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ALBERTO G. SETTI

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Eigenvalue Estimates for The Weighted Laplacian on a Riemannian Manifold.

Alberto G. Setti (*)

Abstract - Given a complete Riemannian manifold M and a smooth positive function w on M, let $L = -\Delta - \nabla(\log w)$ acting on $L^2(M, w\,dV)$. Generalizing techniques used in the case of the Laplacian, we obtain upper and lower bounds for the first non-zero eigenvalue of L, for M compact, and for the bottom of the spectrum, for M non-compact.

1. - Introduction.

Let M^n be an n-dimensional, complete Riemannian manifold. The Laplace operator on M, $-\Delta$, can be defined as the differential operator associated to the standard Dirichlet form

$$Q(u) = \int_{M^n} |\nabla f|^2 dV, \qquad u \in C_c^\infty(M) \subset L^2(dV),$$

where $|\cdot|$ is the norm induced by the Riemannian inner product $\langle\, ,\rangle$, and dV is the volume element on M^n. Now let w be a given smooth strictly positive function on M^n, that will be referred to as a weight function. If we replace the measure dV with the weighted measure $w\,dV$ in the definition of Q, we obtain a new quadratic form Q_w, and we denote by L the elliptic differential operator on $C_c^\infty(M^n) \subset L^2(w\,dV)$ induced by Q_w. In this sense L arises as a natural generalization of the Laplacian. It is clearly symmetric and positive and extends to a positive self-adjoint op-

(*) Indirizzo dell'A.: Dipartimento di Matematica, Università di Milano, via Saldini 50, 20133 Milano, Italy.

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erator on $L^2(w\,dV)$. By Stokes theorem,

$$Lu = -\Delta u - \langle \nabla(\log w), \nabla u\rangle . \tag{1.1}$$

Thus introducing a weight factor is the first step towards decoupling the leading term and the lower order terms of the operator, which in the case of the Laplace operator are completely determined by the metric of M^n.

Weighted Laplacians arise naturally, for example right-invariant (sub) Laplacians on a Lie group with left Haar measure can be viewed as weighted Laplacians with weight given by the modular function. One might also hope that the study of weighted Laplacians will increase the understanding of differential operators acting on L^p spaces with respect to measures on M^n not so closely related to the metric structure of the manifold, e.g. the sum of squares of left invariant vector fields on a Lie group which satisfy Hörmander condition, and generalisations. Finally, operators of this form are the finite dimensional model of operators on infinite dimensional manifolds («the infinite volume limit») which arise in Statistical Mechanics.

Operators of the form (1.1) have already been studied in [Bk1], [BkE], [Da1], [Da3], [Dl], [DlS], which mainly deal with properties of the heat semigroup generated by L, and in [Bk2] where the Riesz transform defined in terms of L is investigated. In this paper we study estimates for the L^2-spectrum of L, a problem also considered in [Dl] and in [DlS]. Pointwise estimates or the heat kernel of L have been obtained in [Se2].

Our goal is to find upper and lower bounds in terms of the geometry of M^n and of the properties of the weight function, for the first non-zero eigenvalue of L, when M^n is compact, and for the bottom of the spectrum if M^n is not compact.

Several approaches are possible: One could consider L as a perturbation of $-\Delta$ and apply perturbative methods. Or one could observe that under the map

$$w^{1/2}\cdot : L^2(w\,dV) \to L^2(dV)$$

L is unitarily equivalent to the Schrödinger operator $H = -\Delta + (w^{-1/2}\Delta w^{1/2})(\cdot)$ acting on $L^2(dV)$, and deduce estimates for L by Schrödinger operator techniques. Both approaches have the disadvantage of requiring hypotheses that depend separately upon the geometry of M^n (typically via the curvature) and on the function w (via uniform bounds for w and its derivatives), without keeping into account the possibility of mutual and perhaps competing interactions.

