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Generalized solutions of time dependent impulsive control systems

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Generalized Solutions of Time Dependent Impulsive Control Systems (*).

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ABSTRACT - This paper is concerned with the impulsive Cauchy problem

$$\dot{x}(t) = f(t, x) + g(t, x)\,\dot{u}(t), \qquad t \in [0, T], \qquad x(0) = \bar{x}$$

where u is a possibly discontinuous control function and the vector fields f, g: $\mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ are measurable in t and Lipschitz continuous in x. If g is smooth w.r.t. the variable x and satisfies $\|g(t, \cdot) - g(s, \cdot)\|_{\mathcal{C}^2} \leq \phi(t) - \phi(s)$, for some increasing function ϕ and every $s < t$, we show that the above Cauchy problem is well posed as u ranges in the space $\boldsymbol{L}^1(d\phi)$.

1. Introduction.

Consider the Cauchy problem for an impulsive control system of the form

$$\dot{x}(t) = f(t, x) + g(t, x)\,\dot{u}(t), \qquad t \in [0, T], \qquad x(0) = \bar{x} \in \mathbb{R}^n, \tag{1.1}$$

where u is a scalar control function and the dot denotes a derivative w.r.t. time. We assume that the vector fields f, g: $\mathbb{R} \times \mathbb{R}^n \mapsto \mathbb{R}^n$ are bounded, measurable in t and Lipschitz continuous in x, so that

$$|f(t, x)| \leq M, \qquad |g(t, x)| \leq M, \tag{1.2}$$

$$|f(t, x) - f(t, y)| \leq L|x - y|, \qquad |g(t, x) - g(t, y)| \leq L|x - y|, \tag{1.3}$$

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for some constants M, L. Under these conditions, for any continuously differentiable scalar function u, the right hand side of (1.1) is measurable in t and Lipschitz in x. Therefore, a well known theorem of Carathéodory [1] provides the existence and uniqueness of the corresponding solution $t \mapsto x(t, u)$. Aim of this paper is to show that, under suitable assumptions on g, the map $u \mapsto x(T, u)$ can be continuously extended to a much larger space of (possibly discontinuous) control functions. Besides (1.2)-(1.3), let g be twice continuously differentiable w.r.t. x, say

$$(1.4) \qquad \|g(t, \cdot)\|_{\mathcal{C}^2} \doteq$$

$$\doteq \sup_x \left\{ |g(t, x)| + \sum_{i=1}^{n} \left| \frac{\partial g(t, x)}{\partial x_i} \right| + \sum_{i,j=1}^{n} \left| \frac{\partial^2 g(t, x)}{\partial x_i \partial x_j} \right| \right\} \leq M .$$

Moreover, we shall assume that the total variation of g w.r.t. time is bounded:

$$(1.5) \qquad \|g(t, \cdot) - g(s, \cdot)\|_{\mathcal{C}^2} \leq \phi(t) - \phi(s), \qquad 0 \leq s < t \leq T$$

for some non-decreasing function ϕ. Observe that, if u is a $\mathcal{C}^1$ function, the solution of (1.1) is not affected by changing g on a set of times of measure zero. For simplicity, we shall thus assume that both g and ϕ are right continuous functions of time. By possibly replacing ϕ with

$$\tilde{\phi}(t) \doteq \begin{cases} 0 & \text{if } \ t < 0 , \\ 1 + t + \phi(t) & \text{if } \ 0 \leq t < T , \\ 2 + T + \phi(T) & \text{if } \ t \geq T , \end{cases}$$

it is not restrictive to assume that

$$(1.6) \qquad \phi(0+) - \phi(0-) \geq 1 , \quad \phi(T+) - \phi(T-) \geq 1 , \quad \dot{\phi}(t) \geq 1 \ \text{a.e.} .$$

By (1.6), the positive Radon measure $d\phi$ contains an atom at $t = 0$ and at $t = T$, and satisfies $d\phi \geq dx$, where dx denotes the standard Lebesgue measure. We can now state the main result of this paper.

THEOREM 1.1. *Consider a set of bounded, measurable control functions of the form* $\mathcal{U} \doteq \{u : [0, T] \mapsto [-M_1, M_1] \mid u \in \mathcal{C}^1\}$. *For* $u \in \mathcal{U}$, *call* $x(t, u)$ *the corresponding solution of the Cauchy problem* (1.1). *Then,* *under the assumptions* (1.2)-(1.6), *the* map $u \mapsto x(T, u)$ *satisfies*

$$(1.7) \qquad |x(T, u) - x(T, v)| \leq C \int_0^T |u(t) - v(t)| \, d\phi(t),$$

for some constant C *and all* $u, v \in \mathcal{U}$.

As a consequence, the map $x(T, u)$ can be uniquely extended by continuity to the closure of $\mathcal{U}$ in the space $L^1(d\phi)$. This provides a natural definition of solution of (1.1) also for a discontinuous control u,

$$x(T, u) \doteq \lim_{n \to \infty} x(T, v_n),$$

where $\{v_n\}_{n \geq 1}$ is any bounded sequence of $\mathcal{C}^1$ functions, tending to u in the space $L^1(d\phi)$.

REMARK 1.2. In the case where g is a piecewise smooth function of t, x, with finitely many jumps at times $0 = t_0 < t_1 < \ldots < t_n = T$, one can always construct a function ϕ such that (1.5) holds. Indeed, for suitable constants C_1, C_2, one can take

$$\phi(t) \doteq C_1 t + C_2 \cdot \sup \{k; \, t_k \leq t\}.$$

REMARK 1.3. Our results can be extended to systems of the form

$$\dot{x} = f(t, x, u) + g(t, x, u) \dot{u}.$$

Indeed, the dependence on u is easily removed by introducing an additional coordinate $x_0 = u$, with $\dot{x}_0 = \dot{u}$.

In the case where the vector fields f, g do not depend on time, solutions of the impulsive Cauchy problem (1.1) were studied in [2]. For a special class of Lagrangean systems with piecewise continuous dependence on a time-like variable, the impulsive control problem was recently considered in [6]. The present approach is simpler than [6], since it does not require any smoothing approximation of the vector field g.

