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Free Products with Amalgamation of Finite Groups and Finite Outer Automorphism Groups of Free Groups.

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ABSTRACT - Our main result is a characterization of groups which contain a free product with amalgamation of two finite groups as a subgroup of finite index. As application, we present a method to construct maximal finite subgroups (that is not contained in a larger finite subgroup) of the outer automorphism group $Out F_r$ of a free group F_r. We construct these groups in a geometric way as automorphism groups of finite graphs, and algebraically as finite quotients of free products with amalgamation of finite groups.

1. - Introduction.

It has been a problem of considerable interest in geometric group theory and topology to characterize the finite extensions of groups belonging to various reasonable classes of groups. For example, every torsionfree extension of a free group or of the fundamental group of a closed surface or a Haken-3-manifold is again of the same type; also, arbitrary finite extensions of such groups can be characterized. Our main result is the following characterization of finite extensions of certain free products with amalgamation of two finite groups.

THEOREM 1. *Let $E_0 = A *_U B$ be a nontrivial free product with amalgamation of two finite groups where U has different indices p and*

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q in A resp. B (we call E_0 an amalgam of index type (p, q) or just a (p, q)-amalgam). Suppose that U is a maximal finite subgroup in both A and B (for example if p and q are prime numbers). Let E be a group containing E_0 as a subgroup of finite index. Then $E = \tilde{A} *_{\tilde{U}} \tilde{B}$ where A, B resp. U are subgroups of the same index in $\tilde{A}$, $\tilde{B}$ resp. $\tilde{U}$ such that $A \cap \tilde{U} = B \cap \tilde{U} = U$. In particular, also E is a (p, q)-amalgam.

At the end of section 3 we give some examples showing that Theorem 1 does not remain true in this strong form if p and q are not different or if U is not maximal in A or B.

As an application of Theorem 1, we present a method how to recognize and construct maximal finite subgroups of the outer automorphism group of a free group. Let F_r denote the free group of rank $r > 1$ and $Out\,F_r := Aut\,F_r/Inn\,F_r$ its outer automorphism group (automorphisms modulo inner automorphisms). The maximal order of a finite subgroup of $Out\,F_r$ is $2^r r!$, for $r > 2$, and for $r > 3$ there is, up to conjugation, a unique subgroup of $Out\,F_r$ of this order, isomorphic to the semidirect product $(\mathbf{Z}_2)^r \ltimes S_r$ ([WZ]). The possible isomorphism types of finite subgroups of $Out\,F_r$ have been determined in [M]. Up to conjugation, the *maximal* finite subgroups (that is not contained in a larger finite subgroup) of $Out\,F_3$ have been determined in [Z1]; the method can be applied to other small values of r but in general not much is known about maximal finite subgroups of $Out\,F_r$.

Finite subgroups of $Out\,F_r$ are most conveniently given in one of the two following ways:

i) in a geometric way as automorphism groups of finite graphs; each finite subgroup of $Out\,F_r$ comes from an action of the group on a finite graph, by taking the induced action on the fundamental group of the graph; conversely each automorphism group of a finite graph of rank $r > 1$ (the rank of its free fundamental group) without vertices of valence one injects into $Out\,F_r$ and thus defines a finite subgroup of $Out\,F_r$, see [WZ], [Z1]; for example, the above finite subgroup of maximal possible order $2^r r!$ comes from the full automorphism group of the graph with a single vertex and r edges (a bouquet of r circles);

ii) in an algebraic way as finite quotients of fundamental groups of finite (effective) graphs of finite groups, by torsionfree subgroups (which are free groups); in fact each finite subgroup of $Out\,F_r$ defines a finite effective extension of the free group F_r, and by [KPS1] the finite extensions of f.g. free group are exactly the fundamental groups of finite graphs of finite groups (see also [Z2]); conversely, every finite effective extension E of a free group F_r defines, by taking conjugations of F_r with elements in E, a finite subgroup of $Out\,F_r$ (isomorphic to E/F_r).

The link between the two presentations is given by the Bass-Serre theory of groups acting on trees which associates to each group acting on a tree (or on a graph) a graph of groups by taking the quotient graph and associating stabilizers of the action to its vertices and edges; conversely each graph of groups comes in this way from an action on a graph (see [S], [Z2]).

Using Theorem 1 we are able to construct various infinite series of maximal finite subgroups of $Out\,F_r$. If the full automorphism group of a finite bipartite (p, q)-valent graph, where p and q are different prime numbers, or of a finite p-valent graph operates transitively on the edges resp. on the oriented edges then it induces a maximal finite subgroup of $Out\,F_r$. For example, the automorphism groups of the complete graph on n vertices and of the graph with 2 vertices and $r+1$ connecting edges define maximal finite subgroups of $Out\,F_r$; these groups are isomorphic to the symmetric group S_n, where $r = (n-1)(n-2)/2$, resp. to the direct product $S_r \times Z_2$.

