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ABSTRACT - This article has two main points. The first is to construct a resolution of the hypergeometric differential module $\mathcal{M}_a$ (see [9]) that can be used to compute, for example, higher formal and analytic solutions of $\mathcal{M}_a$ at any point $\lambda^{(0)} \in \boldsymbol{C}^N$. The second is to establish a connection between the higher formal solutions and a de Rham-type cohomology theory defined by Dwork [5]. By standard results in the theory of D-modules, this implies the finite-dimensionality of Dwork cohomology.

1. - Introduction.

This article completes the work [1] by proving some complementary results that we were unable to establish at that time. Let $\mathcal{D} = \boldsymbol{C}[\lambda_1, \ldots, \lambda_N, \partial_1, \ldots, \partial_N]$ be the N-variable Weyl algebra. Given N nonzero lattice points $d^{(1)}, \ldots, d^{(N)} \in \boldsymbol{Z}^n$, $d^{(j)} = (d_1^{(j)}, \ldots, d_n^{(j)})$, spanning $\boldsymbol{R}^n$ as $\boldsymbol{R}$-vector space and a parameter $a = (a_1, \ldots, a_n) \in \boldsymbol{C}^n$, there is an associated hypergeometric $\mathcal{D}$-module $\mathcal{M}_a$ defined as follows [9]. For $i = 1, \ldots, n$, put

$$Z_{i,a} = \sum_{j=1}^{N} d_i^{(j)} \lambda_j \partial_j + a_i \in \mathcal{D}.$$

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For any $l = (l_1, \ldots, l_N) \in \boldsymbol{Z}^N$, define $\square_l \in \mathcal{D}$ by

$$\square_l = \prod_{l_j > 0} (\partial_j)^{l_j} - \prod_{l_j < 0} (\partial_j)^{-l_j}$$

(where the empty product equals 1). Let L be the lattice of relations among $d^{(1)}, \ldots, d^{(N)}$:

$$L = \left\{ (l_1, \ldots, l_N) \in \boldsymbol{Z}^N \mid \sum_{j=1}^{N} l_j d^{(j)} = 0 \right\}.$$

Let $\mathfrak{I}_a$ be the left ideal of $\mathcal{D}$ defined by

$$\mathfrak{I}_a = \sum_{i=1}^{n} \mathcal{D} Z_{i,a} + \sum_{l \in L} \mathcal{D} \square_l$$

and define $\mathcal{M}_a = \mathcal{D}/\mathfrak{I}_a$, a left $\mathcal{D}$-module. In [9] it was proved that $\mathcal{M}_a$ is holonomic if all $d^{(j)}$ lie on a common primitive affine hyperplane, and in [1] we eliminated this restriction. Thus $\mathcal{M}_a$ is always holonomic.

Our goal is to say something about the higher solutions of $\mathcal{M}_a$ and, in particular, to connect these higher solutions with «algebraic Dwork cohomology.» Note that $Z_{i,a} Z_{j,a} = Z_{j,a} Z_{i,a}$ for all $i, j \in \{1, \ldots, n\}$. Fix $l \in L$ and put

$$b = a + \sum_{l_j > 0} l_j d^{(j)} \quad \left(= a - \sum_{l_j < 0} l_j d^{(j)} \text{ since } l \in L \right).$$

It is straightforward to check that

$$\square_l \circ Z_{i,a} = Z_{i,b} \circ \square_l ,$$

hence right multiplication by $Z_{i,a}$ maps the left ideal $\sum_{l \in L} \mathcal{D} \square_l$ into itself. Put $\mathcal{P} = \mathcal{D} / \sum_{l \in L} \mathcal{D} \square_l$, a left $\mathcal{D}$-module. Then right multiplication by $Z_{i,a}$, $i = 1, \ldots, n$, is a family of commuting endomorphisms of $\mathcal{P}$ (as left $\mathcal{D}$-module), so it makes sense to consider the Koszul complex $K.(\mathcal{P}, \{Z_{i,a}\}_{i=1}^{n})$ they determine:

$$0 \to \mathcal{P}^{\binom{n}{n}} \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_2} \mathcal{P}^{\binom{n}{1}} \xrightarrow{\delta_1} \mathcal{P}^{\binom{n}{0}} \to 0 ,$$

a complex of left $\mathcal{D}$-modules. More precisely,

$$K_m(\mathcal{P}) = \bigoplus_{1 \leq i_1 < \ldots < i_m \leq n} \mathcal{P} e_{i_1} \wedge \ldots \wedge e_{i_m},$$

where the e_{i_j} are a collection of formal symbols, and the boundary map

$\delta_m: K_m(\mathcal{P}) \to K_{m-1}(\mathcal{P})$ is defined by

$$\delta_m(\xi e_{i_1} \wedge \ldots \wedge e_{i_m}) = \sum_{k=1}^{m} (-1)^{k-1} \xi Z_{i_k, a} e_{i_1} \wedge \ldots \wedge \hat{e}_{i_k} \wedge \ldots \wedge e_{i_m}.$$

