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Continuity Results for Solutions of Certain Degenerate Parabolic Equations.

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ABSTRACT - In this paper we prove the local continuity for essentially bounded local weak solutions of a large class of degenerate parabolic equations, with principal part characterized by non standard growth conditions.

1. - Introduction.

The present section is devoted to introduce a class of quasilinear degenerate parabolic equations of the type

$$u_t = \mathrm{Div}\,(F(u, Du)\,Du) \quad \text{in } \mathcal{D}'(\Omega_T) \tag{1.1}$$

where $\Omega_T = \Omega \times (0, T)$, Ω is a bounded domain in R^N and $0 < T < \infty$, $\mathcal{D}'$ is the space of distributions on Ω_T and Du denotes the gradient respect only to the space variable. Here we assume:

$$\begin{cases} F\colon R \times R^N \to R\,, \quad (\eta, z) \mapsto F(\eta, z) \\ \text{is continuous in } \eta, \text{ uniformly continuous in } z\,. \end{cases} \tag{1.2}$$

There exists two functions C_1, C_2, such that for all $\eta \in R$, $z \in R^N$, $|z| \geq 1$

$$\begin{cases} C_1(|\eta|)\,|z|^q \leq F(\eta, z)\,, \\ |F(\eta, z)| \leq C_2(|\eta|)\,|z|^p\,, \end{cases} \tag{1.3}$$

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for a.e. $(x, t) \in \Omega_T$. Here $p > q \geq 0$, $s \mapsto C_1(s)$, $s > 0$, is a non increasing and positive function, while $s \mapsto C_2(s)$, $s > 0$ is a non decreasing function. For $0 < |z| < 1$, we shall assume that $F(\eta, z)$ is uniformly continuous and bounded, say for example there exists some constants $0 < \varepsilon_1 < \varepsilon_2$ such that

$$\varepsilon_1 \leq F(\eta, z), \qquad |F(\eta, z)| \leq \varepsilon_2 . \tag{1.4}$$

A measurable function u is a local distributional solution (supersolution, subsolution) of (1.1) in Ω_T if it satisfies the following conditions:

$$\begin{cases} u \in C_{\mathrm{loc}}(0, T; L^2_{\mathrm{loc}}(\Omega)) \cap W^{1, q+2}_{\mathrm{loc}}(\Omega_T), \\ F(u, Du)\, Du \in L^1_{\mathrm{loc}}(\Omega_T) \end{cases} \tag{1.5}$$

and for every compact subset $\mathcal{K}$ of Ω and every subinterval $[t_1, t_2]$ of $(0, T]$

$$\int_{\mathcal{K}} [u\varphi]_{t_1}^{t_2}\, dx + \int_{t_1}^{t_2} \int_{\mathcal{K}} \{ -u\varphi_t + F(u, Du)\, Du \cdot D\varphi \}\, dx\, dt = 0 \quad (\geq 0, \leq 0) \tag{1.6}$$

for every non negative local testing functions $\varphi \in \mathcal{D}_{\mathcal{K}}(\Omega_T)$, $\mathcal{D}_{\mathcal{K}}(\Omega_T) = \{\varphi \in \mathcal{D}(\Omega_T)$: φ with support in $\mathcal{K} \times [t_1, t_2]\}$.

We can also assume that

$$\|u\|_{\infty, \Omega_T} \leq M \tag{1.7}$$

for some positive and finite constant M. Moreover

(1.8) The maximum principle holds for boundary value problems associated to equation (1.1).

Lastly, we assume that a solution of (1.1) can be constructed as the weak limit in the norm (1.5) of local smooth solutions of regularized problems where F satisfies the following monotonicity condition:

$$\langle F(\eta, z_1) z_1 - F(\eta, z_2) z_2, z_1 - z_2 \rangle \geq 0 \tag{1.9}$$

for all $z_1, z_2 \in R^N$, $\eta \in R$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in R^N.

We always refer to $((C_1(M), C_2(M), M, N, p, q)$ as the *data* and we say $C = C(data)$ if C is a constant which we can determinate a priori in terms of the previous quantities.

The aim of this paper is to get a regularity result for a solution of (1.1). More precisely we want to state the following theorem:

MAIN THEOREM. *Under the assumptions* (1.2)-(1.9), *every essentially bounded distributional solution of* (1.1) *is locally continuous in* Ω_T. *Moreover, there exists a nondecreasing, non negative function*

$$\omega = \omega_{\text{data}}, \qquad \omega(0) = 0$$

such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \omega(|x_1 - x_2| + |t_1 - t_2|^{1/(p+2)})$$

for all $(x_i, t_i) \in K \times (\varepsilon, T)$, *$K$ a compact subset of* Ω, $i = 1, 2$.

The idea of the proof may be euristically discussed as follows: for every $(x_0, t_0) \in \Omega_T$, there is a sequence of shrinking cylinders around (x_0, t_0) such that the essential oscillation of u in these sets decreases to zèro as they tend to zero. The crucial step is to obtain from (1.1) some basic inequalities. This kind of equations can describe the evolution of nonlinear elastic phenomena. The stationary case has been widely studied (see for istance the contributions due to Marcellini [9, 10, 11] for a wide literature on this subject). Here our approach is completely different: we will apply the new techniques mainly developped by Di Benedetto and usually applied for some nonlinear parabolic problems—(see [4, 5, 6, 14, 16])—even in this case, where the main difficulty is given by the non standard structure condition (1.3). The way we overcome this obstacle is fully explained in the next section. Section 3 is devoted to other basic inequalities necessary to prove—in the last section—the main theorem.

REMARK 1.1. For the proof of the main theorem it is only necessary to assume $u \in L^\infty_{\text{loc}}(\Omega_T)$ instead of (1.7). We can also assume

$$\|u\|_{\infty, \Omega_T} = M \leq 1. \tag{1.10}$$

REMARK 1.2. Main teorem is still valid for the operator

$$F = F(t, \eta, z), \qquad 0 < t \leq T$$

which satisfies (1.2)-(1.8) and it is also continuous in t.

2. – Notation and local integral inequalities.

Let $(x_0, t_0) \in R^{N+1}$ be fixed. In what follows, we always assume that θ and ϱ are numbers such that:

$$[(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})] \subseteq \Omega_T$$

where $Q(\varrho, \theta\varrho^{p+2})$ is the cylinder centered in the origin, with height $\theta\varrho^{p+2}$ and cross section K_ϱ,

$$K_\varrho = \{x \in R^N : \max |x_i| < \varrho,\ i = 1, \ldots, N\}.$$

For $k \in R$ we define the truncations

$$(u-k)_+ = \max(u-k, 0), \qquad (u-k)_- = \max(-u+k, 0).$$

We will choose levels k satisfying

$$\|(u-k)_\pm\|_{L^\infty((x_0,t_0)+Q(\varrho,\theta\varrho^{p+2}))} = H_k^\pm \leq \delta, \tag{2.1}$$

where δ is a positive number to fix later. Since F satisfies (1.9), for all $k \in R$ the functions $(u-k)_\pm$ are subsolutions of (1.1) in the sense of (1.6) (see [5, chapter II, section 1]). Let $H_k^\pm$ be defined as in (2.1). We introduce the logarithmic function

$$\Psi(H_k^\pm, (u-k)_\pm, a) = \max\left\{0, \ln \frac{H_k^\pm}{H_k^\pm - (u-k)_\pm + a}\right\}, \tag{2.2}$$

$$0 < a < H_k^\pm,$$

which we briefly denote by $\Psi(u)$ and a piecewise smooth cutoff function ξ, defined in the cylinder $[(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})]$ and such that

$$\begin{cases} \xi \in [0,1], \quad \xi = 0 \quad \text{outside } [x_0 + K_\varrho], \\ 0 < |D\xi| < \infty, \quad 0 \leq \xi_t < \infty, \\ \xi = 1, \quad \forall (x,t) \in [(x_0,t_0) + Q(\sigma\varrho, \sigma\theta\varrho^{p+2})], \quad \sigma \in (0,1). \end{cases} \tag{2.3}$$

We also define the sets

$$A_{k,\varrho}^\pm = \{(x,t) \in [(x_0,t_0) + Q(\varrho, \theta\varrho^{p+2})] : (u-k)_\pm > 0\}.$$