As another application, suppose that the finite group G_0 is a surjective image, with torsionfree kernel isomorphic to F_r, of the modular group $PSL(2, Z) \cong Z_2 * Z_3$ (ismorphic to the fundamental group of the obvious graph of groups with a single edge). Then we show that the corresponding subgroup G_0 of $Out\,F_r$ is contained in a unique maximal finite subgroup G of $Out\,F_r$; moreover the index of G_0 in G is small being of the form 2^i where $0 \leq i \leq 4$. Here we use in a crucial way also the classification of all effective free products with amalgamation of two finite groups such that the amalgam has indices 3 and 3 resp. 2 and 3 in the two factors ([G], see also [DM]).

2. - Preliminaries.

By [KPS1] (see also [Z2, Theorem 2.3.1]) any finite extension of a f.g. free group is isomorphic to the fundamental group $\pi_1(\Gamma, \mathcal{G})$ of a finite graph of finite groups $(\Gamma, \mathcal{G})$. Here Γ denotes a finite graph, and to the vertices and edges of Γ are associated finite groups (the *vertex* resp. *edge groups* of the graph of groups) together with monomorphisms (inclusions) of the edge groups into adjacent vertex groups. We call a graph of groups *minimal* if it has no trivial edges; an edge is called *trivial* if it has distinct vertices and the monomorphism from the edge group into one of the two adjacent vertex groups is an isomorphism. By contracting trivial edges, we can assume that the finite graph of groups $(\Gamma, \mathcal{G})$ is minimal (this does not change the fundamental group of the graph of groups which is an iterated free product with amalgamation and HNN-extension of the vertex groups over the edge groups).

The Bass-Serre theory of groups acting on trees associates to any graph of groups $(\Gamma, \mathcal{G})$ an action without inversions of its fundamental group $E := \pi_1(\Gamma, \mathcal{G})$ on a tree T such that the action has $(\Gamma, \mathcal{G})$ as its associated graph of groups.

To any action without inversions of a group E on a tree T is associated a graph of groups in the following way. The underlying graph is $\Gamma := T/E$. Suppose for simplicity that Γ is a tree (this will be sufficient for the cases we consider in the present paper). Then Γ can be lifted isomorphically to T, and we associate to the edges and vertices of Γ their stabilizers in E. Note that we have also canonical inclusions of the edge groups into adjacent vertex groups. The result is a graph of groups $(\Gamma, \mathcal{G})$, and the main result of the Bass-Serre theory says that $E \cong \pi_1(\Gamma, \mathcal{G})$.

In the following, we shall write also $T/E = (\Gamma, \mathcal{G})$ meaning that the graph of groups $(\Gamma, \mathcal{G})$ is associated to the action of E on T.

LEMMA 1. *Let E be the fundamental group of a finite minimal graph of finite groups, associated to an action of E on a tree T. Then each maximal finite subgroup of E has a unique fixed point in T. The conjugacy classes of maximal finite subgroups of E correspond bijectively to the vertex groups of the graph of groups.*

PROOF. Each finite group acting on a tree has a fixed point (consider the invariant subtree generated by an orbit and delete externel edges in an equivariant way). A maximal finite subgroup M of E has a unique fixed point in T because otherwise the edges of the unique edge path in the tree T connecting two different fixed points would also be fixed by M; then M would occur also as an edge group and the associated graph of groups would not be minimal. By a similar argument, each vertex group is a maximal finite subgroup of M. This finishes the proof of the Lemma.

The *Euler number* of a finite graph of finite groups $(\Gamma, \mathcal{G})$ is definied as

$$\chi(\Gamma, \mathcal{G}) := \sum 1/|G_v| - \sum 1/|G_e|$$

where the sum is extended over all vertex groups G_v resp. edge groups G_e of $(\Gamma, \mathcal{G})$.

By [SW, Lemma 7.4], [Z2, Proposition 2.1.1] the fundamental group of a finite graph of finite groups $(\Gamma, \mathcal{G})$ has a free group F_r as a subgroup of some finite index n. Then one has the formula

$$(1-r)=n\chi(\Gamma,\mathcal{G}),$$

see [KPS1], [Z2, Prop. 2.3.3]. In particular, the Euler number depends only on the fundamental group of $(\Gamma,\mathcal{G})$ and not on $(\Gamma,\mathcal{G})$ itself.

3. - Proof of Theorem 1.

