t and one has:*

$$\begin{aligned}(3.2)\qquad \partial_t C = {} & \langle \omega_{t,s,y}(dl_1, d\lambda_1, d\lambda_2, d\lambda), l_1 \otimes 1_{\lambda_1} \otimes 1_{\lambda_2} \otimes e^{-2\pi i \xi \lambda} \rangle - \\ & - 2\pi i \xi \langle \omega_{t,s,y}(dl_1, d\lambda_1, d\lambda_2, d\lambda), 1_{l_1} \otimes (\lambda_1)^2 e^{-2\pi i \xi \lambda} \rangle,\end{aligned}$$

$$\begin{aligned}(3.3)\qquad \partial_t H = {} & \langle \omega_{t,s,y}(dl_1, d\lambda_1, d\lambda_2, d\lambda), l_1 \otimes 1_{\lambda_1} \otimes \lambda_2 e^{-2\pi i \xi \lambda} \rangle - \\ & - 2\pi i \xi \langle \omega_{t,s,y}(dl_1, d\lambda_1, d\lambda_2, d\lambda), 1_{l_1} \otimes (\lambda_1)^2 \lambda_2 e^{-2\pi i \xi \lambda} \rangle\end{aligned}$$

whereas $\partial_t K(t, s, y; \xi)$ *is defined by the equation:*

$$(3.4) \qquad \partial_t K(t, s, y; \xi) - 2\pi i\xi \int_s^t \partial_t C(t, \tau, y; \xi)\, K(\tau, s, y; \xi)\, d\tau = \partial_t H(t, s, y; \xi).$$

(See Lemma 2.3.1 in [15]).

Let denote $\widetilde{\partial_t C u} = \int_0^t e^{-2\pi i\xi \int_s^t a(\sigma, y) d\sigma} \partial_t C(t, s, y; \xi)\, \widehat{u}(s, \xi, y)\, ds$ and the same for $\widetilde{\partial_t H u}$.

In view of Theorem 2.4 in [15] the problem (2.5) has the following kinetic formulation, in which z is the function defined in (2.11) and $k_0(t, y)$ is given by (2.10):

PROPOSITION 3.1. *For any* $\xi \in \boldsymbol{R}$, *the Cauchy problem related to the homogenized equation* (2.5) *is equivalent to the following system:*

$$(3.5) \qquad \begin{cases} \partial_t \widehat{u} + 2\pi i\xi a(t, y)\, \widehat{u} - 2\pi i\xi \widehat{z} = 0\,, \\ \partial_t \widehat{z} + 2\pi i\xi a(t, y)\, \widehat{z} - 2\pi i\xi k_0(t, y)\, \widehat{u} - 2\pi i\xi \widetilde{\partial_t C z} - 2\pi i\xi \widetilde{\partial_t H u} = 0\,, \end{cases}$$

in $(0, T) \times \Omega$, *with* $(\widehat{u}, \widehat{z})(0, \xi, y) = (\widehat{u}_0(\xi, y), 0)$ *in* Ω.

As in Proposition 2.4 we can prove that:

LEMMA 3.2. *The functions* $\partial_t C$, $\partial_t H$, $\partial_t K$ *are symbols in* $S^1_{0,0}(\boldsymbol{R})$ *for a.e. t, s, y and the related operators are bounded from* $H^1(\boldsymbol{R})$ *to* $L^2(\boldsymbol{R})$. *Moreover there exists a measure* $k_{t,s,y}(dm_1, d\mu_1, d\mu_2, d\mu)$ *parametrized in t, s, y with support in* $\Lambda' \times \Lambda \times \Lambda \times \Lambda_T$, *such that*

$$(3.6) \qquad \partial_t K(t, s, y; \xi) = \langle k_{t,s,y}(dm_1, d\mu_1, d\mu_2, d\mu), m_1 \mu_2 e^{-2\pi i\xi\mu} \rangle - 2\pi i\xi \langle k_{t,s,y}(d\mu_1, d\mu_2, d\mu), (\mu_1)^2 \mu_2 e^{-2\pi i\xi\mu} \rangle.$$

Let denote by $\partial_t K$ the operator

$$(3.7) \qquad \partial_t K u = \int_0^t \int e^{2\pi i\xi\left(x - \int_s^t a(\sigma, y) d\sigma\right)} \partial_t K(t, s, y; \xi)\, \widehat{u}(s, \xi, y)\, d\xi\, ds$$

(the same notation is used for the operators defined by $\partial_t C(t, s, y; \xi)$,

$\partial_t H(t, s, y; \xi))$ and let introduce the spaces $X_0 = H^2(\boldsymbol{R}; L^2(\Omega))$, $Y_0 = L^2(\boldsymbol{R} \times \Omega)$, $X = W^{1,\infty}(0, T; X_0)$, $Y = L^\infty(0, T; Y_0)$.