Now we can state the fundamental inequalities necessary to prove the main theorem. First we need the following lemma:

LEMMA 2.1. *Let* (1.2)-(1.4) *and* (1.7) *hold. There exists constants* $\widetilde{C}_i = \widetilde{C}_i(C_i(M))$, $i = 1, 2$ *such that for all* $L > 1$, *for all* $\gamma > 0$, *there* *exists* $\delta \in (0, 1/2)$, *which can be determinated a priori in terms of the data and* $\beta \in [q, p]$ *such that for all* $z_0 \in R^N$, $|z_0| \in [1, L]$, *for all* $\eta_0 \in R$, $|\eta_0| \in [\gamma, M]$,

$$\tilde{C}_1 |z|^\beta \leq |F(\eta, z)| \leq \tilde{C}_2 |z|^\beta, \quad \forall (\eta, z) \in I_\delta(\eta_0, z_0) \tag{2.4}$$

where

$$I_\delta(\eta_0, z_0) = \{(\eta, z) \in R \times R^N,\ |\eta - \eta_0|,\ |z - z_0| \leq \delta\}.$$

PROOF. If (1.4) holds, (2.4) follows with $\beta = 0$. If (1.3) holds, we get

$$\begin{cases} C_1(M)|z|^q \leq F(\eta, z), \\ |F(\eta, z)| \leq C_2(M)|z|^p, \end{cases}$$

for a.e. $(x, t) \in \Omega_T$. By (1.2), (1.3) and (1.7), we get the existence of a number $\beta \in [q, p]$ and of a constant $C \in [C_1(M), C_2(M)]$ such that

$$|F(\eta_0, z_0)| = C|z_0|^\beta$$

where $|z_0|$ is arbitrarily fixed in $[1, L]$ and $|\eta_0| \in [\gamma, M]$, $\gamma > 0$. Since (1.2) holds, for all $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that $\forall z$, $z_0 \in R^N$, $|z|$, $|z_0| \in [1, L]$, $|z - z_0| < \delta$ and $\forall \eta$, $\eta_0 \in R$, $|\eta|$, $|\eta_0| \in [\gamma, M]$, $|\eta - \eta_0| \leq \delta$

$$|F(\eta_0, z_0)| - \varepsilon \leq |F(\eta, z)| \leq |F(\eta_0, z_0)| + \varepsilon.$$

We fix $\varepsilon = C_1(M)/2$. Collecting the previous inequalities we obtain

$$\frac{C_1(M)}{2} |z_0|^\beta \leq |F(\eta, z)| \leq \frac{3}{2} C_2(M)|z_0|^\beta$$

for all $(\eta, z) \in I_\delta(\eta_0, z_0)$. We choose $\delta = \delta(C_1(M)) \in (0, 1/2)$. Since $|z_0| \leq |z - z_0| + |z| < 3/2|z|$ and $|z_0| \geq |z| - |z - z_0| > |z|/2$, we get

$$\frac{C_1(M)}{2^{\beta+1}} |z|^\beta \leq |F(\eta, z)| \leq 2^{\beta+1} C_2(M)|z|^\beta.$$

Lemma will follow choosing $\tilde{C}_1 = 2^{-(p+1)} C_1(M)$ and $\tilde{C}_2 = 2^{p+1} C_2(M)$. ∎

PROPOSITION 2.1. *Let* (1.1)-(1.9) *hold. There exists a constant* $C = C(data)$ *such that for every cylinder* $[(x_0, t_0) + Q(\sigma\varrho, \sigma\theta\varrho^{p+2})] \subset$

$\subset [(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})] \subset \Omega_T$, *where* $\sigma \in (0, 1)$, *and for every level* $k \in R$

$$\text{(2.5)} \qquad \sup_{t_0 - \sigma\varrho^{p+2} \leq t \leq t_0} \int_{[x_0 + K_{\sigma\varrho}]} (u-k)_\pm^2 \, dx + $$

$$+ C \int_{[(x_0, t_0) + Q(\sigma\varrho, \sigma\theta\varrho^{p+2})]} |D(u-k)_\pm|^{q+2} \, dx\, dt \leq$$

$$\leq C \int_{[(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})]} (u-k)_\pm^2 \, \xi^{q+1} \xi_t \, dx\, dt +$$

$$+ C|A_{k,\varrho}^\pm| + \int_{[x_0 + K_\varrho]} (u-k)_\pm^2 \, \xi^{q+2}(x, t_0 - \theta\varrho^{p+2}) \, dx +$$

$$+ C \int_{(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})} |D\xi|^{p+2} (u-k)_\pm^{p+2} \, dx\, dt .$$

PROOF. It would be technically convenient to have a formulation equivalent to (1.6) that involves u_t. Fix $t \in (0, T)$ and let h be so small that $0 < t < t + h < T$. In (1.6) we take $t_1 = t$, $t_2 = t + h$ and a function $\varphi = \varphi(x)$. Dividing by h, we get

$$\int_{\mathcal{K}} \{ \partial_t u_h \varphi \, dx + [F(u, Du) Du]_h \cdot D\varphi \} \, dx = 0$$

where u_h is the Steklov average of u,

$$u(x, t) = \begin{cases} \dfrac{1}{h} \displaystyle\int_t^{t+h} u(x, s) \, ds & \text{if } 0 < t \leq T - h , \\ 0 & \text{otherwise} . \end{cases}$$

Now we fix a subinterval $[t_1, t_2] \subseteq (0, T]$, $t_1 + h \leq T$ and we integrate the previous integral equality over $[t_1, t_2]$, with a testing function

$$\varphi = (u_h^\varepsilon(x, t) - k)_\pm \, \xi^\alpha$$

where u_h^ε is the average of u_h in the x variables,

$$u_h^\varepsilon(x, t) = \int_{|x-y| \leq \varepsilon} u_h(y, t) \, w(|x-y|) \, dy$$

with $w \in C^\infty(R^N)$, $w = 0$ for $|y| \geq 1$, $\int_{|y| \leq 1} w(y)\, dy = 1$. Such a choice of φ is admissable since $\operatorname{supp} \varphi = Q$, $Q = \mathcal{K}' \times [t_1, t_2]$, with $\mathcal{K}' \subsetneq \mathcal{K}$, $\partial_t \varphi \in L^2(Q)$, $D^n \varphi \in C(Q)$, for any order n. Up to a translation we may assume $(x_0, t_0) = (0, 0)$ and we may choose ϱ sufficiently small in order to have

$$Q(\varrho, \theta\varrho^{p+2}) \subsetneq Q .$$

(see [7, chapter II] for more details). We will state (2.5) only when the structure conditions (1.2), (1.3), (1.5)-(1.9) hold since in the case (1.4) we have nothing to prove. By Lemma 2.1, for all $(x, t) \in Q(\varrho, \theta\varrho^{p+2})$, for all u satisfying (1.7), whatever is the value which the gradient of u assumes, there exists an interval of the type I_δ—whom the gradient belongs to—such that (2.4) holds. Moreover, for all $j \geq 1$, we can define the following sets:

$A_j = \{(x, t) \in Q(\varrho, \theta\varrho^{p+2})$: there exists $\beta_j \in [q, p]$ such that

$$\left.\tilde{C}_1 |D(u-k)_\pm|^{\beta_j} \leq |F(u, D(u-k)_\pm)| \leq \tilde{C}_2 |D(u-k)_\pm|^{\beta_j}\right\}$$

where $\tilde{C}_1$, $\tilde{C}_2$ are fixed constants for every interval I_δ. Obviously

$$A_j \subsetneq Q(\varrho, \theta\varrho^{p+2})$$

moreover, $\forall (x, t) \in Q(\varrho, \theta\varrho^{p+2})$ there exists a number $j = \bar{j}$, such that $(x, t) \in A_{\bar{j}}$, so we get

$$\bigcup_{j \geq 1} A_j = Q(\varrho, \theta\varrho^{p+2}), \qquad A_i \cap A_j = \emptyset , \qquad \forall i \neq j . \tag{2.6}$$