Now let E_0 and E be as in Theorem 1. By [SW, Lemma 7.4], [Z2, Prop. 2.3.3] the group E_0 is a finite extension of a f.g. free group, and consequently also E is a finite extension of a f.g. free group. Let $(\Gamma,\mathcal{G})$ be a finite minimal graph of finite groups such that $E=\pi_1(\Gamma,\mathcal{G})$. Then we have an action without inversions of E on a tree T such that $T/E = (\Gamma,\mathcal{G})$. Now also the action of $E_0\subset E$ defines a finite graph of finite groups $(\Gamma_0,\mathcal{G}_0):=T/E_0$ such that $E_0=\pi_1(\Gamma_0,\mathcal{G}_0)$. We have projections

$$T\to\Gamma_0\to\Gamma.$$

The abelianized group $E_{ab}=E/[E,E]$ is finite because the same is true for E_0 and E_0 has finite index in E. It follows that the graph Γ is a tree (otherwise in $E=\pi_1(\Gamma,\mathcal{G})$ there would be at least one HNN-generator and therefore E_{ab} would be infinite). For the same reason also Γ_0 is a tree. Therefore we can lift Γ isomorphically to a subtree of Γ_0, and then Γ_0 isomorphically to a subtree of T. We denote the lifted trees by the same symbols $\Gamma\subset\Gamma_0\subset T$ and use these lifted trees for the construction of the graphs of groups $(\Gamma,\mathcal{G})$ and $(\Gamma_0,\mathcal{G}_0)$.

By assumption the graph of groups $(\Gamma,\mathcal{G})$ is minimal; in particular it has no trivial vertices of valence one, that is vertices of valence one such that the vertex group coincides with the single adjacent edge group. This implies that the tree T has no vertices of valence one, and then also the graph of groups $(\Gamma_0,\mathcal{G}_0)$ has no trivial vertices of valence one.

In general, the graph of groups $(\Gamma_0,\mathcal{G}_0)$ will not be minimal; denote by $(\Gamma_1,\mathcal{G}_1)$ the minimal graph of groups obtained by contracting all trivial edges of $(\Gamma_0,\mathcal{G}_0)$, with $E_0=\pi_1(\Gamma_1,\mathcal{G}_1)$. Note that the vertices of valence one in Γ_0 remain different vertices in Γ_1 because they are non-trivial. By Lemma 1, $E_0=A*_U B$ has exactly two conjuacy classes of maximal finite subgroups (represented by A and B), and consequently Γ_1 is a graph with a single edge (applying Lemma 1 to $E_0 = \pi_1(\Gamma_1,\mathcal{G}_1)$). It follows that Γ_0 has exactly two vertices of valence one and therfore is a subdivided segment. The two vertex groups of $(\Gamma_1,\mathcal{G}_1)$ are conjugates $\bar{A}$ and $\bar{B}$ of A and B, respectively. These conjugates are the stabilizers of the two vertices of valence one in $\Gamma_0 \subset T$ which we denote by a resp $\bar{b}$; by choosing $\Gamma_0 \subset T$ appropriately we can assume $A = \bar{A}$. By Lemma 1 resp. its proof a and $\bar{b}$ are the unique fixed points of A resp. $\bar{B}$ in T (because the two vertices of valence one in $(\Gamma_0, \mathcal{G}_0)$ are nontrivial).

In particular, we get a presentation

$$E_0 = \pi_1(\Gamma_1, \mathcal{G}_1) = A *_{\bar{U}} \bar{B},$$

where $\bar{U} = A \cap \bar{B}$.

For each vertex of valence two of Γ_0 («interior vertex»), at least one of the indices of the two adjacent edge groups in the corresponding vertex group is equal to one (otherwise the vertex would survive in Γ_1 together with the two vertices of valence one in Γ_0).

Now consider the projection $\pi: \Gamma_0 \to \Gamma$.

LEMMA 2. *The sum of the indices of all adjacent edge groups in a given vertex group is preserved by the projection π. Moreover if π is injective on this set of edges (a local homeomorphism at the vertex) then π preserves the index of each edge group in the given vertex group.*

PROOF. The Lemma follows from the following two observations. For both Γ_0 and Γ, the index of an edge group in a vertex group gives the number of edges in T equivalent to the given edge under E_0 resp. E and adjacent to the given vertex. The sum of the indices at a vertex is equal to the number of all edges in T adjacent to the vertex.

Continuing with the proof of the Theorem it follows now from Lemma 2 that all edges of Γ_0 are mapped to the same edge of Γ (because one of the indices at each interior vertex of $(\Gamma_0, \mathcal{G}_0)$ is equal to one and $(\Gamma, \mathcal{G})$ is minimal), and consequently Γ has only a single edge. Thus

$$E = \pi_1(\Gamma, \mathcal{G}) = \tilde{A} *_{\tilde{U}} \tilde{B}$$

is also a free product with amalgamation of two finite groups.

Consider the presentations

$$E_0 = A *_U B = A *_{\bar{U}} \bar{B}.$$

Because the Euler number of E_0 does not depend on the special presentation, U and $\bar{U}$ have the same order.

The conjugate B of $\bar{B}$ fixes a unique vertex b in T; recall that a is the vertex of $\Gamma_0 \subset T$ fixed by A. The vertices a and b of T are connected by a unique edge path γ_0 in the tree T. Note that $U = A \cap B$ fixes all edges of γ_0. Moreover, as U is a maximal subgroup of A (note that we use this here for the first time), it follows that the stabilizer of the first edge of γ_0 (that is the edge which has a as a vertex) is equal to U.

