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Splitting in locally compact abelian groups

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Splitting in Locally Compact Abelian Groups.

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Introduction.

Let $\mathfrak{L}$ denote the class of all LCA (locally compact abelian) groups and let $\mathfrak{C}$ be the class of all compact abelian groups. Throughout this paper, G will always denote an arbitrary LCA group, unless otherwise stated. If H is a closed subgroup of G, then we will say that H *splits in* G provided G contains a closed subgroup K such that $H \cap K = 0$ and that the map $(h, k) \mapsto h + k$ is a homeomorphism of $H \times K$ onto G.

As is well known, the groups splitting in every discrete abelian group in which they are contained as subgroups are precisely the divisible groups (see [F] Theorem 24.5). Within the class $\mathfrak{L}$ (resp. $\mathfrak{C}$) the groups splitting in every group in which they are contained as closed subgroups are of the form $\boldsymbol{R}^n \times (\boldsymbol{R}/\boldsymbol{Z})^{\mathfrak{m}}$ (resp. $(\boldsymbol{R}/\boldsymbol{Z})^{\mathfrak{m}}$) where $n < \omega$ and $\mathfrak{m}$ is a cardinal number (cf. [AJ]).

In the first part of this paper, we scrutinize the splitting of the identity component G_0 in G. It is well known that the groups of the form as just described are exactly the connected groups in $\mathfrak{L}$ (resp. $\mathfrak{C}$) splitting in every group in $\mathfrak{L}$ (resp. $\mathfrak{C}$) in which they are contained as identity components (cf. [FG]). The description of all groups LCA groups H satisfying $\text{Ext}(H, C) = 0$ for all connected LCA groups C (see [FG]) leads to the description of the totally disconnected groups D in $\mathfrak{L}$ (resp. $\mathfrak{C}$) having the property that every group G in $\mathfrak{L}$ (resp. $\mathfrak{C}$) with $G/G_0 \cong D$ contains G_0 as a splitting subgroup. Within the class of LCA groups, we obtain exactly the groups D possessing a totally disconnected compact open subgroup K which is a direct sum of a compact torsion group and a compact torsion-free group. Within the class of compact abelian groups, D is sim-

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ply of the form K as just described (Theorem 1.2). In [L] we used Pontrjagin duality and dualized May's [M] characterization of discrete abelian groups with splitting torsion part. A corresponding characterization of compact abelian groups with splitting identity component was obtained. It turned out that the largest class of LCA groups where this characterization of groups with splitting identity component applies, is exactly the class of LCA groups G which are topological torsion groups modulo G_0 (see [L] Theorem 4). However, a slight modification of this characterization leads immediately to a characterization of all LCA groups G with splitting G_0: let bG denote the closed subgroup consisting of all compact elements of G. Then G_0 splits in G if and only if G contains compact subgroups $B^1 \supseteq \ldots \supseteq B^n \supseteq \ldots$ having trivial intersection so that each factor group $(G_0 + bG)/(G_0 + B^n)$ is torsion and $(B^n)_0 = B^n \cap G_0$ for all $n < \omega$ (Theorem 1.5). We will show some more characterizations of LCA groups with splitting identity component (Theorems 1.6, 1.7 and Corollaries 1.8-1.10).

Let $\widehat{G}$ denote the Pontrjagin dual group of G. For $H \subseteq G$, let $(\widehat{G}, H)$ be the annihilator of H in $\widehat{G}$. The group $(\widehat{G}, bG)$ coincides with the identity component $\widehat{G}_0$, and a closed subgroup of G splits in G if and only if its annihilator splits in $\widehat{G}$. Hence, corresponding results to those in the first part are obtained, involving the group bG. For instance, by dualizing Theorem 1.2 we get a characterization of groups C consisting of compact elements having the property that every group G in $\mathfrak{L}$ with $bG \cong C$ contains bG as a splitting subgroup: these are precisely the groups C in $\mathfrak{L}$ possessing a compact open subgroup K such that C/K is a direct sum of a bounded group and a divisible torsion group (Theorem 2.1). We establish then characterizations of groups G in $\mathfrak{L}$ having bG as a splitting subgroup generalizing May's [M] characterization of discrete abelian groups with splitting torsion part. For instance, bG splits in G if and only if G has a compact subgroup K contained in open subgroups $A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ such that $\sum A_n$ is dense in G, bA_n/K is a bounded torsion group, and $(bG + A_n)/A_n = b(G/A_n)$ for all $n < \omega$ (Theorem 2.3). Finally, some more characterizations of LCA groups G with splitting subgroup bG are obtained (Theorem 2.4 and Corollaries 2.5-2.7).