For semplicity we set

$$Q_t = K_\varrho \times (-\theta\varrho^{p+2}, t), \qquad Q_t^\sigma = K_{\sigma\varrho} \times (-\theta\sigma\varrho^{p+2}, t),$$

$$Q_{j,t}^\sigma = A_j \cap (K_{\sigma\varrho} \times (-\sigma\theta\varrho^{p+2}, t)), \qquad Q_{j,t} = A_j \cap (K_\varrho \times (-\theta\varrho^{p+2}, t)),$$

for $t \in (-\theta\varrho^{p+2}, 0)$. We integrate by parts over Q_t and estimate the terms separately. By the regularity of u_h^ε and its convergence to u_h in the norm $L^2(Q)$, letting $\varepsilon \to 0$ we get

$$\int_{Q_t} \partial_s u_h^\varepsilon \varphi \, dx \, ds \to \frac{1}{2} \int_{K_\varrho} (u_h - k)_\pm^2 \, \xi^\alpha(x, s) \big|_{s=-\theta\varrho^{p+2}}^{s=t} dx - \\ - \frac{\alpha}{2} \int_{Q_t} (u_h - k)_\pm^2 \, \xi_t \xi^{\alpha-1} \, dx \, ds$$

and then, by the convergence of u_h to u in $L^2(Q)$, for $h \to 0$ we get

$$\int_{Q_t} \partial_s u_h \varphi \, dx \, ds \to \frac{1}{2} \int_{K_\varrho} (u - k)_\pm^2 \, \xi^\alpha(x, s) \big|_{s=-\theta\varrho^{p+2}}^{s=t} dx - \\ - \frac{\alpha}{2} \int_{Q_t} (u - k)_\pm^2 \, \xi_t \xi^{\alpha-1} \, dx \, ds . \tag{2.7}$$

By (1.5) the regularity of Du_h^ε and u_h^ε and the use of the local smooth approximations andicated in the introduction, we get

$$\begin{cases} |[F(u, Du) Du]_h \cdot D(u_h^\varepsilon - k)_\pm \, \xi^\alpha| \leqslant \\ \qquad \leqslant C |[F(u, Du) Du]_h| \in L^1(Q(\varrho, \theta\varrho^{p+2})) , \\ |[F(u, Du) Du]_h \cdot D\xi (u_h^\varepsilon - k)_\pm \, \xi^{\alpha-1}| \leqslant \\ \qquad \leqslant C |[F(u, Du) Du]_h| \in L^1(Q(\varrho, \theta\varrho^{p+2})) , \end{cases}$$

where C doesn't depend on ε, we can apply Lebesgue theorem: since Du_h^ε and u_h^ε—up to a subsequence—converges to Du_h and u_h a. e. in Q, for $\varepsilon \to 0$ we obtain

$$\int_{Q_t} [F(u, Du) Du]_h \cdot D\varphi \, dx \, ds \to \int_{Q_t} [F(u, Du) Du]_h \cdot D(u_h - k)_\pm \, \xi^\alpha \, dx \, ds + \\ + \alpha \int_{Q_t} [F(u, Du) D(u - k)_\pm]_h \cdot D\xi (u_h - k)_\pm \, \xi^{\alpha-1} \, dx \, ds .$$

For $h \to 0$

$$\int_{Q_t} [F(u, Du) Du]_h \cdot D\varphi \, dx \, ds \to \int_{Q_t} F(u, Du) |D(u - k)_\pm|^2 \, \xi^\alpha \, dx \, ds + \\ + \alpha \int_{Q_t} F(u, Du) D(u - k)_\pm \cdot D\xi (u - k)_\pm \, \xi^{\alpha-1} \, dx \, ds \tag{2.8}$$

(see also [7, chapter II] for the convergence arguments). Combining (2.7) with (2.8), we obtain

$$(2.9)\qquad \int_{K_{\sigma\varrho}} (u-k)_\pm^2\,\xi^\alpha\,dx + 2\int_{Q_t} F(u,Du)|D(u-k)_\pm|^2\xi^\alpha\,dx\,ds \leq$$

$$\leq 2\alpha\int_{Q_t} F(u,Du)|D(u-k)_\pm|(u-k)_\pm\,\xi^{\alpha-1}|D\xi|\,dx\,ds +$$

$$+\int_{K_\varrho}(u-k)_\pm^2\,\xi^\alpha(x,\,-\theta\varrho^{p+2})\,dx + \alpha\int_{Q_t}(u-k)_\pm^2\,\xi_t\xi^{\alpha-1}\,dx\,ds\,.$$

By means of (2.6), we can write

$$(2.10)\qquad \sum_{j=1}^\infty I_{j,t} = 2\alpha\int_{Q_t} F(u,Du)|D(u-k)_\pm|(u-k)_\pm\,\xi^{\alpha-1}|D\xi|\,dx\,ds\,,$$

where

$$I_{j,t} = 2\alpha\int_{Q_{j,t}} F(u,Du)|D(u-k)_\pm|(u-k)_\pm\,\xi^{\alpha-1}|D\xi|\,dx\,ds\,.$$

From the definition of A_j, using Holder's inequality with exponents $(\beta_j+2)/(\beta_j+1)$ and β_j, choosing $\alpha=\beta_j+2$, we get

$$(2.11)\qquad I_{j,t} \leq 2(\beta_j+1)\,\widetilde{C}_2\,\varepsilon^{(\beta_j+2)/(\beta_j+1)}\int_{Q_{j,t}} |D(u-k)_\pm|^{\beta_j+2}\xi^{\beta_j+2}\,dx\,ds +$$

$$+2\,\frac{\widetilde{C}_2}{\varepsilon^{\beta_j+2}}\int_{Q_{j,t}}(|D\xi|(u-k)_\pm)^{\beta_j+2}\,dx\,ds\,.$$

Inserting (2.10) in (2.9) and using (2.11), we obtain

$$(2.12)\qquad \int_{K_{\sigma\varrho}}(u-k)_\pm^2\,\xi^\alpha\,dx + \widetilde{C}_1\sum_{j=1}^\infty\int_{Q_{j,t}}(|D(u-k)_\pm|\xi)^{\beta_j+2}\,dx\,ds +$$

$$+\int_{Q_t^\sigma} F(u,Du)|D(u-k)|_\pm^2\,dx\,ds \leq \int_{K_\varrho}(u-k)_\pm^2\,\xi^{q+2}(x,\,-\theta\varrho^{p+2})\,dx +$$

$$+2\widetilde{C}_2\sum_{j=1}^\infty\frac{1}{\varepsilon^{\beta_j+2}}\int_{Q_{j,t}}((u-k)_\pm|D\xi|)^{\beta_j+2}\,dx\,ds +$$

$$+2\widetilde{C}_2(p+2)\sum_{j=1}^{\infty}\varepsilon^{(\beta_j+2)/(\beta_j+1)}\int_{Q_{j,t}}(|D(u-k)_\pm|\xi)^{\beta_j+2}\,dx\,ds+$$

$$+(p+2)\int_{Q_t}(u-k)_\pm^2\,\xi^{q+1}\xi_t\,dx\,ds\,.$$

We fix the value

$$(2.13)\qquad \begin{cases}\varepsilon=\left(\dfrac{\widetilde{C}_1}{4\widetilde{C}_2(p+2)}\right)^{(q+1)/(q+2)}=\min\limits_{j\geq 1}\varepsilon_j\,,\\[2ex] \varepsilon_j=\left(\dfrac{\widetilde{C}_1}{4\widetilde{C}_2(p+2)}\right)^{(\beta_j+1)/(\beta_j+2)}.\end{cases}$$