The proof of Theorem 1.1 is given in the next two sections. We first introduce a suitable definition of solution of (1.1), valid when u lies in the set

$$\mathcal{U}' \doteq \{u : [0, T] \mapsto [-M_1, M_1] \mid u \text{ is piecewise constant and all of its jumps occur at times } t \neq 0, T \text{ where } \phi \text{ is continuous}\}.$$

For $u \in \mathcal{U}'$, we show that the inequality (1.7) holds, hence the map $u \mapsto x(T, u)$ can be continuously extended to the closure of $\mathcal{U}'$ in the space $L^1(d\phi)$. When $u \in \mathcal{C}^1$, this continuous extension coincides with the usual Carathéodory definition. Since the closures of $\mathcal{U}$ and $\mathcal{U}'$ coincide, the result will be proved.

2. Definition of generalized solutions and preliminary lemmas.

Let $k: [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ be a (time dependent) vector field, and fix a time $\tau \in [0, T]$. Denote by $t \mapsto \exp\{t k(\tau)\}\, \tilde{x}$ the solution of the Cauchy problem

$$\dot{x}(t) = k(\tau, x(t)), \qquad x(0) = \tilde{x}. \tag{2.1}$$

We assume that for every $x \in \mathbb{R}^n$, the map $t \mapsto k(t, x)$ is measurable and for every $t \in [0, T]$, the map $x \mapsto k(t, x)$ is continuously differentiable. Moreover, denote by $t \mapsto \Phi(t, k(\tau), \tilde{x})$ the fundamental matrix solution of the linear differential equation

$$\dot{v}(t) = D_x k(\tau, \exp\{t k(\tau)\}\, \tilde{x}) \cdot v(t), \tag{2.2}$$

with $\Phi(0, k(\tau), \tilde{x})$ the identity matrix. Here $D_x k(\tau, \cdot)$ represents the Jacobian matrix of first order partial derivatives of $k(\tau, \cdot)$ with respect to x.

The matrix $\Phi(t, k(\tau), \tilde{x})$ has the following properties.

LEMMA 2.1. *Let M_2 be a constant such that*

$$|D_x k(\tau, \exp\{t k(\tau)\}\, \tilde{x}) \cdot w| \leqslant M_2 |w|$$

for every $\tilde{x}$, $w \in \mathbb{R}^n$, $\tau \in [0, T]$ and $|t| \leqslant M_1$. Then $|\Phi(t, k(\tau), \tilde{x}) \cdot w| \leqslant |w| e^{M_2 |t|}$.

PROOF. Since $d/dt\, |\Phi(t, k(\tau), \tilde{x}) \cdot w| \leqslant M_2 |\Phi(t, k(\tau), \tilde{x}) \cdot w|$, by Gronwall's inequality $|\Phi(t, k(\tau), \tilde{x}) \cdot w| \leqslant |w| e^{M_2 |t|}$. ∎

LEMMA 2.2. *Let k be twice continuously differentiable w.r.t. x and let $\tau \in [0, T]$. Suppose that for any x, $y \in \mathbb{R}^n$*

$$|k(\tau, x) - k(\tau, y)| \leqslant L |x - y|$$

and $\|k(\tau,\cdot)\|_{C^2} \leqslant M$. *Then for any* $0 \leqslant t \leqslant M_1$ *and* $x_1, x_2, w \in \mathbb{R}^n$,

$$|\Phi(t, k(\tau), x_1)\cdot w - \Phi(t, k(\tau), x_2)\cdot w| \leqslant n^3 M M_1 |x_1 - x_2|\,|w|\, e^{2LM_1 + n^2 MM_1}.$$

PROOF. Let $\tau \in [0, T]$ and $x_1, x_2, w \in \mathbb{R}^n$. We put $v_1(t) = \Phi(t, k(\tau), x_1)\cdot w$ and $v_2(t) = \Phi(t, k(\tau), x_2)\cdot w$. For $i = 1$ and 2, $v_i(t)$ is the value at time t of the solution of the linear differential equation

$$\dot{v}_i(t) = D_x k(\tau, \exp\{t k(\tau)\}\, x_i)\cdot v_i(t), \qquad v_i(0) = w.$$

Observing that for any $v, x, y \in \mathbb{R}^n$,

$$|D_x k(\tau, x)\cdot v| \leqslant n^2 M |v|$$

and

$$|D_x k(\tau, x)\cdot v - D_x k(\tau, y)\cdot v| \leqslant n^3 M |x - y|\,|v|,$$

due to Lemma 2.1

$$\begin{aligned}
\frac{d}{dt}|v_1(t) - v_2(t)| &\leqslant \\
&\leqslant |D_x k(\tau, \exp\{t k(\tau)\}\, x_1)\cdot v_1(t) - D_x k(\tau, \exp\{t k(\tau)\}\, x_2)\cdot v_2(t)| \leqslant \\
&\leqslant |D_x k(\tau, \exp\{t k(\tau)\}\, x_1)\cdot v_1(t) - D_x k(\tau, \exp\{t k(\tau)\}\, x_1)\cdot v_2(t)| + \\
&+ |D_x k(\tau, \exp\{t k(\tau)\}\, x_1)\cdot v_2(t) - D_x k(\tau, \exp\{t k(\tau)\}\, x_2)\cdot v_2(t)| \leqslant \\
&\leqslant n^2 M |v_1(t) - v_2(t)| + n^3 M |x_1 - x_2|\,|w|\, e^{LM_1 + n^2 MM_1}.
\end{aligned}$$

Gronwall's inequality implies that

$$|v_1(t) - v_2(t)| \leqslant n^3 M M_1 |x_1 - x_2|\,|w|\, e^{LM_1 + 2n^2 MM_1}. \qquad ■$$

When $u \in \mathcal{U}'$, the corresponding generalized solution $x(t, u)$ of (1.1) can be defined in a straightforward manner. Indeed, let u have jumps at points t_i, with $0 < t_1 < \ldots < t_n < T$. In this case, $x(t, u)$ is the function which solves the differential equation

$$\dot{x}(t) = f(t, x(t)) \tag{2.3}$$

on each subinterval $]t_{i-1}, t_i[$, together with the boundary conditions

$$x(0) = \bar{x}, \qquad x(t_i+) = \exp\{(u(t_i+) - u(t_i-))\, g(t_i)\}\, x(t_i-), \qquad i = 1, \ldots, n. \tag{2.4}$$

To study the continuous dependence of these solutions on the control $u \in \mathcal{U}'$, it is convenient to introduce an alternative representation, in terms of a new variable ξ, which will remove the discontinuities due to the jumps in u.