The approach presented here is more direct. The interplay between the geometry of M^n and the behaviour of the weight function w is mostly taken into account by means of a modified Ricci curvature defined by

$$R_w = \mathrm{Ric} - w^{-1} \mathrm{Hess}\, w\,.$$

Then, generalising techniques successfully applied in the case of $-\Delta$ by Lichnerowicz ([Lz]), Li and Yau ([LiY]) and Li ([Li]), we obtain upper and lower bounds for the first non-zero eigenvalue of L, if M^n is compact, and for the bottom of its L^2-spectrum, if M^n is non-compact. These bounds will be expressed in terms of bounds for R_w, of the dimension n of M^n and of its diameter.

The paper is organized as follows. In § 2 we study lower bounds for the first non-zero eigenvalue λ_1 of L on a compact manifold. We prove first a generalization of Lichnerowicz theorem (cf. [Lz], p. 135) that gives a lower bound for λ_1 in terms R_w. Then we use the technique of gradient estimates to obtain lower bounds for λ_1 which generalize the bounds obtained by Li and Yau ([LiY]) and Li ([Li]) for the Laplacian. We conclude the section by briefly indicating how Li and Yau's techniques can be used to find lower bounds for the first Dirichlet eigenvalue of L on a domain $\Omega \subset M$ with smooth boundary.

In § 3 we consider the case of a (non-compact) manifold with a pole and we prove a lower bound for the bottom of the spectrum λ_0 of L in terms of a negative upper bound for the radial curvature and an upper bound for $|w^{-1}\partial w/\partial r|$, which is in the same spirit as McKean's lower bound for the bottom of the spectrum of the Laplacian on a negatively curved manifold (cf. [McK]). Assuming further that the radial derivative $\partial w/\partial r$ is everywhere nonnegative we also show that λ_0 can be bounded below in terms of a negative upper bound for the radial component of R_w. We conclude the section with two examples that show that the bounds obtained are sharp.

In the last section we prove a comparison theorem for w-volume (Theorem 4.1) which generalizes Bishop's comparison theorem (cf. [Cl], p. 71 ff.). This allows us to extend to L a version of Cheng's comparison theorem ([Cg]) that gives an upper bound for the first generalized Dirichlet eigenvalue of L on a geodesic ball in M^n in terms of the first Dirichlet eigenvalue of the Laplacian on a geodesic ball of the same radius in a suitable space of constant curvature. Using Cheng's argument, this gives, in the compact case, upper bounds for all the eigenvalues of L in terms of a lower bound for R_w and of the diameter of M, and, in the non-compact case, an upper bound for the bottom of the spectrum in terms of a lower bound for R_w.

2. – The compact case: bounds from below for the first non-zero eigenvalue.

Let $M^n = M$ be an n-dimensional Riemannian manifold with weight function w, and let L denote the differential operator defined as in (1.1) above.

Using elliptic regularity, and an argument as in Strichartz ([St], see also Bakry [Bk2] where the details are carried out), one shows that for $\lambda < 0$ the equation $(L_o)^{\star} u = \lambda u$ has only the zero solution in $L^2(M, w\,dV)$, and consequently ([RSi], pp. 122-137) L is essentially self-adjoint on $C_c^{\infty}(M)$ and extends to a positive self-adjoint operator on $L^2(w\,dV)$.

Since L is a positive self-adjoint operator its spectrum is contained in the positive real axis. Moreover, since the eigenfunctions are smooth by elliptic regularity, 0 is an eigenvalue iff the w-volume of M, defined as the volume with respect to the measure $w\,dV$, is finite. If u is in the domain of L, then $\nabla u \in L^2(M, w\,dV)$ and $\|\nabla u\|^2_{L^2(w\,dV)} = (Lu, u)_{L^2(w\,dV)}$ ([Bk2], Prop. 1.3). It follows that for $\lambda < 0$, $(\lambda - L)^{-1}$ is a continuous map of $L^2(w\,dV)$ into $H^1(w\,dV) = \{u \in L^2(w\,dV): \nabla u \in L^2(w\,dV)\}$. Thus if M is a compact manifold, the compactness of the embedding $H^1(w\,dV) \hookrightarrow L^2(w\,dV)$ ([Au], Theorem 2.34), implies that $(\lambda - L)^{-1}$ is a compact operator on $L^2(w\,dV)$ and, as in the case of the Laplacian, we conclude that the spectrum of L is purely discrete and the eigenvalues can be arranged in a diverging sequence

$$0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \nearrow \infty .$$

The corresponding eigenfunctions $\{u_j\}$ are smooth on M and form a basis of $L^2(w\,dV)$.