By (2.13), (2.12) becomes

$$(2.14)\qquad \int_{K_{\sigma\varrho}}(u-k)_\pm^2\,\xi^{p+2}\,dx+\frac{\widetilde{C}_1}{2}\sum_{j=1}^{\infty}\int_{Q_{j,t}}(|D(u-k)_\pm|\xi)^{\beta_j+2}\,dx\,ds+$$

$$+\int_{Q_t^\sigma}F(u,Du)|D(u-k)|_\pm^2\,dx\,ds\leq\int_{K_\varrho}(u-k)_\pm^2\,\xi^{q+2}(x,-\theta\varrho^{p+2})\,dx+$$

$$+C(\text{data})\sum_{j=1}^{\infty}\int_{Q_{j,t}}((u-k)_\pm|D\xi|)^{\beta^j+2}\,dx\,ds+$$

$$+(p+2)\int_{Q_t}(u-k)_\pm^2\,\xi^{q+1}\xi_t\,dx\,ds\,.$$

In order to estimate the second term of the right hand side of (2.14) we observe that since

$$\sum_{j=1}^{\infty}\int_{Q_{j,t}}((u-k)_\pm|D\xi|)^{\beta_j+2}\,dx\,ds=$$

$$=\sum_{j=1}^{\infty}\int_{Q_{j,t}}((u-k)_\pm|D\xi|)^{\beta_j+2}\chi_{\{(u-k)_\pm>0\}}\,dx\,ds$$

where $\chi_{\{A\}}$ is the characteristic function on the set A. Applying Holder's inequality we get

$$(2.15)\qquad \sum_{j=1}^{\infty} \int_{Q_{j,t}} ((u-k)_\pm |D\xi|)^{\beta_j+2}\,dx\,ds \leq$$

$$\leq \sum_{j=1}^{\infty} \frac{\beta_j+2}{p+2} \int_{Q_{j,t}} (|D\xi|(u-k)_\pm)^{p+2}\,dx\,ds +$$

$$+\sum_{j=1}^{\infty} \frac{p-\beta_j}{p+2} \int_{Q_{j,t}} \chi_{\{(u-k)_\pm>0\}}\,dx\,ds \leq$$

$$\leq \int_{Q(\varrho,\,\theta\varrho^{p+2})} ((u-k)_\pm |D\xi|)^{p+2}\,dx\,dt + \frac{p-q}{p+2}|A_{k,\varrho}^{\pm}|\,.$$

We apply (1.3) on the third term of the left hand side of (2.14) in which also we insert (2.15). This yealds the following estimate:

$$(2.16)\qquad \int_{K_{\sigma\varrho}} (u-k)_\pm^2\,\xi^{p+2}\,dx + \frac{\widetilde{C}_1}{2}\sum_{j=1}^{\infty}\int_{Q_{j,t}} (|D(u-k)_\pm|\xi)^{\beta_j+2}\,dx\,ds +$$

$$+C_1(M)\int_{Q_t^\sigma} |D(u-k)|_\pm^{q+2}\,dx\,ds \leq C(\text{data})|A_{k,\varrho}^{\pm}| +$$

$$+C(\text{data})\int_{Q(\varrho,\,\theta\varrho^{p+2})} ((u-k)_\pm|D\xi|)^{p+2}\,dx\,ds +$$

$$+(p+2)\int_{Q_t}(u-k)_\pm^2\,\xi^{q+1}\xi_t\,dx\,ds + \int_{K_\varrho}(u-k)_\pm^2\,\xi^{q+2}(x,\,-\theta\varrho^{p+2})\,dx\,.$$

(2.5) follows from (2.16), using the fact that t was arbitrary in the interval $(-\theta\varrho^{p+2}, 0)$. ■

PROPOSITION 2.2. *Let* (1.1)-(1.6) *hold. There exists a constant* $C = C(data)$ *such that for every level* k *satisfying* (2.1) *and for* $\delta \leq 1/2$

$$(2.17)\qquad \sup_{t_0-\theta\varrho^{p+2}\leq t\leq t_0} \int_{[x_0+K_{\sigma\varrho}]} \Psi^2(u)\,dx \leq C \int_{[x_0+K_\varrho]} \Psi^2(u)(x,\,t_0-\theta\varrho^{p+2})\,dx +$$

$$+C\ln\left(\frac{H_k^{\pm}}{a}\right)\left(\int_{[(x_0,\,t_0)+Q(\varrho,\,\theta\varrho^{p+2})]} |D\xi|^{p+2}\,dx\,dt + |A_{k,\varrho}^{\pm}|\right).$$

PROOF. We may assume $(x_0, t_0) = (0, 0)$ and we work with the sets A_j and the cylinders $Q^\sigma_{k,t}$, $Q_{k,t}$ and Q_t introduced earlier. We take the cutoff functions ξ defined in (2.3) indipendent of $t \in (-\theta\varrho^{p+2}, 0)$ and we select the testing functions

$$\varphi = (\Psi^2(u_h^\varepsilon))'\,\xi^\alpha$$

where we use the symbol $' = \partial u$ and where u_h^ε is defined as in the proof of Proposition 2.1.

By direct calculation

$$(\Psi^2(u_h^\varepsilon))'' = 2(1+\Psi)\,\Psi'^2 \in L^\infty_{\mathrm{loc}}(\Omega_T)$$

which implies that φ is an admissable function in (1.6). We follow the scheme used in the previous proof: we integrate by parts over $Q_{k,t}$ and consider separately the terms. For $\varepsilon \to 0$ we get

$$\int_{Q_t} \partial_s u_h (\Psi^2(u_h^\varepsilon))'\,\xi^\alpha\, dx\, ds \to \int_{Q_t} \partial_s u_h (\Psi^2(u_h))'\,\xi^\alpha\, dx\, ds =$$

$$= \int_{K_\varrho} \Psi^2(u_h(x,s))\big|_{s=-\theta\varrho^{p+2}}^{s=t}\,\xi^\alpha(x)\, dx$$

then for $h \to 0$

$$\int_{Q_t} \partial_s u_h\, \varphi\, dx\, ds \to \int_{K_\varrho} \Psi^2(u(x,s))\big|_{s=-\theta\varrho^{p+2}}^{s=t}\,\xi^\alpha(x)\, dx\,. \tag{2.18}$$

In the same way, for $\varepsilon \to 0$

$$\int_{Q_t} [F(u, Du)\,Du]_h \cdot D\varphi\, dx\, ds \to \int_{Q_t} [F(u, Du)\,Du]_h \cdot D((\Psi^2(u_h)')\,\xi^\alpha)\, dx\, ds\,.$$

For $h \to 0$, the previous inequality implies

$$\int_{Q_t} [F(u, Du)\,Du]_h \cdot D\varphi\, dx\, ds \to$$

$$\to 2\int_{Q_t} F(u, Du)\,|Du|^2 (1+\Psi)\,\Psi'^2\,\xi^\alpha\, dx\, ds +$$

$$+ \alpha \int_{Q_t} F(u, Du)\,Du(\Psi^2)'\,\xi^{\alpha-1} D\xi\, dx\, ds\,. \tag{2.19}$$

Collecting (2.18) with (2.19), we obtain

$$\int_{K_{\sigma\varrho}} \Psi^2(u)\,dx + 2\int_{Q_t} F(u, Du)|D(u-k)_\pm|^2(1+\Psi)\Psi'^2\xi^\alpha\,dx\,ds \leqslant \tag{2.20}$$

$$\leqslant 2\alpha\int_{Q_t} F(u, Du)|D(u-k)_\pm|(1+\Psi)\Psi'\xi^{\alpha-1}|D\xi|\,dx\,ds +$$

$$+\int_{K_\varrho} \Psi^2(x, -\theta\varrho^{p+2})\,dx\,.$$

We repeat the method used in Proposition (2.1). Define

$$R_{j,t} = 2\alpha\int_{Q_{j,t}} F(u, Du)|D(u-k)_\pm|(1+\Psi)\Psi'\xi^{\alpha-1}|D\xi|\,dx\,ds\,.$$

By Lemma 2.1, using Holder's inequality with exponents $(\beta_j+2)/$ $/(\beta_j+1)$ and β_j+2, choosing $\alpha=\beta_j+2$, we get

$$R_{j,t} \leqslant \tag{2.21}$$

$$\leqslant 2(\beta_j+1)\widetilde{C}_2\,\varepsilon^{(\beta_j+2)/(\beta_j+1)}\int_{Q_{j,t}} |D(u-k)_\pm|^{\beta_j+2}(1+\Psi)\Psi'^2\xi^\alpha\,dx\,ds +$$

$$+\frac{2\widetilde{C}_2}{\varepsilon^{\beta_j+2}}\int_{Q_{j,t}} \Psi\Psi'^{-\beta_j}|D\xi|^{\beta_j+2}\,dx\,ds\,.$$