Augmenting this Koszul complex by $H_0(\mathrm{K}.(\mathcal{P})) = \mathcal{M}_a$ we get the complex

$$(1.1) \qquad 0 \to \mathcal{P}^{\binom{n}{n}} \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_2} \mathcal{P}^{\binom{n}{1}} \xrightarrow{\delta_1} \mathcal{P}^{\binom{n}{0}} \to \mathcal{M}_a \to 0 .$$

We seek conditions that will guarantee that (1.1) is a resolution of $\mathcal{M}_a$ that can be used to compute the higher solutions $\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{M}_a, \mathcal{F})$ for certain left $\mathcal{D}$-modules $\mathcal{F}$.

We describe the hypothesis we shall need. A monoid is called *normal* [10, section 1] if it is finitely generated and if for every u, v, w in the monoid and positive integer k such that $u + kv = kw$, there exists u' in the monoid such that $u = ku'$. The relevant monoid for us will be the monoid M generated by $d^{(1)}, \ldots, d^{(N)}$, i.e.,

$$(1.2) \qquad M = \left\{ \sum_{j=1}^{N} c_j d^{(j)} \mid c_j \in \boldsymbol{Z}, \quad c_j \geq 0 \ \text{ for all } j \right\}.$$

Note that if M equals the intersection of the real cone generated by the $d^{(j)}$'s with a subgroup of $\boldsymbol{Z}^n$, then M is normal. By [7, Appendix], for all the classically studied hypergeometric functions M is the set of all lattice points in the real cone generated by the $d^{(j)}$'s, hence is normal. We shall prove:

THEOREM 1.2. *If M is normal, then the sequence* (1.1) *is exact.*

We shall also show that if $\mathcal{F}$ satisfies a certain condition described below, then $\mathcal{P}$ is an acyclic object for the functor $\mathrm{Hom}_{\mathcal{D}}(-, \mathcal{F})$, hence the sequence (1.1) can be used to compute the higher solutions of $\mathcal{M}_a$ in $\mathcal{F}$. Let $\boldsymbol{C}[y] = \boldsymbol{C}[y_1, \ldots, y_N]$ be a polynomial ring in N variables and let $P(y)$ (resp. $Q(y)$) be an $r \times s$ (resp. $s \times t$) matrix with entries in $\boldsymbol{C}[y]$. Consider the sequence

$$(1.3) \qquad \boldsymbol{C}[y]^t \xrightarrow{Q(y)} \boldsymbol{C}[y]^s \xrightarrow{P(y)} \boldsymbol{C}[y]^r$$

defined by matrix multiplication. Let $\mathcal{D}_0 = \boldsymbol{C}[\partial_1, \ldots, \partial_N] \subset \mathcal{D}$ be the subring of constant coefficient differential operators. There is a $\boldsymbol{C}$-algebra isomorphism $\mathcal{D}_0 \simeq \boldsymbol{C}[y]$ defined by mapping ∂_j to y_j. We regard $\boldsymbol{C}[y]$ as a

$\mathcal{D}_0$-module via this isomorphism. Associated to (1.3) is the sequence

$$\text{(1.4)}\qquad \mathrm{Hom}_{\mathcal{D}_0}(\boldsymbol{C}[y]^r, \mathcal{F}) \overset{\circ P(y)}{\to} \mathrm{Hom}_{\mathcal{D}_0}(\boldsymbol{C}[y]^s, \mathcal{F}) \overset{\circ Q(y)}{\to} \mathrm{Hom}_{\mathcal{D}_0}(\boldsymbol{C}[y]^t, \mathcal{F}).$$

There is a natural identification $\mathrm{Hom}_{\mathcal{D}_0}(\boldsymbol{C}[y]^k, \mathcal{F}) \simeq \mathcal{F}^k$ as $\boldsymbol{C}$-vector spaces, under which the sequence (1.4) becomes

$$\text{(1.5)}\qquad \mathcal{F}^r \overset{P(\partial)^t}{\longrightarrow} \mathcal{F}^s \overset{Q(\partial)^t}{\longrightarrow} \mathcal{F}^t,$$

where $P(\partial)$ and $Q(\partial)$ are the matrices with entries in $\mathcal{D}_0$ obtained by replacing y_j by ∂_j for all j and $P(\partial)^t$ (resp. $Q(\partial)^t$) denotes the transpose of $P(\partial)$ (resp. $Q(\partial)$).