Although we are mainly interested in the closed problem for L, in some instances we shall also consider the Dirichlet problem for L on an open relatively compact domain $\Omega \subset M$ with not-necessarily smooth boundary $\partial\Omega$. In this case the operator L with (generalized) Dirichlet boundary conditions on $\partial\Omega$ is the Friedrichs extension of the differential operator $L_o = -\Delta - \nabla(\log w)$ that acts as a positive symmetric operator on $C_c^\infty(M) \subset L^2(w\,dV)$ (cf. [R-Si], p.177). If the boundary $\partial\Omega$ is piecewise smooth, elliptic regularity and the compactness of the embedding $H_o^1(\Omega, w\,dV) \hookrightarrow L^2(\Omega, w\,dV)$, where $H_o^1(\Omega, w\,dV)$ is the closure of $C_c^\infty(\Omega)$ in the norm $\|u\|_1^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2$, imply, as above, that the spectrum of L is purely discrete and the eigenvalues can be arranged in a diverging sequence $0 < \lambda_0(\Omega) < \lambda_1(\Omega) \ldots \nearrow +\infty$.

At this point it is perhaps worth noticing that, even though $Q_w(u, u) = \int_M |\nabla u|^2 w\,dV$ can be written as the standard Dirichlet form relative to the metric $\hat{g} = w^{2/n-2} g$, this does not imply that L is unitarily equivalent to the Laplacian $-\hat{\Delta}$ induced by $\hat{g}$. Indeed L acts on $L^2(w\,dV)$ while $-\hat{\Delta}$ acts on $L^2(d\hat{V})$, where $d\hat{V} = w^{2/n-2}dV$ is the volume element induced by $\hat{g}$. Therefore we cannot reduce the problem of estimating the eigenvalues of L to the same problem for the Laplacian relative to a conformally changed metric. On the other hand Prof. C. Herz ([Hz]) pointed out that if N is a compact 1-dimensional manifold (typically $N = S^1$), the action of the Laplacian of the warped product $\widehat{M} = M \times_w N$ on functions constant on N coincides with the action of L on functions defined on M. Indeed, since $\widehat{M}$ is the product manifold $M \times N$ endowed with the metric

$$\hat{g} = \pi_M^* g_M \oplus (w \circ \pi_M)^2 \pi_N^* g_N,$$

where π_M, π_N are the projections and g_M, g_N the metrics respectively on M and N, a simple computation in local coordinates shows that, for $f = f(x, \theta) \in C^\infty(\widehat{M})$,

$$-\hat{\Delta} f(x, \theta) = (L_x f)(x, \theta) - w^{-2}(\Delta_N f)(x, \theta).$$

In particular, if f_o is an eigenfunction of L belonging to the eigenvalue λ, then $f(x, \theta) = f_o(x)$ is an eigenfunction of $-\hat{\Delta}$ belonging to the same eigenvalue. This fact can be used to obtain lower bounds for the first non-zero eigenvalue of L by appropriately adapting some of the results which hold for the Laplacian.

Assume now that M is compact. As observed above, the spectrum of L is purely discrete. The first eigenvalue being always zero, we are interested in lower bounds for the first non-zero eigenvalue of L. These lower bounds will necessarily depend on the interplay between the geometry of M and the behavior of the weight function w. In many instances this interplay can be effectively taken into account by means of a modified

Ricci curvature defined by

$$R_w = \mathrm{Ric} - w^{-1} \mathrm{Hess}\, w\,. \tag{2.1}$$

Following Bakry ([Bk2]) we also put

$$S_w = \mathrm{Ric} - \mathrm{Hess}\,(\log w) = R_w + d(\log w) \otimes d(\log w)\,. \tag{2.2}$$

We start with a generalization of the classical Bochner-Lichnerowicz-Weitzenböch formula for functions ([BGM], p. 131 ff.). In a more general and slightly different form, the formula appears in [BkE] and [Bk2]. The version which we present is better suited to our purposes and admits a simpler proof.