By virtue of (2.2) and (2.1)

$$\Psi \leqslant \ln\left(\frac{H_k^\pm}{a}\right), \qquad \Psi' \leqslant \frac{1}{a}\,, \qquad (\Psi')^{-1} \leqslant 2\delta\,. \tag{2.22}$$

Therefore, inserting (2.22) in (2.21), we get

$$R_{j,t} \leqslant \tag{2.23}$$

$$\leqslant 2(p+2)\widetilde{C}_2\,\varepsilon^{(\beta_j+2)/\beta_j+1)}\int_{Q_{j,t}} |D(u-k)_\pm|^{\beta_j+2}(1+\Psi)\Psi'^2\xi^\alpha\,dx\,ds +$$

$$+\frac{2\widetilde{C}_2}{\varepsilon^{\beta_j+2}}\ln\left(\frac{H_k^\pm}{a}\right)\int_{Q_{j,t}} |D\xi|^{\beta_j+2}\chi_{\{(u-k)_\pm>0\}}\,dx\,ds\,.$$

Inserting (2.23) in (2.20), using (1.3) together with (1.7) and (2.4), we obtain

$$\int_{K_{\sigma\varrho}} \Psi^2\,dx + C_1(M)\int_{Q_t^\varrho} |D(u-k)_\pm|^{q+2}(1+\Psi)\,\Psi'^2\,dx\,ds + \tag{2.24}$$

$$+\tilde{C}_1\sum_{j\geq 1}\int_{Q_{j,t}} |D(u-k)_\pm|^{\beta_j+2}(1+\Psi)\,\Psi'^2\xi^\alpha\,dx\,ds \leq$$

$$\leq 2(p+2)\tilde{C}_2\sum_{j\geq 1}\int_{Q_{j,t}} |D(u-k)_\pm|^{\beta_j+2}(1+\Psi)\,\Psi'^2\xi^\alpha\,dx\,ds +$$

$$+\sum_{d\geq 1}\frac{2\tilde{C}_2}{\varepsilon^{\beta_j+2}}\ln\left(\frac{H_k^\pm}{a}\right)\int_{Q_{j,t}}|D\xi|^{\beta_j+2}\chi_{\{(u-k)_\pm>0\}}\,dx\,ds + \int_{K_\varrho}\Psi^2(x,-\theta\varrho^{p+2})\,dx\,.$$

We fix ε as in (2.13) and we omit the third term on the left hand side; then (2.24) becomes

$$\int_{K_{\sigma\varrho}} \Psi^2\,dx + C_1(M)\int_{Q_t^\varrho} |D(u-k)_\pm|^{q+2}(1+\Psi)\,\Psi'^2\,dx\,ds \leq \tag{2.25}$$

$$\leq 2\tilde{C}_2\left(\frac{(p+2)4\tilde{C}_2}{\tilde{C}_1}\right)^{q+1}\ln\left(\frac{H_k^\pm}{a}\right)\sum_{j\geq 1}\int_{Q_{j,t}}|D\xi|^{\beta_j+2}\chi_{\{(u-k)_\pm>0\}}\,dx\,dt +$$

$$+\int_{K_\varrho}\Psi^2(x,-\theta\varrho^{p+2})\,dx\,.$$

Arguing as in the conclusion of the Proposition 2.1

$$\sum_{j\geq 1}\int_{Q_{j,t}}|D\xi|^{\beta_j+2}\chi_{\{(u-k)_\pm>0\}}\,dx\,ds \leq \tag{2.26}$$

$$\leq \int_{Q(\varrho,\,\theta\varrho^{p+2})}|D\xi|^{p+2}\,dx\,dt + \frac{p-q}{p+2}\,|A_{k,\varrho}^\pm|\,.$$

Combining (2.26) with (2.25), (2.17) follows. ∎

3. - Basic results.

Let introduce the following subsets of $Q(\varrho, \theta\varrho^{p+2})$

$$A^{\pm}_{k,\varrho}(t) = \{x \in [x_0 + K_\varrho]: (u-k)_{\pm} > 0\}$$

for $t \in (t_0 - \theta\varrho^{p+2}, t_0)$.

Define

$$\mu^+ = \text{ess sup}\, u \qquad \forall (x,t) \in [(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})],$$

$$\mu^- = \text{ess inf}\, u \qquad \forall (x,t) \in [(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})]$$

and

$$\omega = \text{ess osc}\, u = \mu^+ - \mu^- \qquad \forall (x,t) \in [(x_0, t_0) + Q(\varrho, \theta\varrho^{p+2})].$$

The previous definitions imply

$$\omega \leq 2M \tag{3.1}$$

where M is the constant introduced in (1.7). In what follows, we always consider the values $k \in R$ of the type

$$k = \mu^+ - \zeta^+\omega, \qquad k = \mu^+ - \zeta^+\omega$$

where $\zeta^{\pm} \in (0, 1/2)$ and when we want to point out the dependence of k to $\zeta^{\pm}$, we shall write $A^{\pm}_{\zeta^{\pm},\varrho}$ instead of $A^{\pm}_{k,\varrho}$. For sake of semplicity, we also introduce the following notation:

let $m, p > 1$ and consider the Banach spaces

$$V^{m,p}(\Omega_T) = L^\infty(0, T; L^m(\Omega)) \cap L^p(0, T; W^{1,p}(\Omega)),$$

$$V^{m,p}_0(\Omega_T) = L^\infty(0, T; L^m(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega)),$$

both equipped with the norm

$$\|v\|_{V^{m,p}(\Omega_T)} = \operatorname*{ess\,sup}_{0<t<T} \|v(.,t)\|_{m,\Omega} + \|Dv\|_{p,\Omega_T}.$$

When $m = p$ we set $V^{p,p}(\Omega_T) = V^p(\Omega_T)$. Without loss of generality, we may assume

$$0 < \varrho \leq 1. \tag{3.2}$$

We apply the inequalities stated in section 2 to get an estimate of $|A^{+}_{\zeta^{+},\varrho}|$.

PROPOSITION 3.1. *There exists a number* $\nu^+ = \nu^+ \in (0, 1)$, $\nu^+ = \nu^+(\omega, \zeta^+, \theta, \text{data})$, *such that if*

$$|A^+_{\zeta^+, \varrho}| < \nu^+ \, |Q(\varrho, \theta\varrho^{p+2})| \tag{3.3}$$

then

$$u(x, t) \le \mu^+ - \frac{2}{3}\zeta^+\omega \qquad \forall (x, t) \in \left[(x_0, t_0) + Q\left(\frac{\varrho}{2}, \theta\,\frac{\varrho^{p+2}}{2}\right)\right]. \tag{3.4}$$

There exists a number $\nu^+ \in (0, 1)$, $\nu^+ = C(\text{data})/\theta$ *such that if* (3.3) *and also*

$$u(x, t_0 - \theta\varrho^{p+2}) \le \mu^+ - \zeta^+\omega \qquad \forall x \in [x_0 + K_\varrho] \tag{3.5}$$

hold, then

$$u(x, t) \le \mu^+ - \frac{2}{3}\zeta^+\omega \qquad \forall (x, t) \in \left[(x_0, t_0) + Q\left(\frac{\varrho}{2}, \theta\varrho^{p+2}\right)\right]. \tag{3.6}$$

Analogous result for the set $|A^-_{\zeta^-, \varrho}|$.