THEOREM 1.6. *Suppose the left $\mathcal{D}$-module $\mathcal{F}$ has the property that* (1.5) *is exact whenever* (1.3) *is exact. Then* $\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{P}, \mathcal{F}) = 0$ *for all* $k > 0$.

When the hypotheses of Theorems 1.2 and 1.6 are satisfied, $\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{M}_a, \mathcal{F})$ is thus the k-th homology of the sequence

$$\text{(1.7)}\quad 0 \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}^{\binom{n}{0}}, \mathcal{F}) \overset{\delta'_0}{\to} \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}^{\binom{n}{1}}, \mathcal{F}) \overset{\delta'_1}{\to} \dots \overset{\delta'_{n-1}}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}}(\mathcal{P}^{\binom{n}{n}}, \mathcal{F}) \overset{\delta'_n}{\to} 0,$$

where δ'_k denotes composition with δ_{k+1}.

The question of which $\mathcal{D}$-modules $\mathcal{F}$ satisfy the hypothesis of Theorem 1.6 has been extensively studied (see [8, 13]). In particular, $\mathcal{O}_{\lambda^{(0)}}$, the ring of convergent power series at a point $\lambda^{(0)} \in \boldsymbol{C}^N$, and $\widehat{\mathcal{O}}_{\lambda^{(0)}}$, the ring of formal power series at $\lambda^{(0)} \in \boldsymbol{C}^N$, satisfy this condition. Thus when M is normal, the sequence (1.1) is a resolution of $\mathcal{M}_a$ which can be used to compute the higher formal solutions and higher analytic solutions of $\mathcal{M}_a$ at any point $\lambda^{(0)} \in \boldsymbol{C}^N$.

We now turn to the connection with algebraic Dwork cohomology. In [5], Dwork constructed a p-adic cohomology theory for smooth hypersurfaces over finite fields by taking the homology spaces of a certain Koszul complex. This Koszul complex is defined by the action of a finite collection of commuting differential operators on a space of p-adic power series whose coefficients satisfy certain growth conditions. This theory was modified in [2] to give a good p-adic cohomology theory for nondegenerate exponential sums on a torus. If one replaces the space of power series considered in [2] by the space of finite sums of the same type and uses $\boldsymbol{C}$-coefficients instead of p-adic coefficients, one obtains the ring $R = \boldsymbol{C}[x^{d^{(1)}}, \dots, x^{d^{(N)}}]$, where $x^{d^{(j)}} = x_1^{d_1^{(j)}} \dots x_n^{d_n^{(j)}}$. The commuting differential operators on R are ($a \in \mathbf{C}^n$, $\lambda^{(0)} \in \mathbf{C}^N$, $i = 1, \ldots, n$)

$$D_{i, a, \lambda^{(0)}} = x_i \frac{\partial}{\partial x_i} + a_i + \sum_{j=1}^{N} d_i^{(j)} \lambda_j^{(0)} x^{d^{(j)}}.$$

Thus the Koszul complex $\mathrm{K}.(R, \{D_{i, a, \lambda^{(0)}}\}_{i=1}^n)$ is an algebraic analogue over $\mathbf{C}$ of the (p-adic analytic) Koszul complex considered in [2]. Our second main result is that this complex can also be used to compute the higher formal solutions of $\mathcal{M}_a$ at $\lambda^{(0)}$, in fact, it will be seen that this complex is dual to the complex (1.7) with $\mathcal{F} = \widehat{\mathcal{O}}_{\lambda^{(0)}}$. For any $\mathbf{C}$-vector space V, we denote its dual space $\mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ by $V^\vee$.

THEOREM 1.8. *If M is normal, then*

$$\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{M}_a, \widehat{\mathcal{O}}_{\lambda^{(0)}}) \simeq H_k(\mathrm{K}.(R, \{D_{i, a, \lambda^{(0)}}\}_{i=1}^n))^\vee.$$

From the general theory of $\mathcal{D}$-modules [4], one then deduces:

COROLLARY 1.9. *If M is normal, then for all $a \in \mathbf{C}^n$, $\lambda^{(0)} \in \mathbf{C}^N$, and $k \geq 0$,*

$$\dim_{\mathbf{C}} H_k(\mathrm{K}.(R, \{D_{i, a, \lambda^{(0)}}\}_{i=1}^n)) < \infty.$$

Finiteness results of this type were first proved by Dwork [6] in a study of the cohomology of singular hypersurfaces over finite fields. Similar results were proved by Monsky [12], who used them to deduce the finiteness of the algebraic de Rham cohomology of a smooth variety over $\mathbf{C}$ without using resolution of singularities. It seems likely that Corollary 1.9 could also be proved by the methods of those authors.