Choose points c_i with $c_1 = 0, c_{n+1} = T$, such that $c_i < t_i < c_{i+1}$ for each $i = 1, \ldots, n$. Since u is constant outside the points t_i, on each subinterval $I_i = [c_i, c_{i+1}]$, the function $x(t, u)$ provides a solution to

$$(2.5) \qquad \dot{x}(t) = f(t, x(t)) + g(t_i, x(t))\, \dot{u}(t), \qquad c_i \leqslant t \leqslant c_{i+1}.$$

Defining the auxiliary variable $\xi(t) \doteq \exp\{-u(t)\, g(t_i)\}\, x(t)$, it is known [2,3] that ξ is an absolutely continuous function which satisfies

$$(2.6) \qquad \dot{\xi}(t) = F^*(t, t_i, \xi(t), u(t)), \qquad \text{a.e. on } [c_i, c_{i+1}],$$

where $F^* : [0, T] \times [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is defined by

$$(2.7) \quad F^*(t, \tau, \xi, u) = \Phi(-u, g(\tau), \exp\{u\, g(\tau)\}\, \xi) \cdot f(t, \exp\{u\, g(\tau)\}\, \xi).$$

For $u \in \mathcal{U}'$, the corresponding solution $t \mapsto x(t, u)$ can thus be obtained by setting

$$(2.8) \qquad x(t, u) = \exp\{u(t)\, g(t_i)\}\, \xi(t, u), \qquad t \in I_i,$$

where $t \mapsto \xi(t, u)$ is the piecewise continuous function such that

$$(2.9) \qquad \dot{\xi}(t) = F^*(t, t_i, \xi(t), u(t)), \qquad t \in I_i,$$

$$(2.10) \qquad \begin{cases} \xi(0) = \exp\{-u(0)\, g(t_1)\}\, \bar{x}, \\ \xi(c_i +) = \exp\{-u(c_i)\, g(t_i)\} (\exp\{u(c_i)\, g(t_{i-1})\}\, \xi(c_i -)). \end{cases}$$

The main advantage of the representation (2.9)-(2.10) is the following. The total variation of u, and hence of x, can be arbitrarily large. On the other hand, the total variation of ξ is related to the total variation of g, which by (1.5) is bounded in terms of ϕ. For this reason, it is convenient to study the solution of (1.1) in terms of the variable ξ, which is much better behaved than u or x.

From now on, we assume that f and g satisfy all the hypotheses in Theorem 1.1. The following lemma shows that the map F^* defined in (2.7) is Lipschitz continuous w.r.t. both variables ξ, u.

LEMMA 2.3. *There exists* $L_1 > 0$ *such that for any* $t, \tau \in [0, T]$, ξ_1, $\xi_2 \in \mathbb{R}^n$ *and* $|u_1|$, $|u_2| \leqslant M_1$,

$$(2.11) \qquad |F^*(t, \tau, \xi_1, u_1) - F^*(t, \tau, \xi_2, u_2)| \leqslant L_1(|\xi_1 - \xi_2| + |u_1 - u_2|).$$

PROOF. By (1.4), we can easily see that for any ξ, $w \in \mathbb{R}^n$ and $|t| \leqslant M_1$, $|D_x g(\tau, \exp\{tg(\tau)\}\,\xi)\cdot w| \leqslant n^2 M|w|$. By Lemma 2.1 and Lemma 2.2,

$$\begin{aligned}
&|F^*(t, \tau, \xi_1, u_1) - F^*(t, \tau, \xi_2, u_1)| \leqslant \\
&\qquad \leqslant |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_1)\cdot f(t, \exp\{u_1 g(\tau)\}\,\xi_1) - \\
&\qquad - \Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_1 g(\tau)\}\,\xi_1)| + \\
&\qquad + |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_2)\cdot ft, \exp\{u_1 g(\tau)\}\,\xi_1) - \\
&\qquad - \Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_1 g(\tau)\}\,\xi_2)| \leqslant \\
&\qquad \leqslant n^3 M^2 M_1 e^{2LM_1 + 2n^2 MM_1}|\xi_1 - \xi_2| + Le^{LM_1 + n^2 MM_1}|\xi_1 - \xi_2| = \\
&\qquad\qquad = C_1|\xi_1 - \xi_2|
\end{aligned}$$

and

$$\begin{aligned}
&|F^*(t, \tau, \xi_2, u_1) - F^*(t, \tau, \xi_2, u_2)| \leqslant \\
&\qquad \leqslant |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_1 g(\tau)\}\,\xi_2) - \\
&\qquad - \Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_2 g(\tau)\}\,\xi_2)| + \\
&\qquad + |\Phi(-u_1, g(\tau), \exp\{u_1 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_2 g(\tau)\}\,\xi_2) - \\
&\qquad - \Phi(-u_1, g(\tau), \exp\{u_2 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_2 g(\tau)\}\,\xi_2)| + \\
&\qquad + |\Phi(-u_1, g(\tau), \exp\{u_2 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_2 g(\tau)\}\,\xi_2) - \\
&\qquad - \Phi(-u_2, g(\tau), \exp\{u_2 g(\tau)\}\,\xi_2)\cdot f(t, \exp\{u_2 g(\tau)\}\,\xi_2)| \leqslant \\
&\qquad \leqslant LMe^{n^2 MM_1}|u_1 - u_2| + n^3 M^3 M_1 e^{LM_1 + 2n^2 MM_1}|u_1 - u_2| + \\
&\qquad\qquad + n^2 M^2 e^{n^2 MM_1}|u_1 - u_2| = C_2|u_1 - u_2|
\end{aligned}$$

where $C_1 = n^3 M^2 M_1 e^{2LM_1 + 2n^2 MM_1} + Le^{LM_1 + n^2 MM_1}$ and $C_2 = e^{n^2 MM_1}(n^3 M^3 M_1 e^{LM_1 + n^2 MM_1} + LM + n^2 M^2)$. Thus (2.11) holds for $L_1 = C_1 + C_2$. ■