1. – Splitting of the identity component.

The connected groups C in $\mathfrak{L}$ splitting in every group G in $\mathfrak{L}$ with $C \cong G_0$ are topologically isomorphic to $\boldsymbol{R}^n \times (\boldsymbol{R}/\boldsymbol{Z})^{\mathfrak{m}}$ for some $n < \omega$ and cardinal number $\mathfrak{m}$ (see [FG]), and the proof in [FG] shows that the connected groups C in $\mathfrak{C}$ splitting in every group G in $\mathfrak{C}$ with $C \cong G_0$ are of the form $(\boldsymbol{R}/\boldsymbol{Z})^{\mathfrak{m}}$. To determine the structure of the totally disconnected (locally) compact abelian groups D having the property that every (locally) compact abelian group G with $G/G_0 \cong D$ contains G_0 as a direct summand, consider the following theorem:

THEOREM 1.1 (Fulp and Griffith [FG]). *A group H in $\mathfrak{L}$ has the property that* $\operatorname{Ext}(H, C) = 0$ *for all connected groups C in $\mathfrak{L}$ if and only if $H \cong \boldsymbol{R}^n \oplus M$ where M contains a compact open subgroup having cotorsion dual.*

THEOREM 1.2. *The totally disconnected groups D in $\mathfrak{L}$ having the property that every group G in $\mathfrak{L}$ with $G/G_0 \cong D$ contains G_0 as a direct summand are exactly the groups in $\mathfrak{L}$ which have a totally disconnected compact open subgroup K which is a direct sum of a compact torsion group and a compact torsion-free group, that is,*

$$K \cong \prod_{i \in I} \boldsymbol{Z}/p_i^{r_i}\boldsymbol{Z} \times \prod_{p\ prime} \Delta_p^{\mathfrak{n}_p}$$

(Δ_p denotes the group of p-adic integers) where only finitely many distinct primes p_i and positive integers r_i occur and $\mathfrak{n}_p$ are cardinal numbers.

Furthermore, the direct products as above are precisely the totally disconnected groups D in $\mathfrak{C}$ having the property that every group G in $\mathfrak{C}$ with $G/G_0 \cong D$ contains G_0 as a direct summand.

PROOF. Let D be a totally disconnected group in $\mathfrak{L}$ and suppose that every group G in $\mathfrak{L}$ with $G/G_0 \cong D$ contains G_0 as a direct summand. Then $\operatorname{Ext}(D, C) = 0$ for all connected groups C in $\mathfrak{L}$ and D contains a compact open subgroup K having cotorsion dual by Theorem 1.1. Since K is compact and totally disconnected, its dual group is torsion. By [F] Corollary 54.4, a torsion group is cotorsion if and only if it is a direct sum of a bounded and a divisible group, hence K is a totally disconnected group which is a direct sum of a compact torsion group and a compact torsion-free group. Conversely, assume that D is in $\mathfrak{L}$ and has a compact open subgroup K as above. Then D is totally disconnected and since $\widehat{K}$ is cotorsion, we have $\operatorname{Ext}(D, C) = 0$ for all connected groups C in $\mathfrak{L}$ by Theorem 1.1, hence every group G in $\mathfrak{L}$ with $G/G_0 \cong D$ contains G_0 as a direct summand.

Finally, suppose that D is compact. As in the proof of [FG] Theorem 3.6, $\operatorname{Ext}(D, C) \cong \operatorname{Ext}(\widehat{C}, \widehat{D})$ yields $\operatorname{Ext}(D, C) = 0$ for all connected groups C in $\mathfrak{C}$ if and only if $\operatorname{Ext}(F, \widehat{D}) = 0$ for all discrete torsion-free groups F, that is, $\widehat{D}$ is a cotorsion group. If D is totally disconnected such that every group G in $\mathfrak{C}$ with $G/G_0 \cong D$ contains G_0 as a direct summand, then the torsion group $\widehat{D}$ is cotorsion, hence D has the desired form.