PROPOSITION 2.1. *Let M be a (not necessarily compact) manifold with weight function w. If $u \in C^2(M)$ we have*

$$-\frac{1}{2} L(|\nabla u|^2) = |\mathrm{Hess}\, u|^2 - \langle \nabla u, \nabla(Lu)\rangle + S_w(\nabla u, \nabla u) = \tag{2.3}$$

$$= |\mathrm{Hess}\, u|^2 - \langle \nabla u, \nabla(Lu)\rangle + R_w(\nabla u, \nabla u) + \langle \nabla(\log w), \nabla u\rangle^2.$$

PROOF. It suffices to prove the first equality, which follows immediately by combining the BLW formula,

$$\frac{1}{2} \Delta(|\nabla u|^2) = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u),$$

and the following formula for the drift term

$$1/2\langle \nabla(\log w), \nabla(|\nabla u|^2)\rangle = \langle \nabla u, \nabla(\langle \nabla(\log w), \nabla u\rangle)\rangle - \mathrm{Hess}\, w(\nabla u, \nabla u),$$

which, in turn, is a consequence of

$$\langle \nabla u, \nabla(\langle \nabla(\log w), \nabla u\rangle)\rangle = \langle \nabla_{\nabla u}(\nabla(\log w)), \nabla u\rangle + \langle \nabla(\log w), \nabla_{\nabla u} \nabla u\rangle =$$

$$= \mathrm{Hess}\, w(\nabla u, \nabla u) + \langle \nabla u, \nabla_{\nabla(\log w)} \nabla u\rangle.$$

The next theorem extends to L a classical result by Lichnerowicz ([Lz], p. 135]).

THEOREM 2.2. *Let M be a compact, n-dimensional Riemannian manifold with weight function w, and let λ_1 be the first non-zero eigen-*

value of L. Assume that

$$R_w \geqslant \kappa \quad \textit{with } \kappa > 0\,.$$

Then

$$\lambda_1 \geqslant \frac{n+1}{n}\kappa\,. \tag{2.4}$$

PROOF. The proof follows closely that of Lichnerowicz: Let u be an eigenfunction belonging to λ_1. Since $\Delta u = \mathrm{trace}(\mathrm{Hess}\, u)$, we have

$$|\mathrm{Hess}\, u|^2 \geqslant \frac{1}{n}(\Delta u)^2 = \frac{1}{n}(\lambda_1 u + \langle \nabla(\log w), \nabla u\rangle)^2 \geqslant \tag{2.5}$$

$$\geqslant \frac{1}{n+1}\lambda_1^2 u^2 - \langle \nabla(\log w), \nabla u\rangle^2,$$

where the last inequality follows from $(x-y)^2 \geqslant ax^2 - a(1-a)^{-1}y^2$ $\forall a \in (0, 1)$, with $a = n/(n+1)$. Substituting this into (2.3), and using $Lu = \lambda_1 u$ and the assumption $R_w \geqslant \kappa$, we find

$$-\frac{1}{2}L(|\nabla u|^2) \geqslant \frac{1}{n+1}\lambda_1^2 u^2 - (\lambda_1 - \kappa)|\nabla u|^2.$$