LEMMA 2.4. *Let $\tilde{x} \in \mathbb{R}^n$ and $0 \leqslant \tau_1 < \tau_2 \leqslant T$. Then for any $t \in \mathbb{R}$,*

(2.12) $|\exp\{tg(\tau_1)\}\tilde{x} - \exp\{tg(\tau_2)\}\tilde{x}| \leqslant e^{L|t|}(\phi(\tau_2) - \phi(\tau_1))|t|$.

PROOF. Replacing t with $-t$, it is not restrictive to assume $t > 0$. Observing that

$$\frac{d}{dt}|\exp\{tg(\tau_1)\}\tilde{x} - \exp\{tg(\tau_2)\}\tilde{x}| \leqslant$$

$$\leqslant |g(\tau_1, \exp\{tg(\tau_1)\}\tilde{x}) - g(\tau_2, \exp\{tg(\tau_2)\}\tilde{x})| \leqslant$$

$$\leqslant |g(\tau_1, \exp\{tg(\tau_1)\}\tilde{x}) - g(\tau_1, \exp\{tg(\tau_2)\}\tilde{x}))| +$$

$$+ |g(\tau_1, \exp\{tg(\tau_2)\}\tilde{x}) - g(\tau_2, \exp\{tg(\tau_2)\}\tilde{x})| \leqslant$$

$$\leqslant L|\exp\{tg(\tau_1)\}\tilde{x} - \exp\{tg(\tau_2)\}\tilde{x}| + (\phi(\tau_2) - \phi(\tau_1)),$$

Gronwall's inequality implies

$$|\exp\{tg(\tau_1)\}\tilde{x} - \exp\{tg(\tau_2)\}\tilde{x}| \leqslant e^{L|t|}(\phi(\tau_2) - \phi(\tau_1))|t|. \quad \blacksquare$$

Let $t_1, t_2 \in [0, T]$ and $p, q, v \in \mathbb{R}^n$. From (1.4) and (1.5), we can easily see that

$$|(D_x g(t_1, p) - D_x g(t_2, p)) \cdot v| \leqslant n^2 |\phi(t_2) - \phi(t_1)| |v|, \tag{2.13}$$

$$|g(t_1, p) - g(t_1, q)| \leqslant n^2 M|p - q|, \tag{2.14}$$

$$|D_x g(t_1, p) \cdot v| \leqslant n^2 M|v| \tag{2.15}$$

and

$$|(D_x g(t_1, p) - D_x g(t_1, q)) \cdot v| \leqslant n^3 M|p - q| |v|. \tag{2.16}$$

We define a map

$$k(t_i, \tau) = g(t_i, (1 - \tau) q + \tau p), \quad \tau \in [0, 1], \quad i = 1, 2.$$

Then k is differentiable w.r.t. τ and we have

$$|g(t_2, p) - g(t_2, q) - g(t_1, p) + g(t_1, q)| = \tag{2.17}$$

$$= |k(t_2, 1) - k(t_2, 0) - k(t_1, 1) + k(t_1, 0)| = \left|\int_0^1 \frac{d}{d\tau}(k(t_2, \tau) - k(t_1, \tau))\, d\tau\right| =$$

$$= \left| \int_0^1 (D_x g(t_2, (1-\tau)\, q + \tau p) \cdot (p-q) - D_x g(t_1, (1-\tau)\, q + \tau p) \cdot (p-q))\, d\tau \right| \leqslant$$

$$\leqslant n^2 \, |\phi(t_2) - \phi(t_1)| \cdot |p-q| \, .$$

In the similar way, we have that

$$|(D_x g(t_2, p) - D_x g(t_2, q) - D_x g(t_1, p) + D_x g(t_1, q)) \cdot v| \leqslant \tag{2.18}$$

$$\leqslant n^3 \, |\phi(t_2) - \phi(t_1)| \, |p-q| \, |v| \, .$$

PROPOSITION 2.5. *Let $x_0, y_0 \in \mathbb{R}^n$ and let t_1 and t_2* be points on $[0, T]$. Define a map $K: [-M_1, M_1] \rightarrow \mathbb{R}^n$ by

$$K(s) = \tag{2.19}$$

$$= \exp\{-s\, g(t_2) \exp\{s\, g(t_1)\}\, x_0 - \exp\{-s\, g(t_2)\} \exp\{s\, g(t_1)\}\, y_0 \, .$$

Then there exists $B_1 > 0$ such that for any $s \in [-M_1, M_1]$,

$$|K(s)| \leqslant |x_0 - y_0| \, e^{B_1 |\phi(t_2) - \phi(t_1)|} \, . \tag{2.20}$$

PROOF. Let

$$p_1 = \exp\{s\, g(t_1)\}\, x_0 \, , \qquad p_2 = \exp\{-s\, g(t_2)\}\, p_1 \, ,$$

$$q_1 = \exp\{s\, g(t_1)\}\, y_0 \text{ and } q_2 = \exp\{-s\, g(t_2)\}\, q_1 \, .$$

Then

$$K(s) = p_2 - q_2$$

and

$$K'(s) = -g(t_2, p_2) + \Phi(-s, g(t_2), p_1) \cdot g(t_1, p_1) + g(t_2, q_2) -$$

$$- \Phi(-s, g(t_2), q_1) \cdot g(t_1, q_1) = -g(t_2, p_2) + g(t_2, q_2) +$$

$$+ \Phi(-s, g(t_2), \exp\{s\, g(t_2)\}\, p_2) \cdot g(t_1, \exp\{s\, g(t_2)\}\, p_2) -$$

$$- \Phi(-s, g(t_2), \exp\{s\, g(t_2)\}\, q_2) \cdot g(t_1, \exp\{s\, g(t_2)\}\, q_2) \, .$$