The scheme of this paper is as follows. In section 2, we prove a generalization of Theorem 1.6 (Theorem 2.6) and give an equivalent formulation of Theorem 1.2 (Theorem 2.5). In section 3, we prove Theorem 2.5. Finally, in section 4, we prove Theorem 1.8 and Corollary 1.9.

2. - Proof of Theorem 1.6.

We begin by recalling some results of [1], which form the basis for this article. Let $R = \mathbf{C}[x^{d^{(1)}}, \ldots, x^{d^{(N)}}]$ and put $R[\lambda] = R[\lambda_1, \ldots, \lambda_N]$. The ring $R[\lambda]$ is a free $\mathbf{C}[\lambda]$-module on $\{x^u \mid u \in M\}$. We make $R[\lambda]$ into a left $\mathcal{D}$-module by defining for $\xi = \sum_u g_u(\lambda)\, x^u \in R[\lambda]$

$$\partial_j(\xi) = \sum_u \frac{\partial g_u}{\partial \lambda_j} x^u + \sum_u g_u(\lambda)\, x^{u+d^{(j)}}, \tag{2.1}$$

i.e., ∂_j acts as $\partial/\partial\lambda_j + x^{d^{(j)}}$. For $i = 1, \dots, n$, define differential operators $D_{i,a}$ on $R[\lambda]$ by

$$D_{i,a} = x_i \frac{\partial}{\partial x_i} + a_i + \sum_{j=1}^{N} d_i^{(j)} \lambda_j x^{d^{(j)}}. \tag{2.2}$$

As noted in [1, section 4], the $D_{i,a}$ are $\mathcal{D}$-module endomorphisms of $R[\lambda]$, hence $R[\lambda]/\sum_{i=1}^{n} D_{i,a}R[\lambda]$ has a structure of left $\mathcal{D}$-module. Since $\mathcal{D}$ is a free $\boldsymbol{C}[\lambda]$-module with basis $\{\partial^b = \partial_1^{b_1}\dots\partial_N^{b_N} \mid b_j \in \boldsymbol{Z},\ b_j \geq 0 \text{ for all } j\}$, we can define a surjective homomorphism of $\boldsymbol{C}[\lambda]$-modules $\phi\colon \mathcal{D} \to R[\lambda]$ by

$$\phi(\partial_1^{b_1}\dots\partial_N^{b_N}) = x^{\sum_{j=1}^{N} b_j d^{(j)}}.$$

One checks easily that ϕ is actually a homomorphism of $\mathcal{D}$-modules.

THEOREM 2.1 ([1, Theorem 4.4]). *The map ϕ induces isomorphisms of $\mathcal{D}$-modules*

$$\mathcal{P} \simeq R[\lambda], \tag{2.3}$$

$$\mathcal{M}_a \simeq R[\lambda]/\sum_{i=1}^{n} D_{i,a}R[\lambda]. \tag{2.4}$$

For $\xi = \sum_b g_b(\lambda)\,\partial^b \in \mathcal{D}$, we have

$$\phi(\xi Z_{i,a}) = \xi\phi(Z_{i,a}) = \xi D_{i,a}(1) = D_{i,a}(\xi(1)) = D_{i,a}(\phi(\xi)). \tag{2.5}$$

Thus we have:

COROLLARY 2.4. *The isomorphism* (2.3) *identifies the Koszul complex* $\mathrm{K}.(\mathcal{P}, \{Z_{i,a}\}_{i=1}^{n})$ *with the Koszul complex* $\mathrm{K}.(R[\lambda], \{D_{i,a}\}_{i=1}^{n})$.

Theorem 1.2 is therefore equivalent to:

THEOREM 2.5. *If M is normal, then the Koszul complex* $\mathrm{K}.(R[\lambda], \{D_{i,a}\}_{i=1}^{n})$ *is acyclic in positive dimension.*

Theorem 2.5 will be proved in the next section. We turn our attention now to Theorem 1.6. Note that since the $\Box_l$ lie in $\mathcal{D}_0$ for all $l \in \boldsymbol{Z}^N$,

$$\mathcal{P} = \mathcal{D} / \sum_{l \in L} \mathcal{D}\Box_l \simeq \mathcal{D} \bigotimes_{\mathcal{D}_0} \Big(\mathcal{D}_0 / \sum_{l \in L} \mathcal{D}_0 \Box_l \Big) \tag{2.6}$$

as left $\mathcal{D}$-modules, where we regard $\mathcal{D}$ as a right $\mathcal{D}_0$-module. More generally, let $\mathcal{N}_0$ be a finitely-generated left $\mathcal{D}_0$-module and put $\mathcal{N} = \mathcal{D} \otimes_{\mathcal{D}_0} \mathcal{N}_0$, a left $\mathcal{D}$-module. Then Theorem 1.6 is a special case of the following.