PROOF. We can assume $(x_0, t_0) = (0, 0)$. We will work within the cylinders

$$\tilde{Q}_n = K_{\tilde{\varrho}_n} \times (-\theta\tilde{\varrho}_n^{p+2}, 0), \qquad Q_n = K_{\varrho_n} \times (-\theta\varrho_n^{p+2}, 0),$$

where

$$\varrho_n = \frac{\varrho}{2} + \frac{\varrho}{2^{n+1}}, \qquad \tilde{\varrho}_n = \frac{\varrho}{2} + \frac{3\varrho}{2^{n+2}}.$$

In Q_n we define a sequence of cutoff functions ξ_n such that $\xi_n = 1$ over $\tilde{Q}_n$, $\xi_n = 0$ on the parabolic boundary of Q_n, $|D\xi_n| \le 2n/\varrho$, $0 \le \partial_t \xi_n \le 2^n/\theta\varrho^{p+2}$. Then we choose

$$k_n = \mu^+ - \zeta_n\omega, \qquad \zeta_n = \frac{2}{3}\zeta + \frac{1}{3}\frac{\zeta}{2n}.$$

First we observe that

$$(u - k_n)_+ = \zeta_n\omega \le \zeta\omega < \frac{\omega}{2}$$

so that (3.1) and (1.10) imply

$$(u - k_n)_+ \le \delta < 1, \qquad \forall n \ge 1.$$

We can apply (2.5): using the fact that $(u - k_n)_+^2 \geq (u - k_n)_+^{q+2}$ and $(u - k)_+^{p+2} \leq (u - k)_+^{q+2}$, we get

$$\sup_{-\theta\tilde{\varrho}_n^{p+2} < t < 0} \int_{K_{\tilde{\varrho}_n}} (u - k_n)_+^{q+2} \xi_n^{q+2}\, dx + \tag{3.7}$$

$$+ C_1 \int_{\tilde{Q}_n} |D(u - k_n)_+|^{q+2} \xi_n^{q+2}\, dx\, dt \leq$$

$$\leq C \frac{2^{nq}}{\varrho^{p+2}} (\zeta\omega)^{q+2} \left(1 + \frac{(\zeta\omega)^{-q}}{\theta} \right) |A^+_{\xi_n, \varrho_n}| + C |A^+_{\xi_n, \varrho_n}| .$$

Let $x \mapsto \xi_n(x)$ be a non negative piecewise smooth cutoff function defined in $K_{\tilde{\varrho}_n}$, which equals one on $K_{\varrho_{n+1}}$ and which satisfies

$$|D\xi_n| \leq \frac{2^n}{\varrho}$$

then $(u - k_n)_+ \xi_n \in V_0^{q+2}(\tilde{Q}_n)$. Since

$$\int_{\tilde{Q}_n} |D((u - k_n)_+ \xi_n)|^{q+2}\, dx\, dt \leq$$

$$\leq \int_{\tilde{Q}_n} |D(u - k_n)_+|^{q+2} \xi_n^{q+2}\, dx\, dt + C \frac{2^{nq}}{\varrho^{p+2}} (\zeta\omega)^{q+2} |A^+_{\xi_n, \varrho_n}|$$

we can rewrite (3.7) in a more concise way:

$$(1 \wedge C_1) \|(u - k_n)_+ \xi_n\|^{q+2}_{V_0^{q+2}(\tilde{Q}_n)} \leq \tag{3.8}$$

$$\leq C \frac{2^{nq}}{\varrho^{p+2}} (\zeta\omega)^{q+2} \left(2 + \frac{(\zeta\omega)^{-q}}{\theta} \right) |A^+_{\xi_n, \varrho_n}| + C |A^+_{\xi_n, \varrho_n}| .$$

By means of Corollary 3.1 in chapter I of [5], Holder's inequality and the fact that $(u - k_n)_+ \xi_n \in (0, 1]$, we get

$$\int_{\tilde{Q}_n} ((u - k_n)_+ \xi_n)^{q+2}\, dx\, dt \leq \tag{3.9}$$

$$\leq \left(\int_{\tilde{Q}_n} ((u - k_n)_+ \xi_n)^{(q+2)((N+p+2)/N)} \right)^{N/(N+p+2)} |A^+_{\xi_n, \varrho_n}|^{(p+2)/(N+p+2)} \leq$$

$$\leqslant \left(\int_{\tilde{Q}_n} (D(u-k_n)_+ \xi_n)^{q+2} \right)^{N/(N+p+2)} \cdot$$

$$\cdot \left(\operatorname*{ess\,sup}_{-\theta\tilde{\varrho}_n^{p+2} \leqslant t < 0} \int_{K_{\tilde{\varrho}_n}} ((u-k_n)^+ \xi_n)^{p+2} \right)^{(q+2)/(N+p+2)} |A^+_{\zeta_n, \varrho_n}|^{(p+2)/(N+p+2)} \leqslant$$

$$\leqslant \left(\int_{\tilde{Q}_n} (D(u-k_n)_+ \xi_n)^{q+2} \right)^{N/(N+p+2)} \cdot$$

$$\cdot \left(\operatorname*{ess\,sup}_{-\theta\tilde{\varrho}_n^{p+2} \leqslant t < 0} \int_{K_{\tilde{\varrho}_n}} ((u-k_n)_+ \xi_n)^{q+2} \right)^{(q+2)/(N+p+2)} |A^+_{\zeta_n, \varrho_n}|^{(p+2)/(N+p+2)} .$$

By means of (3.8), (3.9) becomes

$$\int_{\tilde{Q}_n} ((u-k_n)_+ \xi_n)^{q+2} \, dx \, dt \leqslant$$

$$\leqslant C \left(\frac{2^{nq}}{\varrho^{p+2}} (\zeta\omega)^{q+2} \left(2 + \frac{(\zeta\omega)^{-q}}{\theta} \right) |A^+_{\zeta_n, \varrho_n}| + C |A^+_{\zeta_n, \varrho_n}| \right)^{(N+q+2)/(N+p+2)} \cdot$$

$$\cdot |A^+_{\zeta_n, \varrho_n}|^{(p+2)/(N+p+2)} .$$

Next we observe that

$$2^{nq} \frac{|A^+_{\zeta_n, \varrho_n}|}{\varrho^{p+2}} \geqslant 1$$

indeed, if not, we would immediately get (3.4) and the continuity theorem will easily follows. Thus

$$\int_{\tilde{Q}_n} ((u-k_n)_+ \xi_n)^{q+2} \, dx \, dt \leqslant \tag{3.10}$$

$$\leqslant C \frac{2^{nq}}{\varrho^{p+2}} \left((\zeta\omega)^{q+2} \left(2 + \frac{(\zeta\omega)^{-q}}{\theta} \right) + 1 \right) |A^+_{\zeta_n, \varrho_n}|^{1+(p+2)/(N+p+2)} .$$

The left hand side of (3.10) is estimated below by

$$(3.11)\qquad \int_{\tilde{Q}_n} ((u-k_n)_+\xi_n)^{q+2}\,dx\,dt \geq \int_{Q_{n+1}} ((u-k_n)_+\xi_n)^{q+2}\,dx\,dt \geq$$

$$\geq |k_n - k_{n+1}|^{q+2}\,|A^+_{\zeta_{n+1},\varrho_{n+1}}| \geq \frac{(\zeta\omega)^{q+2}}{2^{(q+2)(n+2)}}\,|A^+_{\zeta_{n+1},\varrho_{n+1}}|\,.$$

Combining (3.10) with (3.11), we obtain

$$(3.12)\qquad |A^+_{\zeta_{n+1},\varrho_{n+1}}| \leq C\frac{4^{qn}}{\varrho^{p+2}}\left(2+\frac{(\zeta\omega)^{-q}}{\theta}+(\zeta\omega)^{-(q+2)}\right)|A^+_{\zeta_n,\varrho_n}|^{1+(p+2)/(N+p+2)}.$$

We divide (3.12) by $|Q_{n+1}|$ and introduce the quantities $Y_n = |A^+_{\zeta_n,\varrho_n}|/|Q_n|$ so that (3.12) becames

$$Y_{n+1} \leq C b^n Y_n^{1+(p+2)/(N+p+2)}$$

where

$$b=4^q,\quad C=C(\text{data})\left(2+\frac{(\zeta\omega)^{-q}}{\theta}+(\zeta\omega)^{-(q+2)}\right)\theta^{(p+2)/(N+p+2)},\quad \alpha=\frac{p+2}{N+p+2}.$$

By [5, Lemma 4.1, chapt. I], (3.4) follows if (3.3) holds with

$$\nu^+ = \frac{C}{\theta}\left(3+\frac{(\zeta\omega)^{-q}}{\theta}\right)^{-(N+p+2)/(p+2)}.$$

(3.6) follows in a similar way choosing in (2.5) a sequence of testing functions depending only on x and working within the cylinders

$$\tilde{R}_n = K_{\tilde{\varrho}_n}\times(-\theta\varrho^{p+2},0),\qquad R_n = K_{\varrho_n}\times(-\theta\varrho^{p+2},0).$$

The main difference is that in this case

$$\nu^+ = \frac{C(\text{data})}{\theta}. \qquad \blacksquare$$

PROPOSITION 3.2. *Let $\zeta_0^+ \in (0, 1/2)$ be a fixed number such that*

$$u(x, t_0 - \theta\varrho^{p+2}) \leqslant \mu^+ - \zeta_0^+ \omega, \qquad \forall x \in [x_0 + K_\varrho], \tag{3.13}$$

then for every $\nu^+ \in (0, 1)$, there exists a number $\zeta^+ \in (0, \zeta_0^+/4)$, where $\zeta^+ = \zeta^+(\zeta_0^+, \theta$, data), such that

$$|A^+_{\zeta^+, \varrho}(t)| \leqslant \nu^+ |K_{\varrho/2}|, \qquad \forall t \in (t_0 - \theta\varrho^{p+2}, t_0). \tag{3.14}$$

Analogous result for the set $|A^-_{\zeta^-, \varrho}(t)|$.