Define a map $H: [-M_1, M_1] \rightarrow \mathbb{R}^n$ by

$$H(s) = \Phi(-s, g(t_2), \exp\{s\, g(t_2)\}\, p_2) \cdot g(t_1, \exp\{s\, g(t_2)\}\, p_2) - \tag{2.21}$$

$$- \Phi(-s, g(t_2), \exp\{s\, g(t_2)\}\, q_2) \cdot g(t_1, \exp\{s\, g(t_2)\}\, q_2) \, .$$

Observing that for any $p \in \mathbb{R}^n$,

$$\frac{d}{ds}\Phi(-s, g(t_2), \exp\{s g(t_2)\} p)\cdot g(t_1, \exp\{s g(t_2)\} p) =$$

$$= -\Phi(-s, g(t_2), \exp\{s g(t_2)\} p)\cdot D_x g(t_2, \exp\{s g(t_2)\} p)\cdot$$

$$\cdot g(t_1, \exp\{s g(t_2)\} p) + \Phi(-s, g(t_2), \exp\{s g(t_2)\} p)\cdot$$

$$\cdot D_x g(t_1, \exp\{s g(t_2)\} p)\cdot g(t_2, \exp\{s g(t_2)\} p),$$

we have

$$(2.22) \quad H'(s) = \Phi(-s, g(t_2), \exp\{s g(t_2)\} p_2)\cdot$$

$$\cdot[g(t_2, \cdot), g(t_1, \cdot)](\exp\{s g(t_2)\} p_2) -$$

$$-\Phi(-s, g(t_2), \exp\{s g(t_2)\} q_2)\cdot[g(t_2, \cdot), g(t_1, \cdot)](\exp\{s g(t_2)\} q_2),$$

where $[g(t_2, \cdot), g(t_1, \cdot)](p) = D_x g(t_1, p)\cdot g(t_2, p) - D_x g(t_2, p)\cdot g(t_1, p)$. If there exists $B_2 > 0$ such that

$$(2.23) \quad |K'(s)| \leq B_2 |\phi(t_2) - \phi(t_1)| \cdot |p_2 - q_2|,$$

then by Gronwall's inequality

$$(2.24) \quad |K(s)| \leq |x_0 - y_0| e^{B_1 |\phi(t_2) - \phi(t_1)|},$$

where $B_1 = M_1 \cdot B_2$. We thus only have to show that inequality (2.23) holds for some $B_2 > 0$. Since

$$(2.25) \quad K'(s) = H(s) - g(t_2, p_2) + g(t_2, q_2) =$$

$$= H(s) - (g(t_1, p_2) - g(t_1, q_2)) +$$

$$+ (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) =$$

$$= H(s) - H(0) + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)) =$$

$$= \int_0^s H'(\tau)d\tau + (g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2))$$

and by (2.17)

$$|g(t_1, p_2) - g(t_1, q_2) - g(t_2, p_2) + g(t_2, q_2)| \leq n^2 |\phi(t_2) - \phi(t_1)| \cdot |p_2 - q_2|,$$

to claim inequality (2.23) we shall show that there exists $B_3 > 0$ such that for any $\tau \in [-M_1, M_1]$

$$|H'(\tau)| \leqslant B_3 \, |\phi(t_2) - \phi(t_1)| \, |p_2 - q_2| \, . \tag{2.26}$$

We fix $\tau \in [-M_1, M_1]$ and define maps $v_1, v_2 : [-M_1, M_1] \to \mathbb{R}^n$ by

$$v_1(\sigma) = \Phi(\sigma, g(t_2), \exp\{\tau g(t_2\} \, p_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp\{\tau g(t_2\} \, p_2)$$

and

$$v_2(\sigma) = \Phi(\sigma, g(t_2), \exp\{\tau g(t_2)\} \, q_2) \cdot [g(t_2, \cdot), g(t_1, \cdot)](\exp\{\tau g(t_2)\} \, q_2) \, .$$

Then

$$H'(\tau) = v_1(-\tau) - v_2(-\tau) \tag{2.27}$$

and v_1, v_2 satisfy

$$\begin{cases} \dfrac{d}{d\sigma} v_1(\sigma) = D_x g(t_2, \exp\{\sigma g(t_2)\} \, p_3) \cdot v_1(\sigma), \\ v_1(0) = [g(t_2, \cdot), g(t_1, \cdot)](p_3), \end{cases} \tag{2.28}$$

$$\begin{cases} \dfrac{d}{d\sigma} v_2(\sigma) = D_x g(t_2, \exp\{\sigma g(t_2)\} \, q_3) \cdot v_2(\sigma), \\ v_2(0) = [g(t_2, \cdot), g(t_1, \cdot)](q_3), \end{cases} \tag{2.29}$$

where $p_3 = \exp\{\tau g(t_2)\} \, p_2$ and $q_3 = \exp\{\tau g(t_2)\} \, q_2$. We compute a bound for $|v_1(0) - v_2(0)|$ to get

$$\begin{aligned} |v_1(0) - v_2(0)| &= |D_x g(t_1, p_3) \cdot g(t_2, p_3) - D_x g(t_2, p_3) \cdot g(t_1, p_3) - \\ &- D_x g(t_1, q_3) \cdot g(t_2, q_3) + D_x g(t_2, q_3) \cdot g(t_1, q_3)| = \\ &= \big| (D_x g(t_1, p_3) - D_x g(t_2, p_3)) \cdot (g(t_2, p_3) - g(t_2, q_3)) + \\ &+ (D_x g(t_2, q_3) - D_x g(t_2, p_3)) \cdot (g(t_1, p_3) - g(t_2, p_3)) + \\ &+ (D_x g(t_1, p_3) - D_x g(t_2, p_3) - D_x g(t_1, q_3) + D_x g(t_2, q_3)) \cdot g(t_2, q_3) + \\ &+ D_x g(t_2, q_3)(g(t_2, p_3) - g(t_1, p_3) - g(t_2, q_3) + g(t_1, q_3)) \big| \leqslant \\ &\leqslant 4n^4 M |\phi(t_2) - \phi(t_1)| \, |p_3 - q_3| \leqslant 4n^4 M e^{LM_1} |\phi(t_2) - \phi(t_1)| \, |p_2 - q_2| \, . \end{aligned} \tag{2.30}$$