Conversely, if D is a direct product of groups as asserted, then D is totally disconnected and the dual of a cotorsion group, hence $\mathrm{Ext}(D, C) = 0$ for all connected groups in $\mathfrak{C}$. Thus every group G in $\mathfrak{C}$ with $G/G_0 \cong D$ contains G_0 as a direct summand. ■

The following fact is due to May [M]:

THEOREM 1.3 (May [M]). *Let A be an abelian group. Then the torsion part tA of A splits in A if and only if A contains an ascending chain $A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of subgroups such that*

(i) $\sum A_n = A$;

(ii) *tA_n is bounded for all n;*

(iii) *$t(A/A_n) = (tA + A_n)/A_n$ for all n.*

Recall that a locally compact abelian group G is said to be a *topological torsion group* provided $(n!)x \to 0$ for each $x \in G$ (see [R]). In [L] we dualized the above characterization and obtained the following result:

THEOREM 1.4. *Let G be an LCA group. If G/G_0 is a topological torsion group, then G_0 splits in G if and only if G contains a descending chain $B^1 \supseteq \ldots \supseteq B^n \supseteq \ldots$ of compact subgroups such that*

(i) $\bigcap B^n = 0$;

(ii) *$G/(G_0 + B^n)$ is a torsion group for all n;*

(iii) *$(B^n)_0 = G_0 \cap B^n$ for all n.*

If G contains a descending chain as above, then G/G_0 is a topological torsion group.

Condition (ii) in the above statement implies that G is a topological torsion group modulo G_0 (see [L]) and we will replace it by a suitable condition to obtain a characterization of all locally compact abelian groups with splitting identity component.

THEOREM 1.5. *Let G be an LCA group. Then G_0 splits in G if and only if G contains a descending chain $B^1 \supseteq \ldots \supseteq B^n \supseteq \ldots$ of compact subgroups such that*

(i) $\bigcap B^n = 0$;

(ii) *$(G_0 + bG)/(G_0 + B^n)$ is a torsion group for all n;*

(iii) *$(B^n)_0 = G_0 \cap B^n$ for all n.*

PROOF. Let $H = G_0 + bG$. Since $(H/H_0)^\wedge \cong (\widehat{G}, G_0)/(\widehat{G}, G_0 + bG) = b\widehat{G}/(b\widehat{G} \cap \widehat{G}_0)$ is totally disconnected, H/H_0 is a topological torsion group (see [R] Theorem 3.15). If G_0 splits in G, then clearly H_0 splits in H, hence conditions (i) - (iii) hold by Theorem 1.4. Conversely, suppose that the stated conditions (i) - (iii) are satisfied. Then H_0 splits in H by Theorem 1.4, hence there is a continuous homomorphism $f\colon H \to H_0$ with $f|H_0 = \mathrm{id}$. Now $H_0 = G_0$ is divisible and H is an open subgroup of G (cf. [HR] 9.26(a)), hence f can be extended to a continuous homomorphism $f'\colon G \to G_0$. Therefore G_0 splits in G. ■

Next we show a similar characterization involving bounded torsion groups.

THEOREM 1.6. *G_0 splits in G if and only if G has an open subgroup K containing a descending chain $B^1 \supseteq \ldots \supseteq B^n \supseteq \ldots$ of compact subgroups such that*

(i) $\bigcap B^n = 0$;

(ii) $K/(G_0 + B^n)$ *is a bounded torsion group for all* n;

(iii) $(B^n)_0 = G_0 \cap B^n$ *for all* n.

PROOF. Again, let $H = G_0 + bG$. If G_0 splits in G, then we let $B^1 \supseteq \supseteq \ldots \supseteq B^n \supseteq \ldots$ be a sequence of compact subgroups as in Theorem 1.5. Then H/H_0 contains a compact open subgroup K'/H_0, hence $K = K' + B^1$ is an open subgroup of G. It follows that each factor group $K/(G_0 + B^n)$ is a compact torsion group and is therefore bounded. Conversely, suppose that the stated conditions are satisfied. Then $K \cap H$ is an open subgroup of G and $(K \cap H)/G_0$ is a topological torsion group. By Theorem 1.4, G_0 splits in $K \cap H$. Now we use the same argument as in the proof of Theorem 1.5 and conclude that G_0 splits in G. ■

Note that by the structure theorem for locally compact abelian groups, we can write $G = V \oplus \widetilde{G}$ where V is a maximal vector subgroup and $\widetilde{G}$ contains a compact open subgroup. V and $\widetilde{G}$ are uniquely determined up to topological isomorphism (see [AA]), so if V' is a maximal vector subgroup of G, then we have $\overline{G} = G/V \cong G/V'$, and $\overline{G}$ contains a compact open subgroup.