THEOREM 2.6. *Let* $\mathcal{N}$ *be as above and suppose the left* $\mathcal{D}$*-module* $\mathcal{F}$ *has the property that* (1.5) *is exact whenever* (1.3) *is exact. Then* $\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{N}, \mathcal{F}) = 0$ *for all* $k > 0$.

PROOF. Since $\mathcal{D}_0$ is isomorphic to the polynomial ring in N variables over $\boldsymbol{C}$, the Hilbert syzygy theorem gives a resolution of $\mathcal{D}_0$-modules

$$0 \to (\mathcal{D}_0)^{m_N} \to \ldots \to (\mathcal{D}_0)^{m_1} \to (\mathcal{D}_0)^{m_0} \to \mathcal{N}_0 \to 0\,. \tag{2.7}$$

Regarding $\mathcal{D}$ as a right $\mathcal{D}_0$-module and tensoring over $\mathcal{D}_0$ with $\mathcal{D}$ gives a resolution of $\mathcal{D}$-modules

$$0 \to \mathcal{D}^{m_N} \to \ldots \to \mathcal{D}^{m_1} \to \mathcal{D}^{m_0} \to \mathcal{N} \to 0 \tag{2.8}$$

since $\mathcal{D}$ is a free $\mathcal{D}_0$-module. We need to show that the complex

$$0 \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{m_0}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{m_1}, \mathcal{F}) \to \ldots \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{m_N}, \mathcal{F}) \to 0 \tag{2.9}$$

is acyclic except possibly at the $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{m_0}, \mathcal{F})$-term. But

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^m, \mathcal{F}) \simeq \mathcal{F}^m \simeq \mathrm{Hom}_{\mathcal{D}_0}((\mathcal{D}_0)^m, \mathcal{F})$$

as $\boldsymbol{C}$-vector spaces and the sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_{\mathcal{D}_0}((\mathcal{D}_0)^{m_0}, \mathcal{F}) \to \\ \to \mathrm{Hom}_{\mathcal{D}_0}((\mathcal{D}_0)^{m_1}, \mathcal{F}) \to \ldots \to \mathrm{Hom}_{\mathcal{D}_0}((\mathcal{D}_0)^{m_N}, \mathcal{F}) \to 0 \end{aligned} \tag{2.10}$$

is exact, except possibly at the $\mathrm{Hom}_{\mathcal{D}_0}((\mathcal{D}_0)^{m_0}, \mathcal{F})$-term, by the exactness of (2.7) and our hypothesis on $\mathcal{F}$.

3. - Proof of Theorem 2.5.

Let $\Delta \subset \boldsymbol{R}^n$ be the convex hull of the origin and the points $d^{(1)}, \ldots, d^{(N)}$. Let $C(\Delta)$ be the real cone generated by $d^{(1)}, \ldots, d^{(N)}$. For all $u \in C(\Delta)$ we define the *weight* $w(u)$ to be the smallest nonnegative real number w such that $u \in w\Delta$. It is easily seen that there is a positive integer e such that $w(M) \subset e^{-1}\mathbf{Z}$.

This weight function can be used to define a filtration F. on the rings R and $R[\lambda]$, namely, let $F_{k/e}R$ (resp. $F_{k/e}R[\lambda]$) be the $\mathbf{C}$-span (resp. $\mathbf{C}[\lambda]$-span) of all x^u, $u \in M$, such that $w(u) \leqslant k/e$. We filter the Koszul complex $\mathrm{K}.(R[\lambda], \{D_{i,a}\}_{i=1}^{n})$ so that the boundary maps preserve the filtration. Namely, we define

$$(3.1) \qquad F_{k/e}K_m(R[\lambda]) = \bigoplus_{1 \leqslant i_1 < \dots < i_m \leqslant n} F_{k/e-m}R[\lambda]\, e_{i_1} \wedge \dots \wedge e_{i_m},$$

so that $\delta_m(F_{k/e}K_m(R[\lambda])) \subseteq F_{k/e}K_{m-1}(R[\lambda])$.