PROOF. We may assume $(x_0, t_0) = (0, 0)$ and choose in (2.2) the values

$$k = \mu^+ - \zeta_0^+ \omega, \qquad a = \zeta^+ \omega,$$

where $0 < \zeta^+ < \zeta_0^+/4$ will be chosen later. In this way $(u - k)_+ \leqslant \zeta_0^+ \omega \leqslant 1/2$ so that (2.1) is satisfied with $\delta \leqslant 1/2$. We can apply proposition 2.2: we choose a testing function ξ such that

$$|D\xi| \leqslant 2/\varrho. \tag{3.15}$$

Using (3.13), (3.15) and the fact that $0 < \varrho \leqslant 1$, we get that the right hand side of (2.17) is majorised by

$$2C\theta \ln\left(\frac{\zeta_0^+}{2\zeta^+}\right) |K_{\varrho/2}|. \tag{3.16}$$

We minorise the left hand side of (2.17) by

$$\left(\ln\left(\frac{\zeta_0^+}{2\zeta^+}\right)\right)^2 |A^+_{\zeta, \varrho/2}(t)|. \tag{3.17}$$

(3.17) together with (3.16) gives

$$|A^+_{\zeta, \varrho/2}(t)| \leqslant \frac{C\theta}{\ln(\zeta_0^+/2\zeta^+)} |K_{\varrho/2}|.$$

To prove (3.14), we choose ζ^+ from the relation

$$\zeta^+ = \frac{\zeta_0^+}{2} \exp(-\theta C/\nu^+). \qquad \blacksquare \tag{3.18}$$

4. – Proof of the continuity theorem.

In this section, we prove the main theorem following the method used in [6, Sections 5-29] and [14, Section 4]. Up to a translation we can assume $(x_0, t_0) = (0, 0)$ and choose the parameter θ equal to 1.

The first alternative.

If (3.3) hold, thanks to (3.4), we can apply proposition 3.2. (3.14) implies again (3.3) from which (3.4) follows in the cylinder $Q(\varrho/4, (\varrho/4)^{p+2})$. This—joined with the analogous result for $A_{\xi^-, \varrho}$—implies

$$\operatorname*{ess\,osc}_{Q(\varrho/8,\, \varrho^{p+2}/8)} u \leq \eta\omega \tag{4.1}$$

where $\eta = \eta(\omega) \in (0, 1)$. Thus, going down from $Q(\varrho, \varrho^{p+2})$ to $Q(\varrho/8, \varrho^{p+2}/8)$, the oscillation of u decreases by a factor η.

The second alternative: the case $N = q + 2$.

Assume that both (3.3) and the analogous for $|A_{\xi^-, \varrho}|$ are violated. Arguing as in [6, sections 5-9], we get the existence of a constant $C = C(\text{data})$ such that:

$$(\delta\varrho)^{N+p-q}\omega \leq C \int_D |Du|^{q+2}\, dx\, dt \tag{4.2}$$

where $D = \{\|x\| \in (\delta\varrho, \varrho) \times (-\varrho^{p+2}, 0)\}$ and $\delta \in (0, 1)$ is an increasing function of ω, $\delta(\omega) \to 0$ for $\omega \to 0$ [6, Propositions 8.1].

We write (4.2) for the family of cylinders

$$Q_n^j = [(0, t_n^j) + Q(\delta^n \varrho, (\delta^n \varrho)^{p+2})], \qquad t_n^j = -j(\delta^n \varrho)^{p+2},$$

$$j = 0, 1, \ldots, \delta^{-(p+2)n} - 1.$$

Adding over j and then over $n = 0, 1, \ldots, n_0 - 1$, we arrive at the following estimate:

$$n_0 \delta^{n^{(N-(q+2))}} \varrho^{N+p-q} \omega^{q+2} \leq C(\text{data}) \int_{Q(\varrho, \varrho^{p+2})} |Du|^{q+2}\, dx\, dt.$$

Thanks to (2.5), we can majorize the right hand side of the previous inequality by $C\omega^2\varrho^N$. For $N = q + 2$ we get:

$$n_0 \leq C(\text{data})\, \omega^{-q} \varrho^{q-p}.$$

We choose a number n_0 such that the reverse inequality holds. The contradiction implies (3.3) must hold and the continuity theorem follows arguing as in the first alternative.

The case $N > q + 2$.

In order to state the main theorem in dimension greater than $q + 2$, we substantially follow [6, Sections 22–29]: first we make a partition of $Q(\varrho, \theta\varrho^{p+2})$ in the following way:

$$(4.3)\quad Q(\varrho, \theta\varrho^{p+2}) = \bigcup_{i=0,1,\ldots,(\theta-1)} [(0, t_i) + Q(\varrho, \varrho^{p+2})], \quad t_i = -i\varrho^{p+2}.$$

We fix one of these box and translate its vertex into the origin. By few changes of Propositions 22.1 and 23.1 of [6], we obtain the following result:

PROPOSITION 4.1. *There exists positive integers m and $\delta \in (0, 1)$, which can be determined a priori in terms of only* data, ω, *such that for each box* $[(0, t_i) + Q(\varrho, \varrho^{p+2})]$, *there exists a subcylinder* $[(x_l, t_h) + Q(2\delta_0\varrho, 2(\delta_0\varrho)^{p+2})]$, *for which either*

$$(4.4)\qquad u(x, t) > \mu^- + \frac{\omega}{18}, \qquad \forall (x, t) \in [(x_l, t_h) + Q(\delta_0\varrho, (\delta_0\varrho)^{p+2})]$$

or

$$(4.5)\qquad u(x, t) < \mu^+ - \frac{\omega}{18}, \qquad \forall (x, t) \in [(x_l, t_h) + Q(\delta_0\varrho, (\delta_0\varrho)^{p+2})]$$

where

$$\delta_0(\omega) = \frac{\delta(\omega)}{4m(\omega)}, \qquad s(\omega) = \left(\frac{4m(\omega)}{\delta(\omega)}\right)^N, \qquad t(\omega) = \left(\frac{4m(\omega)}{\delta(\omega)}\right)^{p+2},$$

$$t_h = (1-h)(2\delta_0\varrho)^{p+2}, \qquad h = 1, \ldots, t(\omega), \qquad l = 1, \ldots, s(\omega).$$

Now assume that one of the previous conditions, say for example (4.4), holds in the cylinder

$$[(x_l, \tau) + Q(\delta_0\varrho, (\delta_0\varrho)^{p+2})]$$

$\tau \in [-h(4m\delta_0)^{p+2} - r^{p+2}, (1-h)(4m\delta_0)^{p+2}]$, $r \in [\delta\varrho/m, \varrho/m]$, contained in the lower half of $Q(\varrho, \theta\varrho^{p+2})$, θ being fixed. From such a box we can construct the long cylinder

$$[(x_l, 0) + Q(4r, 4\bar{\theta} r^{p+2})]$$

where $2\delta_0^{-(p+2)}(\theta - 1) \leq \bar{\theta} \leq 4\delta_0^{-(p+2)}\theta$, $r = \delta_0\varrho$.