THEOREM 1.7. *G_0 splits in G if and only if $\overline{G}$ has a compact open subgroup G' containing a descending chain $D^1 \supseteq \ldots \supseteq D^n \supseteq \ldots$ of closed subgroups such that*

(i) $\bigcap D^n = 0$;

(ii) $G'/(G'_0 + D^n)$ *is a torsion group for all* n;

(iii) $(D^n)_0 = G'_0 \cap D^n$ *for all* n.

PROOF. Suppose that $G = G_0 \oplus C$ and let K be a compact open subgroup of C. Since $G_0 = V \oplus H$ for some compact connected group H, we can write $G = V \oplus \tilde{G}$ where $\tilde{G} = H \oplus C$ contains the compact open subgroup $G' = H \oplus K$. Let $D^n = n!K$ for all n. Then $\bigcap D^n = \bigcap n!K = K_0 = 0$, $G'/(G_0' + D^n) = (H+K)/(H+n!K)$ is torsion and $(D^n)_0 = 0 = G_0' \cap D^n$ for all n. The converse follows immediately from Theorem 1.6. ■

REMARK. If $\overline{G}$ has a compact open subgroup containing a chain as above, then *every* compact open subgroup of $\overline{G}$ contains such a chain: suppose $G = G_0 \oplus C$ and write $G_0 = V \oplus H$ as before. Now let G' be any compact open subgroup of $\tilde{G} = H \oplus C$ and define $D^n = n!(C \cap G')$ for all n. Then it is easy to see that conditions (i)-(iii) in Theorem 1.7 are satisfied.

COROLLARY 1.8. *G_0 splits in G if and only if $\overline{G}$ contains a compact open subgroup G' and a descending chain $D^1 \supseteq \ldots \supseteq D^n \supseteq \ldots$ of closed subgroups such that*

(i) $\bigcap D^n = 0$;

(ii) $G'/[(G_0' + D^n) \cap G']$ *is a torsion group for all* n;

(iii) $(D^n)_0 = G_0' \cap D^n$ *for all* n.

PROOF. Suppose $\overline{G}$ contains a compact open subgroup G' and a descending chain $D^1 \supseteq \ldots \supseteq D^n \supseteq \ldots$ as claimed. Let $B^n = D^n \cap G'$ for all n. Then $\bigcap B^n = 0$ and $G'/(G_0' + B^n) = G'/[G_0' + (D^n \cap G')] = G'/[(G_0' + D^n) \cap G']$ is torsion. Further, $(B^n)_0 \subseteq G_0' \cap B^n = G_0' \cap D^n \cap G' = (D^n)_0 \cap G' = (D^n)_0$ because $D^n \cap G'$ is open in D^n. Hence $G_0' \cap B^n$ is a connected subset of B^n and therefore contained in $(B^n)_0$. Thus G_0' splits in G' which implies that G_0 splits in G. ■

COROLLARY 1.9. *G_0 splits in G if and only if $\overline{G}$ contains a closed subgroup C such that $C \cap \overline{G}_0 = 0$ and $\overline{G}/(\overline{G}_0 + C)$ is a torsion group.*

PROOF. If G_0 splits in G, then obviously $\overline{G}$ contains a closed subgroup C as above. Conversely, assume that $\overline{G}$ has a closed subgroup C as above. Let G' be a compact open subgroup of $\overline{G}$ and define $D^n = n!(C \cap G')$ for all n. Then $\bigcap D^n = 0$. The group $\overline{G}/(G_0' + C) = \overline{G}/(\overline{G}_0 + C)$ is torsion, hence $G'/[G_0' + (C \cap G')] = G'/[(G_0' + C) \cap G']$ is torsion and therefore $G'/(G_0' + D^n)$ is torsion. Finally, $G_0' \cap D^n \subseteq \overline{G}_0 \cap C = 0$ and $(D^n)_0 \subseteq C_0 = 0$. Hence G_0 splits in G. ■

COROLLARY 1.10. *G_0 splits in G if and only if G contains a closed subgroup C such that $C \cap G_0 = 0$ and $G/(G_0 + C)$ is a torsion group.*

PROOF. Suppose G has a closed subgroup C as claimed and let $\varphi: G \to \overline{G}$ be the natural map. Then $\varphi(C) \cap \varphi(G_0) = \varphi(C) \cap \overline{G}_0 = 0$ and $\overline{G}/(\overline{G}_0 + \varphi(C))$ is torsion, so G_0 splits in G by Corollary 1.9. ■

2. – Splitting of the subgroup of all compact elements.