Integrating with respect to $w\,dV$, and using the identities

$$\int_M Lfw\,dV = \int_M L1\, fw\,dV = 0\,, \qquad \text{and} \qquad \|\nabla f\|^2 = \int_M (Lf)\, fw\,dV$$

that hold for all f in the domain of L, one concludes as in [LZ], p. 135, that

$$0 \geqslant \left\{\frac{\lambda_1^2}{n+1} - \lambda_1(\lambda_1 - \kappa)\right\}\|u\|^2$$

and (2.4) follows. ■

REMARK. If $w \equiv 1$ so that $L = -\Delta$ and $R_w = \mathrm{Ric}$, Theorem 2.2 fails to reproduce Lichnerowicz estimate. We are indebted to J. D. Deuschel for the following variation of the argument above. For every $\varepsilon > 0$, let

$$R_\varepsilon = R_w + (1-\varepsilon)\, d(\log w) \otimes d(\log w) = S_w - \varepsilon d(\log w) \otimes d(\log w)\,,$$

and assume that $R_\varepsilon \geqslant \kappa_\varepsilon$. Then, by using $(x-y)^2 \geqslant ax^2 - a(1-a)^{-1}y^2$, with $a = n\varepsilon/(1+n\varepsilon)$ in (2.5) one obtains

$$-\frac{1}{2}L(|\nabla u|^2) \geqslant \frac{\varepsilon}{1+n\varepsilon}\lambda_1^2 u^2 - (\lambda_1 - \kappa_\varepsilon)|\nabla u|^2,$$

and, arguing as above, this yields

$$\lambda_1 \geqslant \frac{n\varepsilon + 1}{(n-1)\varepsilon + 1}\kappa_\varepsilon. \tag{2.6}$$

Notice that:

— For $\varepsilon = 1$ we recover Theorem 2.2;

— If $w \equiv 1$ and $\mathrm{Ric} \geqslant \kappa$, then for all ε we can take $\kappa_\varepsilon = \kappa$ and, letting $\varepsilon \to \infty$ in (2.6) we obtain Lichnerowicz's estimate for the first non-zero eigenvalue of the Laplacian, $\lambda_1 \geqslant n\kappa/(n-1)$;

— Assuming that $S_w \geqslant \kappa$ and letting $\varepsilon \to 0$ in (2.6), we recover the estimate $\lambda_1 \geqslant \kappa$ proven, with a different proof, in [Dl-S]. It is remarkable that the last result holds even for M noncompact. Indeed the hypothesis $S_w \geqslant \kappa$ implies first that the w-volume of M (the volume with respect to the measure $w\,dV$) is finite ([Bk1]), so that 0 is an eigenvalue of L, and then that the gap between 0 and the rest of the spectrum is at least κ (cf. [DlS]).

Adapting Li and Yau's techniques, we shall derive next two gradient estimates for the eigenfunctions of L belonging to a non-zero eigenvalue. Using these gradient estimates we will be able to give bounds from below for the first non-zero eigenvalue of L.

THEOREM 2.3. *Let M be an n-dimensional compact Riemannian manifold with weight function w and let u be an eigenfunction of L with eigenvalue $\lambda > 0$. Assume that $R_w \geqslant \kappa$. Then $\forall \beta \geqslant 1$ we have*

$$|\nabla u|^2 \leqslant 2\left\{(n+1)\frac{\lambda\beta}{\beta-1} - n\kappa\right\}(\beta \sup|u| - u)^2. \tag{2.7}$$

PROOF. The proof is a modification of the proof of Theorem 1 in [LiY]. Assuming without loss of generality that $\sup|u| = 1$ and $\beta > 1$, define a function F by

$$F(x) = \frac{|\nabla u|^2}{(\beta - u)^2},$$

and let x_o be the point where F attains its maximum. Since u is smooth by elliptic regularity, so is F, and, applying the maximum principle, we have

$$\nabla F(x_o) = 0 \qquad -\frac{1}{2} LF(x_o) \leq 0 .$$

Now,

$$\nabla F = \frac{\nabla(|\nabla u|^2)}{(\beta - u)^2} + 2\frac{|\nabla u|^2 \nabla u}{(\beta - u)^3},$$

so that, at x_o,

$$\nabla(|\nabla u|^2) = -2\frac{|\nabla u|^2}{(\beta - u)} \nabla u . \tag{2.8}$$