Now also Γ_0 is an edge path in T from a to the unique fixed point $\bar{b}$ of $\bar{B}$, and $\bar{U} = A \cap \bar{B}$ fixes each edge of Γ_0. Because all edges emanating from a are equivalent under the action of E_0 and U and $\bar{U}$ have the same order it follows that U is conjugate to $\bar{U}$ in E_0. Because also $\bar{B}$ and B are conjugate we are now in the position to apply Lemma 1 in [KPS2] which says that, under the above circumstances, there exists an element x in A such that $xBx^{-1} = \bar{B}$ and $xUx^{-1} = \bar{U}$. In particular we have $x(a) = a$, $x(b) = \bar{b}$ and $x(\gamma_0) = \Gamma_0$. Therefore we can assume that $\gamma_0 = \Gamma_0$, $\bar{B} = B$ and $\bar{U} = U$, and consequently

$$\pi_1(\Gamma_1, \mathcal{G}_1) = A *_U B.$$

Now U is maximal also in B; this implies that all indices at the interior vertices of $(\Gamma_0, \mathcal{G}_0)$ are equal to one (contract $(\Gamma_0, \mathcal{G}_0)$ to $(\Gamma_1, \mathcal{G}_1)$, noting that by the above U is the stabilizer of the first edge of Γ_0 and contained in the stabilizers of all other edges). There cannot be more than one interior vertex in Γ_0 because otherwise by Lemma 2 the group $E = \tilde{A} *_{\tilde{U}} \tilde{B}$ would be a (2, 2)-amalgam and thus E and then also E_0 would have rational Euler number zero which is not the case. If there is exactly one interior vertex in Γ_0 then again by Lemma 2 we have $p = q$ which we excluded.

It follows that also Γ_0 has exactly one edge, therefore $(\Gamma_0, \mathcal{G}_0) = (\Gamma_1, \mathcal{G}_1)$ and

$$\pi_1(\Gamma_0, \mathcal{G}_0) = A *_U B.$$

This implies $A \subset \tilde{A}$, $B \subset \tilde{B}$ and $U \subset \tilde{U}$. By Lemma 2 we have $[\tilde{A} : \tilde{U}] = [A : U]$, $[\tilde{B} : \tilde{U}] = [B : U]$, and therefore also $[\tilde{A} : A] = [\tilde{B} : B] = [\tilde{U} : U]$.

This finishes the proof of Theorem 1.

If $p \neq q$ or U is not maximal in A or B, Theorem 1 does not remain true in the above strong form.

EXAMPLES. *a*) Let $(\Gamma, \mathcal{G})$ be a finite graph of finite groups with fundamental group

$$E = \pi_1(\Gamma, \mathcal{G}) = A *_U B$$

where U is a subgroup of index two in A (so Γ consists of a single edge). Let

$$\phi\colon E = A *_U B \to \mathbf{Z}_2$$

be the surjection such that $\phi(U) = \phi(B) = 0$. Let E_0 be the kernel of ϕ.

As above, the group E acts on a tree T with quotient $T/E = (\Gamma, \mathcal{G})$. Then the action of $E_0 \subset E$ on T defines a graph of groups $T/E_0 = (\Gamma_0, \mathcal{G}_0)$ such that $E_0 = \pi_1(\Gamma_0, \mathcal{G}_0)$. Constructing the graph of groups $(\Gamma_0, \mathcal{G}_0)$ explicitly one finds that Γ_0 has exactly two edges and obtains the presentation

$$E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = B *_U U *_U B .$$

(This is, in a very special case, the proof of the subgroup theorem for free products with amalgamation using group actions on trees, see e.g. [Z2, Theorem 1.4.2]: the vertices of the graph of groups $(\Gamma_0, \mathcal{G}_0)$ correspond bijectively to the double cosets $E_0 x G_v$ in E of E_0 and the vertex groups G_v of $(\Gamma, \mathcal{G})$, with associated vertex groups $E_0 \cap x G_v x^{-1}$, similarly for the edges.) Obviously the graph of groups $(\Gamma_0, \mathcal{G}_0)$ is not minimal. Contracting it to a minimal graph of groups we get

$$E_0 = B *_U B$$

which has index type (p, p), where $p = [B: U]$.

b) Now let $(\Gamma, \mathcal{G})$ be the finite graph of finite groups with fundamental group

$$E = \pi_1(\Gamma, \mathcal{G}) = \boldsymbol{S}_m *_{\boldsymbol{A}_m} \boldsymbol{A}_{m+1} .$$