Let $\mathrm{gr}(R)$ be the graded ring associated to the filtered ring R: $\mathrm{gr}(R) = \bigoplus_{k=0}^{\infty} \mathrm{gr}^{(k/e)}(R)$, where

$$\mathrm{gr}^{(k/e)}(R) = (F_{k/e}R)/(F_{(k-1)/e}R).$$

It is clear that $\mathrm{gr}(R[\lambda])$, the graded ring associated to $R[\lambda]$, is $(\mathrm{gr}(R))[\lambda]$. The ring $\mathrm{gr}(R)$ (resp. $\mathrm{gr}(R[\lambda])$) is identical to R (resp. $R[\lambda]$) as $\mathbf{C}$-vector space (resp. as $\mathbf{C}[\lambda]$-module) but multiplication is given by

$$x^u x^{u'} = \begin{cases} x^{u+u'}, & \text{if } u, u' \text{ lie over a common face of } \Delta, \\ 0, & \text{otherwise}. \end{cases}$$

Put $f_\lambda(x) = \sum_{j=1}^{N} \lambda_j x^{d^{(j)}} \in F_1 R[\lambda]$ and let

$$f_{i,\lambda}(x) = x_i \frac{\partial f_\lambda}{\partial x_i} = \sum_{j=1}^{N} d_i^{(j)} \lambda_j x^{d^{(j)}} \in F_1 R[\lambda]$$

$$\bar{f}_{i,\lambda} = \text{image of } f_{i,\lambda} \text{ in } \mathrm{gr}^{(1)}(R[\lambda]).$$

The graded complex associated to the filtered Koszul complex $\mathrm{K}.(R[\lambda], \{D_{i,a}\}_{i=1}^{n})$ is the Koszul complex $\mathrm{K}.(\mathrm{gr}(R[\lambda]), \{\bar{f}_{i,\lambda}\}_{i=1}^{n})$. By [14, Chapter 9] there is a convergent E^1 spectral sequence with

$$E^1_{k,m} = H_{k+m}(\mathrm{K}.(\mathrm{gr}^{(k)}(R[\lambda]))),$$

$$E^\infty_{k,m} = \mathrm{gr}^{(k)}(H_{k+m}(\mathrm{K}.(R[\lambda]))).$$

Thus to prove Theorem 2.5, it suffices by standard properties of spectral sequences to show:

THEOREM 3.1. *If M is normal, then the Koszul complex* $K_\cdot(\mathrm{gr}(R[\lambda]), \{\bar{f}_{i,\lambda}\}_{i=1}^n)$ *is acyclic in positive dimension.*

The proof of Theorem 3.1 will occupy the remainder of this section. For any face σ of Δ, let $C(\sigma)$ be the real cone generated by σ. Let $M_\sigma = M \cap C(\sigma)$ and let

$$R_\sigma = \boldsymbol{C}[x^u \mid u \in M_\sigma], \qquad R_\sigma[\lambda] = \boldsymbol{C}[\lambda][x^u \mid u \in M_\sigma].$$

Define $f_{i,\sigma,\lambda} = \sum_{d^{(j)} \in \sigma} d_i^{(j)} \lambda_j x^{d^{(j)}} \in R_\sigma[\lambda]$.

As in [11, Proposition 2.6], there is an exact sequence

$$0 \to \mathrm{gr}(R[\lambda]) \to C_{n-1} \to C_{n-2} \to \ldots \to C_0 \to 0, \tag{3.2}$$

where for $0 \leqslant k \leqslant n-1$,

$$C_k = \bigoplus_\sigma R_\sigma[\lambda] \tag{3.3}$$

and the direct sum is over all faces σ of Δ of dimension k that are not contained in any face of Δ that contains the origin. The map $C_k \to C_{k-1}$ is defined (up to sign) as follows. If $\dim \sigma = k$, $\dim \tau = k-1$, and $g = \sum_{u \in M_\sigma} c_u(\lambda)\, x^u \in R_\sigma[\lambda]$, then g maps to $\sum_{u \in M_\tau} c_u(\lambda)\, x^u \in R_\tau[\lambda]$.

To prove Theorem 3.1, we wish to repeat the argument of [11, section 2.12]. To justify using this argument, the only thing to check is that for each σ as above with $\dim \sigma = k$, some subset of $\{f_{i,\sigma,\lambda}\}_{i=1}^n$ of cardinality $k+1$ forms a regular sequence on $R_\sigma[\lambda]$. We begin by observing that R_σ (and hence $R_\sigma[\lambda]$) is a Cohen-Macaulay ring. For by [10, Theorem 1], it suffices to show that M_σ is normal. For that it is enough to show that M_σ is finitely generated, the other condition being obvious since $M_\sigma = M \cap C(\sigma)$ and M is assumed normal. Equivalently, we prove:

LEMMA 3.3. *The ring R_σ is a finitely-generated $\boldsymbol{C}$-algebra.*

PROOF. Let M_1 be the monoid generated by those $d^{(j)}$'s that lie on σ and let $R_1 = \boldsymbol{C}[x^u \mid u \in M_1]$. Let M' be the subgroup of $\boldsymbol{Z}^n$ generated by $d^{(1)}, \ldots, d^{(N)}$ and let $R_2 = \boldsymbol{C}[x^u \mid u \in M' \cap C(\sigma)]$. Then R_1 is a finitely-generated $\boldsymbol{C}$-subalgebra of R_σ and R_σ is a $\boldsymbol{C}$-subalgebra of R_2. We claim that

(3.4) $\qquad R_2$ is a finitely-generated R_1-module.