A straightforward computation that uses (2.3), (2.8) and $Lu = \lambda u$ yields

$$-\frac{1}{2} LF(x_o) = \frac{|\mathrm{Hess}\, u|^2 - \lambda |\nabla u|^2 + R_w(\nabla u, \nabla u) + \langle \nabla(\log w), \nabla u\rangle^2}{(\beta - u)^2} - \\ - \frac{\lambda u |\nabla u|^2}{(\beta - u)^3} - \frac{|\nabla u|^4}{(\beta - u)^4} . \tag{2.9}$$

Now let $\{\partial_i\}_{i=1}^n$ be a local orthonormal frame field in a neighbourhood of x_o. An argument as in [LiY], formula (1.9), yields

$$|\mathrm{Hess}\, u|^2 \geq \mathrm{Hess}\, u(\partial_1, \partial_1)^2 + \frac{1}{n-1}(\Delta u - \mathrm{Hess}\, u(\partial_1, \partial_1))^2 \geq \tag{2.10}$$

$$\geq \left(1 + \frac{1}{2n}\right) \mathrm{Hess}\, u(\partial_1, \partial_1)^2 - \frac{(\Delta u)^2}{(n+1)} \geq$$

$$\geq \left(1 + \frac{1}{2n}\right) \mathrm{Hess}\, u(\partial_1, \partial_1)^2 - \frac{(\lambda u)^2}{n} - \langle \nabla(\log w), \nabla u\rangle^2,$$

where the second inequality follows from $(x-y)^2 \geqslant ax^2 - \alpha(1-\alpha)^{-1}y^2$ with $\alpha = (n-1)/2n$, and the last inequality is a consequence of $\Delta u = - = \lambda u - \langle \nabla(\log w), \nabla u\rangle$ and of $(x+y)^2 \leqslant (1-s)^{-1}x^2 + s^{-1}y^2$ with $s = = 1/(n+1)$. Notice that our choice of α (Li and Yau take $\alpha = 1/2$) is motivated by the desire of giving a unified treatment of cases $n = 2$ and $n \geqslant 2$ here and in the sequel.

The definition of Hessian and (2.8) imply that, at x_o,

$$\text{Hess } u(\partial_i, \nabla u) = -\frac{|\nabla u|^2 \partial_i u}{(\beta - u)},$$

for every i, so that, assuming without loss of generality that, at x_o, $\partial_1 u \neq 0$, and $\partial_i u = 0$, $i > 1$, we obtain

$$\text{Hess } u(\partial_1, \partial_1) = -\frac{|\nabla u|^2}{(\beta - u)} \quad (\text{at } x_0). \tag{2.11}$$

Putting (2.11) into (2.10), substituting the result into (2.9) and simplifying, we get

$$0 \geqslant \frac{1}{2n}\frac{|\nabla u|^4}{(\beta-u)^4} + \frac{R_w(\nabla u, \nabla u)}{(\beta-u)^2} - \frac{\lambda^2}{n}\frac{u^2}{(\beta-u)^2} - \frac{\lambda\beta}{\beta-u}\frac{|\nabla u|^2}{(\beta-u)^2}.$$

Since, by assumption, $\sup |u| = 1$, we have $|u/(\beta-u)| \leqslant 1/(\beta-1)$, and $\beta/(\beta-u) \leqslant \beta/(\beta-1)$. Substituting these into the inequality above, and using the hypothesis $R_w \geqslant \kappa$ and the definition of F, we finally find

$$0 \geqslant \frac{1}{2n}F^2(x_o) - \left(\frac{\lambda\beta}{\beta-1} - \kappa\right)F(x_o) - \frac{1}{n}\left(\frac{\lambda}{\beta-1}\right)^2,$$

which can be rewritten as

$$F^2(x_o) - 2CF(x_o) - 2S^2 \leqslant 0,$$

with

$$C = n\left(\frac{\lambda\beta}{\beta-1} - \kappa\right) \quad \text{and} \quad S = \frac{\lambda}{\beta-1}.$$