Let

$$\phi: E \to \boldsymbol{S}_{m+1}$$

be the canonical map considering $\boldsymbol{S}_m$ and $\boldsymbol{A}_{m+1}$ as subgroups of $\boldsymbol{S}_{m+1}$ in the standard way; let

$$E_0 := \phi^{-1}(\boldsymbol{S}_m)$$

be the preimage of $\boldsymbol{S}_m \subset \boldsymbol{S}_{m+1}$ in E. As above E acts on a tree T such that $T/E = (\Gamma, \mathcal{G})$, and constructing the quotient $T/E_0 = (\Gamma_0, \mathcal{G}_0)$ (or equivalently, applying the subgroup theorem for free products with amalgamation) we get the presentation

$$E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = \boldsymbol{S}_{m-1} *_{\boldsymbol{A}_{m-1}} \boldsymbol{A}_m *_{\boldsymbol{A}_m} \boldsymbol{S}_m .$$

Again $(\Gamma_0, \mathcal{G}_0)$ is not minimal; contracting it we have

$$E_0 = \boldsymbol{S}_{m-1} *_{\boldsymbol{A}_{m-1}} \boldsymbol{S}_m ,$$

and $\boldsymbol{A}_{m-1}$ is not maximal in $\boldsymbol{S}_m$.

4. – Finite maximal groups of outer automorphisms.

For $r > 1$, each finite subgroup $G_0 \subset Out\, F_r$ determines a group extension, unique up to equivalence of extensions,

$$1 \to F_r \hookrightarrow E_0 \to G_0 \to 1$$

which is determined by the following property: the given action of $G_0 \subset$ $\subset Out\, F_r$ on F_r is recovered by taking conjugations of F_r by preimages of elements of G_0 in E_0. The extension E_0 can be defined as the preimage of $G_0 \subset Out\, F_r$ in $Aut\, F_r$ under the canonical projection, noting that $Inn\, F_r \cong F_r$. In particular, the extension E_0 of F_r is *effective*, i.e. no nontrivial element of E_0 operates trivially on F_r by conjugation.

Conversely, any effective extension as above determines a subgroup $G_0 \subset Out\, F_r$, by taking conjugations of F_r by preimages of elements of G_0 in E_0.

The proof of the following Lemma is easy and left to the reader.

LEMMA 3. *A finite extension* $1 \to F_r \hookrightarrow E_0 \to G_0 \to 1$ *of a free group* F_r *of rank* $r > 1$ *is effective (and thus defines* G_0 *as a subgroup of* $Out\, F_r$*) if and only if* E_0 *has no nontrivial finite normal subgroups.*

We start with a geometric method to construct maximal finite subgroups $G_0 \subset Out\, F_r$, by considering group actions on finite graphs. For a proof of the following Lemma, see [WZ] or [Z1].

LEMMA 4. *Let* G_0 *be a finite group acting effectively (faithfully) on a finite hyperbolic graph* Δ, *that is a graph of rank* $r > 1$ *without vertices of valence one. Then, by taking induced actions on the fundamental group,* G_0 *injects into* $Out\, F_r$.

For G_0 and Δ as in the Lemma, let E_0 be the group of automorphisms of the universal covering tree T of Δ consisting of all lifts of elements of G_0 to T. Again we have an extension

$$1 \to F_r \hookrightarrow E_0 \to G_0 \to 1$$

where F_r denotes now the universal covering group of Δ. Note that, purely algebraically, this is the above extension belonging to the subgroup $G_0 \subset Out\, F_r$.

Suppose that G_0 acts without inversions on Δ, by subdividing edges. As in section 2, the action of E_0 on T defines a finite graph of finite groups $\Delta/G = T/E_0 = (\Gamma_0, \mathcal{G}_0)$ such that $E_0 = \pi_1(\Gamma_0, \mathcal{G}_0)$.

In the following, we shall consider the case where the quotient graph $\Gamma_0 = \Delta/G_0 = T/E_0$ consists of a single edge with two different vertices (a *segment*). Then $E_0 = \pi_1(\Gamma_0, \mathcal{G}_0)$ is a free product with amalgamation $A *_U B$ of some index type (p, q). We call also $(\Gamma_0, \mathcal{G}_0)$ a segment of index type (p, q) or just a (p, q)-segment. Note that T is the unique bipartite (p, q)-valent tree $T_{p,q}$; this means that the vertices of T can be partitioned into two disjoint sets of vertices of valences p and q, respectively, and that every edge goes from one set to the other. In particular, also Δ is a bipartite (p, q)-valent graph on which G_0 operates edge-transitively.

THEOREM 2. *Let Δ be a finite hyperbolic graph such that the full automorphism group $G_0 = Aut(\Delta)$ of Δ acts without inversions and such that the quotient Δ/G_0 is a (p, q)-segment, where $p \neq q$. Suppose that the edge group of $\Delta/G_0 = (\Gamma_0, \mathcal{G}_0)$ is maximal in the two vertex groups. Then G_0 induces a maximal finite subgroup of $Out F_r$.*

PROOF. The fundamental group $E_0 = \pi_1(\Gamma_0, \mathcal{G}_0)$ is an amalgam $A *_U B$ of index type (p, q), where $p \neq q$, and by hypothesis U is maximal in A and B.