Since R_σ is an R_1-submodule of R_2, it will follow that R_σ is a finitely-generated R_1-module, hence is a finitely-generated $\boldsymbol{C}$-algebra.

To prove (3.4), let $u \in M' \cap C(\sigma)$. Then

$$u = \sum_{d^{(j)} \in \sigma} c_j d^{(j)}, \qquad c_j \in \mathbf{Q}, \quad c_j \geqslant 0 \text{ for all } j \tag{3.5}$$

$$= \sum_{d^{(j)} \in \sigma} [c_j]\, d^{(j)} + \sum_{d^{(j)} \in \sigma} (c_j - [c_j])\, d^{(j)}, \tag{3.6}$$

where $[c_j]$ denotes the greatest integer in c_j. Let

$$F = \left\{ \sum_{d^{(j)} \in \sigma} \gamma_j d^{(j)} \,|\, 0 \leqslant \gamma_j < 1 \text{ for all } j \right\}.$$

We have $\sum_{d^{(j)} \in \sigma} [c_j]\, d^{(j)} \in M_1$ and $\sum_{d^{(j)} \in \sigma} (c_j - [c_j])\, d^{(j)} \in M' \cap F$. Since F is bounded and M' is discrete, it follows that $M' \cap F$ is finite, say, $M' \cap F = \{\mu_1, \dots, \mu_s\}$. Equation (3.6) implies that for every $u \in M' \cap C(\sigma)$ one has $x^u = x^{u_1} x^{\mu_i}$ for some i, with $u_1 \in M_1$. Thus $x^{\mu_1}, \dots, x^{\mu_s}$ generate R_2 as R_1-module.

Since $R_\sigma[\lambda]$ is Cohen-Macaulay, a collection of elements forms a regular sequence if and only if it can be extended to a system of parameters. Thus we are reduced to showing that some subset of $\{f_{i,\sigma,\lambda}\}_{i=1}^n$ of cardinality $k+1$ can be extended to a system of parameters. We remark here that we are viewing $R_\sigma[\lambda]$ as a graded ring by defining the degree of a monomial $\lambda_1^{v_1} \dots \lambda_N^{v_N} x^u$ to be $V_1 + \dots + V_N + w(u)$. Thus the $f_{i,\sigma,\lambda}$ are homogeneous of degree 2. Note that on the ring $R_\sigma[\lambda]$, exactly $k+1$ of the derivations $x_i \partial/\partial x_i$ are linearly independent, hence we may assume that all $f_{i,\sigma,\lambda}$ are linear combinations of $f_{1,\sigma,\lambda}, \dots, f_{k+1,\sigma,\lambda}$. We claim that these $k+1$ elements can be extended to a system of parameters. Since the Krull dimension of $R_\sigma[\lambda]$ is $k+1+N$, we must show there exist homogeneous elements $g_1, \dots, g_N \in R_\sigma[\lambda]$ such that

$$\dim_{\mathbf{C}} R_\sigma[\lambda]/(g_1, \dots, g_N, f_{1,\sigma,\lambda}, \dots, f_{k+1,\sigma,\lambda}) < \infty,$$

or, equivalently,

$$\dim_{\mathbf{C}} R_\sigma[\lambda]/(g_1, \dots, g_N, f_{1,\sigma,\lambda}, \dots, f_{n,\sigma,\lambda}) < \infty. \tag{3.7}$$

Since $R_\sigma[\lambda]$ is a finitely-generated $\mathbf{C}$-algebra, (3.7) is equivalent to the assertion that the only maximal ideal containing $(g_1, \dots, g_N, f_{1,\sigma,\lambda}, \dots, f_{n,\sigma,\lambda})$ is $(\lambda_1, \dots, \lambda_N, \{x^u \,|\, u \in M_\sigma \backslash (0, \dots, 0)\})$. But the proof of [1, Theorem 3.9], shows that this is the case if we take $g_j = \lambda_j - \alpha_j x^{d^{(j)}}$, $j = 1, \dots, N$, for generic $\alpha_j \in \mathbf{C}^\times$. (The ring $R_\sigma[\lambda]$ is slightly different from the ring considered in the proof of [1, Theorem 3.9]. However, since R_σ is a finitely-generated $\mathbf{C}$-algebra, it is straightforward to check that [1, Lemma 3.12] and hence [1, Lemma 3.13] still hold. To conform with the notation of this article, the symbols «M» and «M'» in [1, Lemmas 3.12 and 3.13] should be replaced by «M_σ» and «M'_σ», respectively, where M'_σ denotes the subgroup of $\boldsymbol{Z}^n$ generated by M_σ.)