By considering $-\sigma$ instead of σ, we assume that $\sigma \geqslant 0$. Observing that

$$(2.31)\quad \begin{aligned} |v_1(\sigma)| &\leqslant |v_1(0)|e^{n^2M\sigma} \leqslant \\ &\leqslant |D_x g(t_1, p_3)\cdot g(t_2, p_3) - D_x g(t_2, p_3)\cdot g(t_1, p_3)|e^{n^2MM_1} \leqslant \\ &\leqslant |D_x g(t_1, p_3)\cdot g(t_2, p_3) - D_x g(t_2, p_3)\cdot g(t_2, p_3)| + \\ &+ |D_x g(t_2, p_3)\cdot g(t_2, p_3) - D_x g(t_2, p_3)\cdot g(t_1, p_3)|) e^{n^2MM_1} \leqslant \\ &\leqslant 2n^2Me^{n^2MM_1}|\phi(t_2) - \phi(t_1)|, \end{aligned}$$

we have a bound for $|v_1(\sigma) - v_2(\sigma)|$ as

$$\begin{aligned} &|v_1(\sigma) - v_2(\sigma)| = \\ &= \Bigg| \int_0^\sigma (D_x g(t_2, \exp\{\eta g(t_2)\} p_3)\cdot v_1(\eta) - D_x g(t_2, \exp\{\eta g(t_2)\} q_3)\cdot v_1(\eta) + \\ &+ D_x g(t_2, \exp\{\eta g(t_2)\} q_3)\cdot v_1(\eta) - \\ &- D_x g(t_2, \exp\{\eta g(t_2)\} q_3)\cdot v_2(\eta))\, d\eta + (v_1(0) - v_2(0)) \Bigg| \leqslant \\ &\leqslant \int_0^\sigma n^3 M|\exp\{\eta g(t_2)\} p_3 - \exp\{\eta g(t_2)\} q_3|\,|v_1(\eta)|\, d\eta + \\ &+ \int_0^\sigma n^2 M|v_1(\eta) - v_2(\eta)|\, d\eta + |v_1(0) - v_2(0)| \leqslant \\ &\leqslant 2n^5M^2M_1 e^{2LM_1 + n^2MM_1}|\phi(t_2) - \phi(t_1)|\,|p_2 - q_2| + \\ &\qquad + 4n^4Me^{LM_1}|\phi(t_2) - \phi(t_1)|\,|p_2 - q_2| + \int_0^\sigma n^2M|v_1(\eta) - v_2(\eta)|\, d\eta\,. \end{aligned}$$

By Gronwall's inequality,

(2.32) $|v_1(\sigma) - v_2(\sigma)| \leqslant B_3\,|p_2 - q_2|\,|\phi(t_2) - \phi(t_1)|$ for any $0 \leqslant \sigma \leqslant M_1$,

where $B_3 = (2n^5M^2M_1e^{2LM_1 + n^2MM_1} + 4n^4Me^{LM_1})e^{n^2MM_1}$. By (2.27),

$$(2.33)\qquad |H'(\tau)| \leqslant B_3\,|p_2 - q_2|\,|\phi(t_2) - \phi(t_1)|\,.$$

We thus have that for any $s \in [-M_1, M_1]$

$$\left| \int_0^s H'(\tau)\, d\tau \right| \leq B_4 \,|p_2 - q_2|\, |\phi(t_2) - \phi(t_1)| \,, \tag{2.34}$$

where $B_4 = B_3 M_1$. By (2.25)

$$|K'(s)| \leq B_2 \,|p_2 - q_2|\, |\phi(t_2) - \phi(t_1)| \,,$$

where $B_2 = B_4 + n^2$. As a consequence, the proposition is proved. ■

3. Proof of the theorem.

Before proving that (1.7) holds for $u, v \in \mathcal{U}$, we show that it holds for $u, v \in \mathcal{U}'$. Let $u, v \in \mathcal{U}'$. Recall that the generalized solutions $x(t, u)$ and $x(t, v)$ can be defined in terms of (2.8)-(2.10). Assume that either u or v jumps at t_i where

$$0 < t_1 < t_2 < \ldots < t_n < T \,,$$

moreover, we may assume that u and v are left continuous since ϕ is continuous at each t_i and the integral $\int_0^T |u(t) - v(t)|\, d\phi(t)$ is not affected by changing the value $|u(t_i) - v(t_i)|$. Let $c_1 = 0$ and $d_n = T$. We choose $c_i, d_{i-1} \in (t_{i-1}, t_i]$ with $c_i < d_{i-1}$ for $i = 2, \ldots, n$. Define the time intervals $I_i = [c_i, d_i]$, $i = 1, \ldots, n$. Since u and v are left continuous, it is not restrictive to assume that $d_{i-1} = t_i$. Define

$$X(t) = \exp\{-u(t)\, g(t)\}\, x(t, u) \,,$$

$$Y(t) = \exp\{-v(t)\, g(t)\}\, x(t, v) \,.$$

Since

$$|x(T, u) - x(T, v)| = |\exp\{u(T)\, g(T)\}\, X(T) - \exp\{v(T)\, g(T)\}\, Y(T)| \,,$$

we need to estimate the increase of $|X(t_i) - Y(t_i)|$ as i increases. On each interval I_i, we define

$$X_i(t) = \exp\{-u(t)\, g(t_i)\}\, x(t, u) \,, \qquad Y_i(t) = \exp\{-v(t)\, g(t_i)\}\, x(t, v) \,.$$