Suppose that G_0, considered as a subgroup of $Out F_r$, is contained in a finite subgroup G of $Out F_r$. Then G defines an extension

$$1 \to F_r \hookrightarrow E \to G \to 1,$$

and, purely algebraically, $E_0 = A *_U B$ is a subgroup of finite index in E. Then also

$$E = \tilde{A} *_{\tilde{U}} \tilde{B}$$

is a (p, q)-amalgam as in Theorem 1.

Now E is the fundamental group of a graph of groups $(\Gamma, \mathcal{G})$ with a single edge. By the Bass-Serre theory of groups acting on trees the group $E = \pi_1(\Gamma, \mathcal{G}) = \tilde{A} *_{\tilde{U}} \tilde{B}$ acts on the bipartite (p, q)-valent tree $T = T_{p,q}$ such that $T/E = (\Gamma, \mathcal{G})$. Then this defines also an action of the subgroup E_0 of E on T, with quotient $T/E_0 = (\Gamma_0, \mathcal{G}_0)$ (as in the proof of Theorem 1). Denoting by $F_r \subset E_0 \subset E$ the universal covering group of Δ, the group $G = E/F_r$ acts on the graph $\Delta = T/F_r$ extending the action of $G_0 = E_0/F_r$. But G_0 was the full automorphism group of Δ, therefore $G = G_0$ and G_0 is a maximal finite subgroup of $Out F_r$. This finishes the proof of Theorem 2.

COROLLARY 1. *a) Let Δ be a finite bipartite (p, q)-valent graph where p and q are different prime numbers. Suppose that the automor-*

phism group $G_0 = Aut(\Delta)$ *operates transitively on the edges of* Δ. *Then* G_0 *induces a maximal finite subgroup of* $Out\, F_r$.

b) Let Δ *be a finite p-valent graph where p is a prime number greater than two. Suppose that* $G_0 = Aut(\Delta)$ *acts transitively on oriented edges of* Δ. *Then* G_0 *induces a maximal finite subgroup of* $Out\, F_r$.

PROOF. Part *a*) is just a specialization of Theorem 2. In the situation of part *b*), each oriented edge of Δ is equivalent to its reverse edge under the action of G_0 (in particular, G_0 acts with inversions). Subdividing each edge of Δ by a new vertex we obtain a bipartite $(2, p)$-valent graph Δ' (the first barycentric subdivision of Δ). Note that G_0 is still the full automorphism group of Δ' and that G_0 acts edge-transitively and without inversions on Δ'. Now part *b*) of the Corollary follows from part *a*).

EXAMPLES. *a*) Let $\Delta = \Delta(r)$ be the $(r+1)$-valent graph of rank r with two vertices and $r+1$ edges connecting these vertices (a *multiple edge of rank r*). Subdividing each edge by a new vertex we obtain the complete bipartite $(2, r+1)$-valent graph which we denote also by Δ. The automorphism group of this graph is

$$G_0 = Aut(\Delta(r)) = \boldsymbol{S}_{r+1} \times \boldsymbol{Z}_2 .$$

To the action of G_0 on Δ (or, equivalentaly, to the action of the lift E_0 of G_0 to the universal covering $T = T_{2, r+1}$ of Δ) is associated a finite graph of finite groups $(\Gamma_0, \mathcal{G}_0)$ where $\Gamma_0 = \Delta/G_0 = T/E_0$ is a graph with a single edge. Considering stabilizers in G_0 (or equivalently, in E_0), we have

$$E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = (\boldsymbol{S}_r \times \boldsymbol{Z}_2) *_{S_r} \boldsymbol{S}_{r+1} .$$

The fundamental group or universal covering group F_r of Δ is isomorphic to the kernel of the canonical projection

$$\phi\colon E_0 = (\boldsymbol{S}_r \times \boldsymbol{Z}_2) *_{S_r} \boldsymbol{S}_{r+1} \to G_0 = \boldsymbol{S}_{r+1} \times \boldsymbol{Z}_2 .$$

By Theorem 2 the group $G_0 = Aut(\Delta)$ induces a maximal finite subgroup of $Out\, F_r$.

b) More generally, let $\Delta = \Delta_{p,q}$ be the complete bipartite (p, q)-valent graph where $p \neq q$ (so there is exactly one edge between each vertex of valence p and each vertex of valence q, in particular Δ has p resp. q vertices of valence q resp. p). The automorphism group G_0 of

$\Delta_{p,q}$ is the product $S_p \times S_q$, and the quotient Δ/G_0 has fundamental group

$$E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = (S_{p-1} \times S_q) *_{(S_{p-1} \times S_{q-1})} (S_p \times S_{q-1}).$$

By Theorem 2 the group $G_0 = Aut(\Delta)$ induces a maximal finite subgroup of $Out\,F_r$, for the appropriate r.