This completes the proof of Theorem 2.5.

4. – Proof of Theorem 1.8 and Corollary 1.9.

We recall that it is well-known (and straightforward to check) that for any finitely-generated $\mathcal{D}$-module $\mathcal{N}$ there is an isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \widehat{\mathcal{O}}_{\lambda^{(0)}}) \simeq (\boldsymbol{C}_{\lambda^{(0)}} \underset{\boldsymbol{C}[\lambda]}{\otimes} \mathcal{M})^{\vee}, \tag{4.1}$$

where $\boldsymbol{C}_{\lambda^{(0)}}$ denotes the complex numbers $\boldsymbol{C}$ regarded as $\boldsymbol{C}[\lambda]$-module via the isomorphism $\boldsymbol{C} \simeq \boldsymbol{C}[\lambda]/(\lambda_1 - \lambda_1^{(0)}, \dots, \lambda_N - \lambda_N^{(0)})$. Since any finitely-generated $\mathcal{D}$-module $\mathcal{M}$ has a resolution by finitely-generated free $\mathcal{D}$-modules (which are also free $\boldsymbol{C}[\lambda]$-modules), it follows from (4.1) that

$$\mathrm{Ext}^k_{\mathcal{D}}(\mathcal{M}, \widehat{\mathcal{O}}_{\lambda^{(0)}}) \simeq \mathrm{Tor}_k^{\boldsymbol{C}[\lambda]}(\boldsymbol{C}_{\lambda^{(0)}}, \mathcal{M})^{\vee}. \tag{4.2}$$

By [4, Chapter V, Theorem 3.4.2], if $\mathcal{M}$ is holonomic then

$$\dim_{\boldsymbol{C}} \mathrm{Tor}_k^{\boldsymbol{C}[\lambda]}(\boldsymbol{C}_{\lambda^{(0)}}, \mathcal{M}) < \infty \tag{4.3}$$

for all $\lambda^{(0)} \in \boldsymbol{C}^N$ and all $k \geq 0$. Thus we have:

LEMMA 4.4. *If $\mathcal{M}$ is a holonomic left $\mathcal{D}$-module, then $\dim_{\boldsymbol{C}} \mathrm{Ext}^k_{\mathcal{D}}(\mathcal{M}, \widehat{\mathcal{O}}_{\lambda^{(0)}}) < \infty$ for all $\lambda^{(0)} \in \boldsymbol{C}^N$ and all $k \geq 0$.*

REMARK. This lemma is no doubt well-known, but we are not aware of a reference for it.

Put $\mathcal{W}_a = R[\lambda] / \sum_{i=1}^{n} D_{i,a} R[\lambda]$ and consider the complex obtained by augmenting the Koszul complex $\mathrm{K}.(R[\lambda], \{D_{i,a}\}_{i=1}^n)$ by $\mathcal{W}_a$:

$$0 \to R[\lambda]^{\binom{n}{n}} \to \dots \to R[\lambda]^{\binom{n}{0}} \to \mathcal{W}_a \to 0. \tag{4.5}$$

Note that $R[\lambda]$ is a free $\boldsymbol{C}[\lambda]$-module and, when M is normal, (4.5) is exact by Theorem 2.5. Thus when M is normal, (4.5) is a resolution of $\mathcal{W}_a$ by free $\boldsymbol{C}[\lambda]$-modules, hence can be used to compute $\mathrm{Tor}_k^{\boldsymbol{C}[\lambda]}(\boldsymbol{C}_{\lambda^{(0)}}, \mathcal{W}_a)$, i.e.,

$\mathrm{Tor}_k^{C[\lambda]}(C_{\lambda^{(0)}}, \mathcal{W}_a)$ is the k-th homology of the sequence

$$(4.6)\qquad 0 \to C_{\lambda^{(0)}} \bigotimes_{C[\lambda]} R[\lambda]^{\binom{n}{n}} \to \ldots \to C_{\lambda^{(0)}} \bigotimes_{C[\lambda]} R[\lambda]^{\binom{n}{0}} \to 0\,.$$

One checks directly that this sequence is just the Koszul complex $K.(R, \{D_{i,a,\lambda^{(0)}}\}_{i=1}^n)$, hence

$$(4.7)\qquad H_k(K.(R, \{D_{i,a,\lambda^{(0)}}\}_{i=1}^n)) \simeq \mathrm{Tor}_k^{C[\lambda]}(C_{\lambda^{(0)}}, \mathcal{W}_a)\,.$$

Since $\mathcal{W}_a \simeq \mathcal{M}_a$ (as $\mathcal{D}$-modules) by (2.4), Theorem 1.8 now follows from (4.2) and Corollary 1.9 follows from Lemma 4.4.

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