We get the following existence and uniqueness result:

THEOREM 3.1. *Under hypothesis* (3.1), *the problem* (2.20) *is equivalent in* $(0, T) \times \boldsymbol{R} \times \Omega$ *to the system:*

$$(3.8) \quad \begin{cases} \partial_t u + a(t, y)\, \partial_x u - \partial_x z = 0 \\ \partial_t z + a(t, y)\, \partial_x z - k_0(t, y)\, \partial_x u - \partial_x(\partial_t Cz) - \partial_x(\partial_t Hu) = 0 \end{cases}$$

with $(u, v)|_{t=0} = (u_0(x, y), 0)$ *in* $\boldsymbol{R} \times \Omega$.

If we assume $k_0(t, y) > 0$ *a.e. in* $(0, T) \times \Omega$, *for any initial data in* $X_0 \times X_0$ *the system* (3.8) *has a unique solution in* $W^{2,\infty}(0, T; X_0) \times X$.

PROOF. In view of Proposition 3.1 we have the equivalence between (2.5) and (3.6). For $U_0 \in X_0 \times X_0$, $F(t, \xi, y) \in L^1(0, T; Y_0)$, by integration in time, the system (3.5) is equivalent to the one in (2.25) for which existence and uniqueness hold in $X \times Y$. The kernel $R(t, s, y; \xi)$ is absolutely continuous in t and clearly the solution $S(t, s, y; \xi)$ of the resolvent equation (2.26) inherits the same property.

From (2.27) we see that the solution U belongs to $W^{2,\infty}(0, T; X_0) \times X$ and (2.27) has a meaning also when $u_0 \in L^\infty(\boldsymbol{R} \times \Omega)$. ∎

4. An example of measure.

We consider a sequence $a_\varepsilon(t, y; \gamma)$ bounded in $L^\infty((0, T) \times \Omega \times]0, 1[)$ with small amplitude oscillations, i.e. such that a.e. for $(t, y) \in (0, T) \times \Omega$ and γ small

$$a_\varepsilon(t, y; \gamma) = a(t, y) + \gamma b_\varepsilon(t, y) + \gamma^2 c_\varepsilon(t, y) + o(\gamma^2).$$

We assume that a_ε, $\partial_t a_\varepsilon \overset{*}{\rightharpoonup} a_\gamma(t, y)$, $\partial_t a_\gamma(t, y)$, b_ε, $\partial_t b_\varepsilon \overset{*}{\rightharpoonup} b$, $\partial_t b$, $c_\varepsilon \overset{*}{\rightharpoonup} c$ in L^∞ weak*.

Moreover, let $(b_\varepsilon(t, y) - b(t, y))(b_\varepsilon(s, y) - b(s, y)) \overset{*}{\rightharpoonup} k_0(t, s, y)$ in $L^\infty((0, T) \times (0, T) \times \Omega)$ weak*.

We suppose that $k_0(t, y) := k_0(t, t, y) > 0$ a.e. t, y.

In this case the measure $k_{t,s,y;\gamma}(d\sigma_1, d\sigma_2, d\sigma)$ given by Proposition 2.4 has the following asymptotic expansion:

LEMMA 4.1. *The measure $k_{t,s,y;\gamma}$ admits for small γ the asymptotic expansion:*

$$k_{t,s,y;\gamma}(d\sigma_1, d\sigma_2, d\sigma)\left(\sigma_1\sigma_2\delta_{x=\int_s^t a_\gamma(\tau,y)d\tau+\sigma}\right)dx =$$

$$= \gamma^2 k_0(t,s,y)\,\delta_{x=\int_s^t a(\tau,y)d\tau}\,dx + o(\gamma^2).$$

PROOF. The expansion is deduced from relation (1.7) when $\omega_{t,s,y;\gamma}$ is the measure associated to $a_\varepsilon - a_\gamma$, where one sees that the zero order moment is given by

$$\langle k_{t,s,y;\gamma}, \sigma_1\sigma_2\rangle = \langle \omega_{t,s,y;\gamma}, \lambda_1\lambda_2\rangle = \gamma^2 k_0(t,s,y) + o(\gamma^2). \qquad \blacksquare$$