c) Now let $\Delta = \Delta_n$ be the n-valent complete graph on $n+1$ vertices or, subdividing all edges, the corresponding bipartite $(2, n)$-valent graph, of rank $r = n(n-1)/2$. Its automorphism group $G_0 = Aut(\Delta_n)$ is the symmetric group S_{n+1}. Again the quotient graph Δ/G_0 has a single edge, and the fundamental group of the corresponding finite graph of finite groups $(\Gamma_0, \mathcal{G}_0)$ is

$$E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = (S_{n-1} \times Z_2) *_{S_{n-1}} S_n .$$

The fundamental or universal covering group F_r of Δ is isomorphic to the kernel of the canonical projection

$$\phi: E_0 = (S_{n-1} \times Z_2) *_{S_{n-1}} S_n \to G_0 = S_{n+1} .$$

By Theorem 2 the group $G_0 = Aut(\Delta_n)$ induces a maximal finite subgroup of $Out\,F_r$.

d) Let Δ be the first barycentric subdivision of the 1-skeleton of the n-dimensional cube, with automorphism group the semidirect product $G_0 = (Z_2)^n \ltimes S_n$. The quotient Δ/G_0 has fundamental group

$$E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = S_n *_{S_{n-1}} (S_{n-1} \times Z_2),$$

and again G_0 induces a maximal finite subgroup of $Out\,F_r$. In a similar way, one may consider the 1-skeletons of other regular polytopes.

COROLLARY 2. *The automorphism groups $S_{r+1} \times Z_2$ and S_n of the multiple edge of rank r resp. the complete graph on n vertices induce maximal finite subgroups of $Out\,F_r$ (where $r = n(n-1)/2$ in the second case).*

We call an amalgam $E_0 = A *_U B$ of two finite groups *effective* if it contains no nontrivial finite normal subgroups. Note that by Lemma 3 any surjection with torsionfree kernel from an effective amalgam E_0 onto a finite group G_0 represents G_0 as a subgroup of $Out\,F_r$ (where F_r denotes the kernel of ϕ).

Let G_0 be the subgroup of $Out\,F_r$ defined by a surjection, with torsionfree kernel F_r, from an effective (p, q)-amalgam $E_0 = \pi_1(\Gamma_0, \mathcal{G}_0) = A *_U B$ onto the finite group G_0, where $p \neq q$ and U is maximal in A and

B. Note that E_0 acts on the tree $T = T_{p,q}$ such that $T/E_0 = (\Gamma_0, \mathcal{G}_0)$, and that $G_0 = E_0/F_r$ acts on the quotient graph $\Delta = T/F_r$.

By the proof of Theorem 2 we have the following

COROLLARY 3. *The group G_0 is contained in a unique maximal subgroup G of $Out\, F_r$ which is induced by the full automorphism group G of the graph $\Delta = T/F_r$.*

Now we discuss a more algebraic method to construct maximal finite subgroups G_0 of $Out\, F_r$. We shall consider the most basic case where the extension E_0 defined by G_0 is a free product with amalgamation $E_0 = A *_U B$ of index type (2, 3).

Suppose that G_0 is contained as a subgroup of index j in the finite subgroup G of $Out\, F_r$. Then also E_0 is a subgroup of index j in the extension E of F_r determined by G. By Theorem 1, the extension E is also a (2, 3)-amalgam.

The effective (2, 3)-amalgams are completely classified. Each effective (2, 3)-amalgam $A *_U B$ contains an effective (3, 3)-amalgam $B *_U B$ as a subgroup of index two (as in example a) in section 2). The effective (3, 3)-amalgams have been classified in [G]; there are exactly 15 such amalgams. From this it is easy to classify the effective (2, 3)-amalgams. There are exactly 7 of them; they are described in detail in [DM, p. 206/7] and are as follows.

$$E_0 := \mathbf{Z}_2 * \mathbf{Z}_3 \cong PSL(2, \mathbf{Z}),$$

$$E_1 := \mathbf{D}_2 *_{\mathbf{Z}_2} \mathbf{D}_3 \cong PGL(2, \mathbf{Z}),$$

$$\overline{E}_1 := \mathbf{Z}_4 *_{\mathbf{Z}_2} \mathbf{D}_3\ ,$$

$$E_2 := \mathbf{D}_4 *_{\mathbf{D}_2} \mathbf{D}_6 \cong Aut\, PGL(2, \mathbf{Z}),$$

$$E_3 := \mathbf{D}_8 *_{\mathbf{D}_4} \mathbf{S}_4\ ,$$

$$\overline{E}_3 := \overline{\mathbf{D}}_8 *_{\mathbf{D}_4} \mathbf{S}_4\ ,$$

$$E_4 := K *_{(\mathbf{D}_4 \times \mathbf{Z}_2)} (\mathbf{S}_4 \times \mathbf{Z}_2),$$

where $\overline{\mathbf{D}}_8$ denotes a quasidihedral group of order 16 and K a group of order 32.