Thus (4.4) implies

$$u(x, -4\bar{\theta}r^{p+2}) > \mu^- + \frac{\omega}{18}, \qquad \forall x \in [x_l + K_{4r}].$$

The last information is the analogous of (3.13); using Proposition 3.1, we get the following result:

PROPOSITION 4.2. *Under the previous assumptions, there exists a number* $\zeta \in (0, 1/18)$, *that can be determined a priori in terms of* data, ω, *such that*

$$u(x, t) > \mu^- + \zeta\omega, \qquad \forall (x, t) \in [(x_l, 0) + Q(r, \bar{\theta}r^{p+2})]. \tag{4.6}$$

PROOF. See [6, Proposition 24.1] for more details. ∎

If $x_l = 0$, we could procede as in the first alternative. Since in general is not so, we have to prove that an estimate similar to (4.6) actually holds in a cylinder with vertex at the origin. We will determine the number θ as the product of a finite increasing sequence of positive integers k_j, i.e.

$$\theta = \prod_{j=1}^{2s(\omega)} k_j .$$

We assume that $k_1, \ldots, k_j$ have been found and determine k_{j+1}. We take contiguous stacks of boxes of the type (4.3), each containing $\tilde{k}_j = \prod_{i=1}^{j} k_i$ of such cylinders, i.e.

$$S_n^j = \bigcup_{i=(n-1)\tilde{k}_j}^{n\tilde{k}_j - 1} [(0, t_i) + Q(\varrho, \varrho^{p+2})].$$

In turn we form larger stacks by taking k_{j+1} contiguous stacks of the form S_n^j, i. e.

$$S_m^{j+1} = \bigcup_{n=(m-1)k_{j+1}+1}^{mk_j+1} S_n^j .$$

Each S_m^{j+1} is the union of k_{j+1} pairwise disjoint stacks S_n^j, By a lemma of sequential selection [6, Lemma 29.1], we find a $j \in \{1, \ldots, s(\omega)\}$ such that among the stacks S_m^{j+1} there exists one where (4.4) holds for the same abscissa x_l, for at least one cube of the type (4.3), within each of the smaller stacks S_n^j (see also [6, section 26]). Now we rewrite (4.4):

within $Q(\varrho, \theta\varrho^{p+2})$, for some $t_{j+1} \in (-\theta, 0)$, there is a cylinder

$$(4.7) \qquad [(x_l, t_{j+1}) + Q(r, k_{j+1}(\delta_0^{-1}\varrho)^{p+2})], \qquad r = \frac{\delta_0 \varrho}{4}$$

such that

$$(4.8) \quad u(x, t) > \mu^- + \zeta_j \omega, \quad \forall (x, t) \in [(x_l, t_{j+1}) + Q(r, k_{j+1}(\delta_0^{-1}\varrho)^{p+2})]$$

As $x_l \neq 0$, we can consider it as the centre of a ball $B_{4\sqrt{N}\varrho}(x_l)$ contained in $K_{4\varrho}$ and including $K_{2\varrho}$. Now we make a change of variables:

$$x \mapsto 4\frac{x - x_l}{|x_l|} \; t \mapsto \frac{4^{p+2}(t^{j+1} - t) + k_{j+1}\varrho^{p+2}}{\varrho^{p+2}} .$$

Since $\delta_0 \varrho/4 \leq \|x\| \leq \sqrt{N}\varrho$, we have the transformation

$$(4.9) \quad \{\delta_0 \varrho/4 \leq \|x - x_l\| \leq \sqrt{N}\varrho\} \mapsto \{1 < \|x\| < 2d\}, \quad d \in (1, 4^3 \delta_0^{-1} \sqrt{N}) .$$

We introduce the new function

$$(4.10) \qquad w = \frac{u - \mu^-}{\zeta_j \omega} .$$

(4.8)-(4.10) enable us to formulate the following parabolic problem:

$$(4.11) \quad \begin{cases} w_t = \operatorname{Div}(G(w, Dw)Dw), & \forall (x, t) \in B_{4d} \times (0, k_{j+1}), \\ w \geq 1, \quad \forall (x, t) \in B_{\varepsilon_0} \times (0, k_{j+1}), & \varepsilon_0 = \dfrac{\delta_0}{4\sqrt{N}} . \end{cases}$$

Here we assume:

$$(4.12) \quad \begin{cases} G : R \times R^N \to R, \quad (w, Dw) \mapsto G(w, Dw) \\ \qquad \qquad \text{is continuous in } w \text{, uniformly continuous in } Dw, \\ z \mapsto G(w, z) \text{ is monotone for all } z \in R^N . \end{cases}$$

There exists two functions

$$a_1 = C_1(|\zeta_j \omega w + \mu^-|)(\zeta_j \omega)^q, \qquad a_2 = C_2(|\zeta_j \omega w + \mu^-|)(\zeta_j \omega)^p .$$

C_i, $i = 1, 2$ defined as in (1.3), such that

$$\begin{cases} a_1(|w|)|Dw|^q \leqslant G(w, Dw), \\ |G(w, Dw)| \leqslant a_2(|w|)|Dw|^p . \end{cases} \tag{4.13}$$

PROPOSITION 4.3. *Consider the parabolic problem*

$$\begin{cases} v_t = \operatorname{Div}(G(v, Dv)Dv), & \forall (x, t) \in \{\varepsilon_0 < |x| < 4d\} \times (0, k), \\ v(x, t) = 1, & \forall |x| = \varepsilon_0 , \\ v(x, t) = 0, & \forall |x| = 4d, \\ v(x, 0) = 0, \end{cases} \tag{4.14}$$

$$v \in V_0^{2, q+2}(B_{4d} \times (0, k)). \tag{4.15}$$

Under assumptions (4.11)-(4.13) *problem* (4.14)-(4.15) *has a unique continuous radial solution*

$$0 \leqslant v \leqslant 1, \qquad \forall (x, t) \in \{\varepsilon_0 < |x| < 4d\} \times (0, k). \tag{4.16}$$

Moreover, there exists a number σ_0 *and a time level* k *which can be determine a priori in terms of the* data, *such that, for every* $y \in \{1 < < |x| < d\}$, *there is a time* $t \in (0, k)$ *such that*

$$v(y, t) > \sigma_0 . \tag{4.17}$$

PROOF. With few modifications, we can make use of the energy estimates contained in sections 2, 3 and the first and the second alternative of the present section in order to prove the continuity of v. In particular, using the scheme of sections 20 and 21 of [6], we get (4.17), while (4.16) is a consequence of assumption (1.7) (see also [6, Propositions 13.1-13.3] for more details). ■

We will use v as a comparison function for the solution of (4.12). Indeed by (4.17), we can state the existence of a positive number $\sigma_{0,j}$ such that

$$v(y, t) > \sigma_{0,j}, \qquad \forall y \in \{1 < |x| < \tilde{d}_j\}, \tag{4.18}$$

where $t \in (0, k_{j+1})$. We fix a point for which (4.18) holds: since $v(\cdot, t)$ is continuous in $\{1 < |x| < 2d\}$, uniformly in t, there exists a ball

$B_{\sqrt{N}\tilde{d}_j}(y)$ such that

$$v(x, t) > \frac{\sigma_{0,j}}{2}, \qquad \forall x \in B_{\sqrt{N}\tilde{d}_j}(y).$$

Since $v(\cdot, t)$ is radial, the previous lower bound still holds at that point of the annulus $\{1 < |x| < 2d\}$ which coincides with the origin of the original coordinates. By (1.8) we get

$$w \geqslant v$$

i.e., using (4.10), coming back to the original variables, there exists a time level t_0,

$$t_0 \in [-\theta\varrho^{p+2}, -\theta\varrho^{p+2}/2]$$

and a number $\tilde{\delta} = \tilde{d}_j \delta_0$, such that

$$u(x, t_0) > \mu^- + \frac{1}{2}\sigma_{0,j}\zeta_j\omega, \qquad \forall x \in K_{\tilde{\delta}\varrho}.$$

By this procedure we can determine k_{j+1} from $k_1, \ldots, k_j$. Analogously if we have started from (4.5). Now the main theorem follows as in [6, Propositions 24.2, 24.3 and section 25].

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