By (2.6), on the interval I_i, X_i and Y_i satisfy the differential equations

$$\dot{X}_i(t) = F^*(t, t_i, X_i(t), u(t)), \qquad \dot{Y}_i(t) = F^*(t, t_i, Y_i(t), v(t)),$$

respectively. Due to Lemma 2.3, on the interval $[t_i, t_{i+1}]$ we have

$$\frac{d}{dt}|X_i(t) - Y_i(t)| \leqslant L_1(|X_i(t) - Y_i(t)| + |u(t) - v(t)|). \tag{3.1}$$

We thus have an estimate by Gronwall's inequality

$$|X_i(t_{i+1}) - Y_i(t_{i+1})| \leqslant \tag{3.2}$$

$$\leqslant |X_i(t_i) - Y_i(t_i)|\, e^{L_1(t_{i+1} - t_i)} + L_1 e^{L_1 T} \int_{t_i}^{t_{i+1}} |u(s) - v(s)|\, ds .$$

Next, we estimate the difference between

$$|X(t_{i+1}) - Y(t_{i+1})| \quad \text{and} \quad |X_i(t_{i+1}) - Y_i(t_{i+1})| .$$

If we put $x_0 = X_i(t_{i+1})$ and $y_0 = Y_i(t_{i+1})$, then

$$|X(t_{i+1}) - Y(t_{i+1})| = \tag{3.3}$$

$$= |\exp\{-u(t_{i+1})\, g(t_{i+1})\} \exp\{u(t_{i+1})\, g(t_i)\}\, x_0 -$$

$$- \exp\{-v(t_{i+1})\, g(t_{i+1})\} \exp\{v(t_{i+1})\, g(t_i)\}\, y_0| \leqslant$$

$$\leqslant |\exp\{-u(t_{i+1})\, g(t_{i+1})\} \exp\{u(t_{i+1})\, g(t_i)\}\, x_0 -$$

$$- \exp\{-u(t_{i+1})\, g(t_{i+1})\} \exp\{u(t_{i+1})\, g(t_i)\}\, y_0| +$$

$$+ |\exp\{-u(t_{i+1})\, g(t_{i+1})\} \exp\{u(t_{i+1})\, g(t_i)\}\, y_0 +$$

$$- \exp\{-v(t_{i+1})\, g(t_{i+1})\} \exp\{v(t_{i+1})\, g(t_i)\}\, y_0| \doteq E_1 + E_2 .$$

If in (3.3) $E_2 \leqslant C_3(\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$ for some $C_3 > 0$, then by Proposition 2.5

$$|X(t_{i+1}) - Y(t_{i+1})| \leqslant \tag{3.4}$$

$$\leqslant |X_i(t_{i+1}) - Y_i(t_{i+1})|\, e^{B_1(\phi(t_{i+1}) - \phi(t_i))} + C_3 \int_{t_i}^{t_{i+1}} |u(s) - v(s)|\, d\phi(s).$$

By (3.2) and (3.4),

(3.5) $|X(t_{i+1}) - Y(t_{i+1})| \leqslant$

$$\leqslant e^{B_1(\phi(t_{i+1})-\phi(t_i))}\left(|X(t_i)-Y(t_i)|e^{L_1(t_{i+1}-t_i)}+L_1 e^{L_1 T}\int_{t_i}^{t_{i+1}}|u(s)-v(s)|\,ds\right)+$$

$$+C_3\int_{t_i}^{t_{i+1}}|u(s)-v(s)|\,d\phi(s).$$

Observing that on the interval $[0, t_1]$ equation (1.1) is $\dot{x}=f(t, x)$, $x(t_1, u)=x(t_1, v)$ and

(3.6) $$|X(t_1)-Y(t_1)| \leqslant M|u(0)-v(0)|\,.$$

Due to (3.5) and (3.6), we can use the induction to obtain

$$|X(T)-Y(T)| \leqslant e^{B_1(\phi(T)-\phi(0))+L_1 T}|X(t_1)-Y(t_1)|+$$

$$+L_1 e^{2L_1 T+B_1(\phi(T)-\phi(0))}\int_0^T|u(s)-v(s)|\,d\phi(s)+$$

$$+C_3 e^{B_1(\phi(T)-\phi(0))+L_1 T}\int_0^T|u(s)-v(s)|\,ds \leqslant e^{B_1(\phi(T)-\phi(0))+L_1 T}\cdot$$

$$\cdot\left(M|u(0)-v(0)|+(L_1 e^{L_1 T}+C_3)\int_0^T|u(s)-v(s)|\,d\phi(s)\right)\leqslant$$

$$\leqslant C_4\left(\int_0^T|u(s)-v(s)|\,d\phi(s)\right),$$

where $C_4=e^{B_1(\phi(T)-\phi(0))+L_1 T}(M+L_1 e^{L_1 T}+C_3)$. We can estimate $|x(T, u)-x(T, v)|$:

$$|x(T, u)-x(T, v)|=|\exp\{u(T)\,g(T)\}\,X(T)-\exp\{v(T)\,g(T)\}\,Y(T)|\leqslant$$

$$\leqslant|\exp\{u(T)\,g(T)\}\,X(T)-\exp\{u(T)\,g(T)\}\,Y(T)|+$$

$$+|\exp\{u(T)\,g(T)\}\,Y(T)-\exp\{v(T)\,g(T)\}\,Y(T)|\leqslant$$

$$\leqslant e^{LM_1}|X(T)-Y(T)|+M|u(T)-v(T)|\leqslant C_5\int_0^T|u(s)-v(s)|\,d\phi(s),$$

where $C_5=C_4 e^{LM_1}+M$. Hence (1.7) holds for u, $v\in\mathcal{U}'$.