Note that also $GL(2, \mathbf{Z})$ is an amalgam of type $\mathbf{D}_4 *_{\mathbf{D}_2} \mathbf{D}_6$ which is not effective however because the center of $GL(2, \mathbf{Z})$ is isomorphic to $\mathbf{Z}_2$. In contrast, the amalgam $E_2 = \mathbf{D}_4 *_{\mathbf{D}_2} \mathbf{D}_6$ is effective, and this determines uniquely the inclusions of the edge group into the vertex groups.

We discuss the case of the amalgam $E_0 = \boldsymbol{Z}_2 * \boldsymbol{Z}_3 \cong PSL(2, \boldsymbol{Z})$.

COROLLARY 4. *Let* G_0 *be the subgroup of* $Out\,F_r$ *defined by a surjection*

$$\phi\colon E_0 = PSL(2, \boldsymbol{Z}) = \boldsymbol{Z}_2 * \boldsymbol{Z}_3 \to G_0\,,$$

with torsionfree kernel F_r. *Then* G_0 *is contained in a unique maximal finite subgroup* G *of* $Out\,F_r$, *and the index* j *of* G_0 *in* G *is equal to* 2^i, *where* $0 \leqslant i \leqslant 4$. *In particular, if* G_0 *is a simple group of order greater than* $(j-1)!$ *than* G_0 *is a normal subgroup of* G.

PROOF. By Corollary 3, the group G_0 is contained in a unique maximal subgroup G of $Out\,F_r$, as a subgroup of some index j. Then also the extension E of F_r belonging to $G \subset Out\,F_r$ contains E_0 as a subgroup of index j. By Theorem 1, the extension E is an effective (2, 3)-amalgam of the form $E = \tilde{A} *_{\tilde{U}} \tilde{B}$ where A, B and U are subgroups of index j in $\tilde{A}$, $\tilde{B}$ and $\tilde{U}$, respectively. Now also E is one of the above effective (2, 3)-amalgams, and consequently $j = 2^i$, where $0 \leqslant i \leqslant 4$.

By considering left multiplication on the left cosets of G_0 in G we get a homomorphism from G to the symmetric group S_j whose kernel is contained in G_0. If G_0 is simple of order greater than $(j-1)!$ then the kernel of this homomorphism is equal to G_0 and consequently G_0 is a normal subgroup of G.

Similar results hold for the other effective (2, 3)-amalgams. For example, any surjection with torsionfree kernel of E_4 onto a finite group G_0 defines G_0 as a maximal finite subgroup of $Out\,F_r$.

There is a rich literature on the finite quotients of the modular group $PSL(2, \boldsymbol{Z})$ and of the extended modular group $PGL(2, \boldsymbol{Z})$. Of course any finite group which is generated by two elements of orders 2 and 3 is a quotient of $PSL(2, \boldsymbol{Z})$. The finite quotients of the modular groups occur in various circumstances as maximal symmetry groups, and in the following we describe some of these.

Let $[2, 3, n]$ denote the extended triangle group generated by the reflections in the sides of a hyperbolic triangle with angles $\pi/2$, $\pi/3$ and π/n (where $n \geqslant 7$), and denote by $(2, 3, n)$ the subgroup of index two consisting of all orientation preserving elements. Then the class of finite groups which are quotients of the modular group coincides with the class of groups which are finite quotients of some triangle group $(2, 3, n)$ (because any map from $PSL(2, \boldsymbol{Z})$ to a finite group obviously factors through one of the groups $(2, 3, n)$), and similar for the extended modular group $PGL(2, \boldsymbol{Z})$ and the extended triangle groups $[2, 3, n]$. The finite quotients of the triangle group $(2, 3, 7)$ are called Hurwitz groups; they occur as automorphism groups of maximal possible order $84(g - 1)$ of closed Riemann surfaces of genus g.

The finite quotients of the extended triangle groups $[2, 3, n]$ are exactly the automorphism groups of the regular (reflexible) triangular maps on closed surfaces. The finite quotients of the group $[2, 3, 7]$ occur as the automorphism groups of maximal possible order of closed Klein surfaces. As shown in [C], for $n > 167$ all alternating groups are finite quotients of the triangle group $(2, 3, 7)$ and of the extended triangle group $[2, 3, 7]$, and thus also also of $PSL(2, \boldsymbol{Z})$ and $PGL(2, \boldsymbol{Z})$. It is shown in [Si] that the projective linear groups $PSL(2, F(q))$ over finite fields $F(q)$, with a few explicitely described exceptions, are quotients of $PGL(2, \boldsymbol{Z})$. The quotients of $PGL(2, \boldsymbol{Z})$ are also exactly the automorphism groups of maximal possible order of compact Klein surfaces with nonemty boundary, and form a subclass of the finite diffeomorphism groups of maximal possible order $12(g - 1)$ of 3-dimensional handlebodies of genus g, see the introduction of [Z3].

We close with the following

QUESTION. Which groups occur as maximal finite subgroups of $Out\,F_r$, for some r?

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