Recall that a torsion group T splits in every discrete abelian group in which T is contained as its torsion part if and only if T is a direct sum of a bounded group and a divisible group (cf. [F] Theorem 100.1). We will now use Pontrjagin duality and obtain some results on LCA groups G involving the group bG. The following theorem describes the groups $C = bC$ splitting in every locally compact abelian group G in which C is contained as the subgroup of all compact elements.

THEOREM 2.1. *The groups C in $\mathfrak{L}$ with $bC = C$ having the property that every group G in $\mathfrak{L}$ with $bG \cong C$ contains bG as a direct summand are exactly the groups in $\mathfrak{L}$ which have a compact open subgroup such that the factor group is a direct sum of a bounded torsion group and a divisible torsion group.*

PROOF. Suppose that $bC = C$ where every group G in $\mathfrak{L}$ with $bG \cong C$ contains bG as a direct summand. Let H be in $\mathfrak{L}$ with $H/H_0 \cong \widehat{C}$. Then $b\widehat{H} \cong [H/(H, b\widehat{H})]^\wedge = (H/H_0)^\wedge \cong C$. By our assumption, $b\widehat{H}$ splits in $\widehat{H}$, hence $H_0 = (H, b\widehat{H})$ splits in H. By Theorem 1.2, $\widehat{C}$ has a totally disconnected compact open subgroup $(\widehat{C}, K)$ which is a direct sum of a compact torsion group and a compact torsion-free group. Hence K is compact and open in C and $C/K = B \oplus D$ where B is bounded and D is a divisible torsion group.

Conversely, assume C has a compact open subgroup K where $C/K = B \oplus D$ as above. Then $(\widehat{C}, K)$ and therefore $\widehat{C}$ is totally disconnected, hence C consists of compact elements. Now let G be in $\mathfrak{L}$ such that $bG \cong C$. Then $\widehat{G}/\widehat{G}_0 = \widehat{G}/(\widehat{G}, bG) \cong \widehat{C}$. Since $(\widehat{C}, K)$ is a totally disconnected compact open subgroup of $\widehat{C}$ which is a direct sum of a compact torsion group and a compact torsion-free group, $\widehat{G}_0$ splits in $\widehat{G}$ by Theorem 1.2. Thus bG splits in G. ■

To characterize all LCA groups G with splitting subgroup bG, we need the following lemma:

LEMMA 2.2. *Let N be a closed subgroup of the group G in $\mathfrak{L}$ and suppose that $bG+N$ is a closed subgroup of G. Then $(bG+N)/N$ is the subgroup of all compact elements of G/N if and only if $\widehat{G}_0 \cap (\widehat{G}, N)$ is the identity component of $(\widehat{G}, N)$.*

PROOF. Let ϱ be the topological isomorphism from $(G/N)^\wedge$ onto $(\widehat{G}, N)$ induced by the natural map $\varphi: G \to G/N$. Then ϱ maps the group $(G/N)_0^\wedge = ((G/N)^\wedge, b(G/N))$ onto $(\widehat{G}, N)_0$. Furthermore, ϱ maps $((G/N)^\wedge, (bG+N)/N)$ onto $(\widehat{G}, bG+N) = \widehat{G}_0 \cap (\widehat{G}, N)$ and therefore $(bG+N)/N$ is equal to $b(G/N)$ if and only if $\widehat{G}_0 \cap (\widehat{G}, N)$ is equal to $(\widehat{G}, N)_0$. ∎

The following characterization of all locally compact abelian groups G with splitting subgroup bG obviously generalizes Theorem 1.3:

THEOREM 2.3. *bG splits in G if and only if G has a compact subgroup K contained in an ascending chain $A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of open subgroups such that*

(i) *$\sum A_n$ is a dense subgroup of G;*

(ii) *bA_n/K is a bounded torsion group for all n;*

(iii) *$(bG+A_n)/A_n = b(G/A_n)$ for all n.*

PROOF. Suppose G contains an ascending chain $K \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of subgroups as above. Then $K' = (\widehat{G}, K)$ is open in $\widehat{G}$ and $B^n = (\widehat{G}, A_n)$ is compact for all n. Further, $\bigcap B^n = (\widehat{G}, \sum A_n) = 0$ and each group $K'/(\widehat{G}_0 + B^n) = (\widehat{G}, K)/(\widehat{G}, bA_n) \cong (bA_n/K)^\wedge$ is a bounded torsion group. Lemma 2.2 shows that $(B^n)_0 = \widehat{G}_0 \cap B^n$ for all n. By Theorem 1.6, $\widehat{G}_0$ splits in $\widehat{G}$ and therefore bG splits in G, as claimed.