Now we need to show that, in (3.3), $E_2 \leq C_3(\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$ for some constant $C_3 > 0$. By Lemma 2.4,

$$|\exp\{u(t_{i+1})\,g(t_i)\}\,y_0 - \exp\{(u(t_{i+1}) - v(t_{i+1}))\,g(t_{i+1})\}\exp\{v(t_{i+1})\,g(t_i)\}\,y_0| \leq$$

$$\leq e^{2LM_1}(\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$$

and

$$|\exp\{-u(t_{i+1})\,g(t_{i+1})\}\exp\{u(t_{i+1})\,g(t_i)\}\,y_0 - \exp\{-v(t_{i+1})\,g(t_{i+1})\}\exp\{v(t_{i+1})\,g(t_i)\}\,y_0| =$$

$$= |\exp\{-u(t_{i+1})\,g(t_{i+1})\}\exp\{u(t_{i+1})\,g(t_i)\}\,y_0 - \exp\{-u(t_{i+1})\,g(t_{i+1})\}\exp\{(u(t_{i+1}) - v(t_{i+1}))\,g(t_{i+1})\}\cdot$$

$$\cdot\exp\{v(t_{i+1})\,g(t_i)\}\,y_0| \leq e^{3LM_1}(\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|\,.$$

Therefore $E_2 \leq C_3(\phi(t_{i+1}) - \phi(t_i))|u(t_{i+1}) - v(t_{i+1})|$ for $C_3 = e^{3LM_1}$.

Next, we claim that (1.7) holds for $u, v \in \mathcal{U}$. Suppose that for any $w \in \mathcal{U}$, there exists a sequence $\{w_n\}$ in $\mathcal{U}'$ such that $w_n \to w$ in $L^1(d\phi)$ and $x(T, w_n) \to x(T, w)$ as $n \to \infty$. For u and $v \in \mathcal{U}$, we have sequences $\{u_n\}$ and $\{v_n\}$ so that $u_n \to u$, $v_n \to v$ in $L^1(d\phi)$, $x(T, u_n) \to x(T, u)$ and $x(T, v_n) \to x(T, v)$ as $n \to \infty$. We thus have that for any $n \in \mathbb{N}$

$$(3.7) \qquad |x(T, u) - x(T, v)| \leq$$

$$\leq |x(T, u) - x(T, u_n)| + |x(T, v) - x(T, v_n)| + |x(T, u_n) - x(T, v_n)| \leq$$

$$\leq |x(T, u) - x(T, u_n)| + |x(T, v) - x(T, v_n)| + C\int_0^T |u_n(s) - v_n(s)|\,d\phi(s)$$

and take $n \to \infty$ in (3.7) to get

$$|x(T, u) - x(T, v)| \leq C\int_0^T |u(s) - v(s)|\,d\phi(s)\,.$$

Hence we only have to show that for any $w \in \mathcal{U}$, there exists a sequence $\{w_n\}$ in $\mathcal{U}'$ such that $w_n \to w$ in $L^1(d\phi)$ and $x(T, w_n) \to x(T, w)$ as $n \to \infty$. Let $w \in \mathcal{U}$. We can construct a sequence $\{w_n\}$ in $\mathcal{U}'$ such that $w_n \to w$ in $L^1(d\phi)$ and $w_n(t) = \sum_{i=1}^{\delta(n)} a_i^n \chi_{I_i^n}(t)$, where $I_1^n = [a_1^n, b_1^n]$, $I_i^n =$

$=(a_i^n, b_i^n]$ for $i=2, \ldots, \delta(n)$,

$$0=a_1^n<b_1^n=a_2^n<b_2^n<\ldots<b_{\delta(n)-1}^n=a_{\delta(n)}^n<b_{\delta(n)}^n=T \tag{3.8}$$

and

$$b_i^n-a_i^n<\frac{1}{2n}\,. \tag{3.9}$$

For $n\in\mathbb{N}$ and $i=2, \ldots, \delta(n)$, we choose $c_i^n\in(a_i^n, b_i^n)$. Put $c_1^n=0$ and $c_{\delta(n)+1}^n=T$. Define the time intervals $J_1^n=[c_1^n, c_2^n]$ and $J_i^n=(c_i^n, c_{i+1}^n]$ for $i=2, \ldots, \delta(n)$.

Before proving that $\lim_{n\to\infty} x(T, w_n)=x(T, w)$, we observe that for a control function $\tilde{w}$, if the solution or the generalized solution of the initial value problem

$$\dot{x}(t)=f(t, x)+g(b_i^n, x)\dot{\tilde{w}}(t) \qquad \text{for } t\in J_i^n\,, \quad i=1, \ldots, \delta(n) \text{ and } x(0)=\bar{x} \tag{3.10}$$

exists, then we denote by $y_n(t, \tilde{w})$ the solution or the generalized solution of equation (3.10) corresponding to a control function $\tilde{w}$. If $\tilde{w}\in\mathcal{U}$, then $y_n(t, \tilde{w})$ is the usual solution of (3.10). If $\xi(t, \tilde{w})$ is the solution of the differential equation such that on each interval J_i^n,

$$\begin{cases}\dot{\xi}(t)=F^*(t, b_i^n, \xi(t), \tilde{w}(t))\,,\\ \xi(c_i^n)=\exp\{-\tilde{w}(c_i^n)\,g(b_i^n)\}\,y_n(c_i^n, \tilde{w})\,,\end{cases} \tag{3.11}$$

then $y_n(t, \tilde{w})$ also satisfies that

$$y_n(t, \tilde{w})=\exp\{\tilde{w}(t)\,g(b_i^n)\}\,\xi(t, \tilde{w})\,, \qquad t\in J_i^n\,. \tag{3.12}$$

On the other hand, if $\tilde{w}\in\mathcal{U}'$, then $y_n(t, \tilde{w})$ is inductively defined by (3.11) and (3.12), in this case $y_n(t, \tilde{w})=x(t, \tilde{w})$ is the generalized solution of (1.1) corresponding to $\tilde{w}$.

Simple computation yields that

$$\lim_{n\to\infty} y_n(T, w)=x(T, w)\,, \tag{3.13}$$

when we take $y_n(t, w)$ as a usual Carathéodory solution of (3.10) corresponding to w. By Theorem 5 in [2],

$$\lim_{n\to\infty} |y_n(T, w_n)-y_n(T, w)|=0\,, \tag{3.14}$$

when $y_n(t, w)$ is defined by (3.11) and (3.12) corresponding to w. As a consequence,

$$\lim_{n \to \infty} x(T, w_n) = x(T, w)$$

and (1.7) holds for $u, v \in \mathcal{U}$. ■

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