The average operator K defined in (2.21) is correspondently expanded in

$$Ku = \gamma^2\int_0^t k_0(t,s,y)\,u\left(s, x - \int_s^t a(\sigma,y)\,d\sigma, y\right)ds + o(\gamma^2).$$

From (3.8) we get that the effective equation behaves up to order γ^2 like the second order integrodifferential equation

$$\partial^2_{tt}u + (a_\gamma + a)\,\partial^2_{tx}u + (aa_\gamma - k_0(t,y))\,\partial^2_{xx}u + \partial_t a_\gamma(t,y)\,\partial_x u -$$

$$-\gamma^2\int_0^t \partial_t k_0(t,s,y)\,\partial^2_{xx}u\left(s, x - \int_s^t a(\tau,y)\,d\tau, y\right)ds = 0\,.$$

REFERENCES

[1] R. ALEXANDRE, *Homogénéisation de certaines équations du type transport*, Thèse de doctorat, Univ. Paris VI (1993).

[2] L. AMBROSIO - P. D'ANCONA - S. MORTOLA, *Γ-convergence and least square methods*, Ann. Mat. Pur. Appl. (IV), Vol. **CLXVI** (1994), pp. 101-127.

[3] Y. AMIRAT - K. HAMDACHE - A. ZIANI, *Homogénéisation d' équations hyperboliques du premier ordre et applications aux écoulements miscibles en milieu poreux*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **6** (1989), pp. 397-417.

[4] Y. AMIRAT - K. HAMDACHE - A. ZIANI, *Homogenization of parametrized families of hyperbolic problems*, Proc. of Roy. Soc. of Edinb., **120A** (1992), pp. 199-221.

[5] Y. AMIRAT - K. HAMDACHE - A. ZIANI, *Some results on homogenization of convection diffusion equations*, Arch. Rat. Mech. Anal., **114** (1991), pp. 155-178.

[6] Y. AMIRAT - K. HAMDACHE - A. ZIANI, *On homogenization of ordinary differential equations and linear transport equations*, Calculus of variations, Homogenization and continuum mechanics, Series on Adv. in Math. for appl. Sc., vol. **18**, pp. 29-50.

[7] Y. AMIRAT - K. HAMDACHE - A. ZIANI, *Kinetic formulation for a transport equation with memory*, Comm. in P.D.E., **16**(8&9) (1991), pp. 1287-1311.

[8] W. E., *Homogenization of linear and nonlinear transport equations*, Comm. on Pure and Applied Math., **XLV**(3) (1992), pp. 301-326.

[9] I. M. GEL'FAND - G. E. SHILOV, *Generalized functions, 1, 2*, Academic Press (1964).

[10] F. GOLSE, *Remarques sur l' homogénéisation pour l'équation de transport*, C. R. A. S. Paris, **305** (1987), pp. 801-804.

[11] L. HÖRMANDER, *The analysis of linear partial differential operators, I*, Springer Verlag ed., N. 256 (1983).

[12] Y. KATZNELSON, *An introduction to harmonic analysis*, Dover publications (1976).

[13] V. LAKSHMIKANTHAM - S. LEELA, *Differential and integral inequalities*, Academic Press (1969).

[14] S. MIGORSKI - S. MORTOLA - J. TRAPLE, *Homogenization of first order differential operators*, Rend. Acc. Naz. Sci. XL Mem. Mat., (5), **16** (1992), pp. 259-276.

[15] M. PETRINI, *Homogenization of linear and nonlinear ordinary differential equations with time depending coefficients*, to appear in Rend. Sem. Mat.

[16] L. TARTAR, *Non local effects induced by homogenization, P.D.E. and Calculus of Variations. Essays in honour of E. De Giorgi*, Vol. **2**, Birkhauser (1989), pp. 925-938.

[17] L. TARTAR, *Memory effects and homogenization*, Arch. Rat. Mech. Anal. (1990), pp. 121-133.

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