Conversely, suppose bG splits in G. Then $\widehat{G}_0$ splits in $\widehat{G}$, hence $\widehat{G}$ has an open subgroup K' containing compact subgroups $B^1 \supseteq \ldots \supseteq B^n \supseteq \ldots$ satisfying the corresponding conditions in Theorem 1.6. Now let $K = (G, K')$ and $A_n = (G, B^n)$ for all n. Then the closure of $\sum A_n$ is equal to G. Further, $K/(G_0 + B^n) \cong (bA_n/K)^\wedge$ is a bounded torsion group for all n and (iii) holds by Lemma 2.2. This completes the proof. ∎

THEOREM 2.4. *bG splits in G if and only if $\overline{G}$ has a compact open subgroup G' contained in an ascending chain $A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of closed subgroups such that*

(i) *$\sum A_n$ is a dense subgroup of $\overline{G}$;*

(ii) *bA_n/G' is a bounded torsion group for all n;*

(iii) *$(b\overline{G}+A_n)/A_n = b(\overline{G}/A_n)$ for all n.*

PROOF. We write $G = V \oplus \widetilde{G}$ and obtain $\widehat{G} = (\widehat{G}, V) \oplus (\widehat{G}, \widetilde{G})$. Since vector groups are self-dual, $V' = (\widehat{G}, \widetilde{G})$ is a maximal vector subgroup of $\widehat{G}$. Now suppose that bG splits in G. Then $\widehat{G}_0$ splits in $\widehat{G}$, so by Theorem 1.7 $(\widetilde{G})^\wedge \cong \widehat{G}/V'$ has a compact open subgroup C containing a descending chain $D^1 \supseteq \ldots \supseteq D^n \supseteq \ldots$ of closed subgroups such that $\bigcap D^n = 0$ and for all n, $C/(C_0 + D^n)$ is a torsion group and $(D^n)_0 = C_0 \cap D^n$. Let $G' = (\widetilde{G}, C)$ and $A_n = (\widetilde{G}, D^n)$ for all n. Then we obtain an ascending chain $G' \subseteq A_1 \subseteq \subseteq \ldots \subseteq A_n \subseteq \ldots$ of closed subgroups where G' is compact and open in $\widetilde{G}$. The closure of $\sum A_n$ coincides with $\widetilde{G}$ and the group $bA_n/G' = (b\widetilde{G} \cap \cap A_n)/G' = (\widetilde{G}, (\widetilde{G})_0^\wedge + D^n)/(\widetilde{G}, C) \cong [C/(C_0 + D^n)]^\wedge$ is a bounded torsion group for all n. Finally, Lemma 2.2 yields the equality $(b\widetilde{G} + A_n)/A_n = b(\widetilde{G}/A_n)$. Hence $\overline{G}$ contains an ascending chain satisfying the desired properties. For the converse, we use similar arguments as in the proof of Theorem 2.3. ■

By the same logic, dualization of Corollary 1.8 yields

COROLLARY 2.5. *bG splits in G if and only if $\overline{G}$ contains a compact open subgroup G' and an ascending chain $A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ of closed subgroups such that*

(i) *$\sum A_n$ is a dense subgroup of G;*

(ii) *$(bA_n + G')/G'$ is a bounded torsion group for all n;*

(iii) *$(b\overline{G} + A_n)/A_n = b(\overline{G}/A_n)$ for all n.*

COROLLARY 2.6. *bG splits in G if and only if $\overline{G}$ contains a closed subgroup D such that bD is a bounded torsion group and $\overline{G} = b\overline{G} + D$.*

PROOF. Again, we write $G = V \oplus \widetilde{G}$ and identify $\widetilde{G}$ with $\overline{G}$. If $\overline{G}$ contains a closed subgroup D as above, then we let $C = ((\overline{G})^\wedge, D)$ and obtain $(\overline{G})_0^\wedge \cap C = ((\overline{G})^\wedge, b\overline{G} + D) = 0$. Moreover, $(\overline{G})^\wedge/[(\overline{G})_0^\wedge + C] \cong \cong (\overline{G}, (\overline{G})_0^\wedge + C)^\wedge = (bD)^\wedge$ is a torsion group. By Corollary 1.9, $\widehat{G}_0$ splits in $\widehat{G}$ and therefore bG splits in G. The converse is trivial. ■

Finally, dualization of Corollary 1.10 yields

COROLLARY 2.7. *bG splits in G if and only if G contains a closed subgroup D such that bD is a bounded torsion group and $G = bG + D$.*

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