Notice that if $\kappa > 0$, then $\lambda \geqslant (n+1)\kappa/n > (n-1)\kappa$, by Theorem 2.2, so that C is always positive. The quadratic formula applied to the last inequality gives

$$\frac{|\nabla u|^2}{(\beta - u)^2} = F(x) \leqslant F(x_o) \leqslant C + \{C^2 + 2S^2\}^{1/2} \leqslant$$

$$\leqslant 2C + \sqrt{2}\,S = 2(n+1)\frac{\lambda\beta}{\beta - 1} - 2n\kappa,$$

and (2.7) follows. ■

REMARK. A similar, but weaker, result can be obtained by using the relationship noted in the introduction between eigenfunctions of L on M and eigenfunctions of the Laplacian of the warped product $\widehat{M} = M \times_w S^1$. Notice that $\widehat{M}$ is a compact, $(n+1)$-dimensional manifold and that, by [ON], Corollary 7.43, we have

$$\widehat{\mathrm{Ric}}(X, Y) = (\mathrm{Ric} - w^{-1}\,\mathrm{Hess}\, w)(X, Y),$$

$$\widehat{\mathrm{Ric}}(X, \Theta) = 0,$$

$$\widehat{\mathrm{Ric}}(\Theta, \Theta) = -w^{-1}\Delta w \cdot |\Theta|^2_{\widehat{M}},$$

for $X, Y \in TM \simeq TM \times \{0\} \subset T(\widehat{M})$, $\Theta \in TS^1$ and where a hat on a symbol indicates that the corresponding object is defined on $\widehat{M}$. For u as in the statement of the theorem define the function $\widehat{u}$ on $\widehat{M}$ by $\widehat{u}(x, \theta) = u(x)$, $(x, \theta) \in \widehat{M}$ so that $-\widehat{\Delta}\widehat{u} = \lambda \widehat{u}$. Assuming that $R_w \geqslant \kappa$ and that $-w^{-1}\Delta w \geqslant \kappa$, [LiY], Theorem 1, yields

$$|\widehat{\nabla}\widehat{u}|^2 \leqslant 2\left\{(n+2)\frac{\lambda\beta}{\beta - 1} - (n+1)\,\kappa\right\}(\beta \sup|\widehat{u}| - \widehat{u})^2, \qquad \forall \beta \geqslant 1,$$

which immediately gives

$$|\nabla u|^2 \leqslant 2\left\{(n+2)\frac{\lambda\beta}{\beta - 1} - (n+1)\kappa\right\}(\beta \sup|u| - u)^2, \qquad \forall \beta \geqslant 1.$$

THEOREM 2.4. *Let M, w and u be as in Theorem 2.3, and assume that $S_w \geqslant \kappa$. Then $\forall \alpha \geqslant 0$ and $\beta^2 \geqslant \sup(\alpha + u)^2$,*

$$(2.12) \qquad |\nabla u|^2 \leqslant$$

$$\leqslant \max_{x \in M}\left\{\frac{\lambda(\beta^2 - (\alpha + u)\,\alpha) - (\beta^2 - (\alpha + u)^2)\,\kappa}{\beta^2}\right\}(\beta^2 - (\alpha + u)^2).$$

REMARK. Notice that, since $S_w = R_w + d(\log w) \otimes d(\log w) \geqslant R_w$, (2.12) certainly holds with the same constant if we assume instead that $R_w \geqslant \kappa$.

PROOF. As in [LiY], Theorem 4, the proof is carried out by applying the arguments used above to the function G defined by

$$G(x) = \frac{|\nabla u|^2}{\beta^2 - (\alpha + u)^2}$$

where $\alpha > 0$ and $\beta^2 > \sup(\alpha + u)^2$. ■

In the next two theorems we use the gradient estimates just derived to obtain lower bounds for the first non-zero eigenvalue of L.

THEOREM 2.5. *Let M be a compact, n-dimensional manifold with weight function w and diameter d, and assume that $R_w \geqslant \kappa$. If $2n\kappa \leqslant d^{-2}$, then the first non-zero eigenvalue λ_1 of L satisfies*

$$\lambda_1 \geqslant \frac{1}{(n+1)d^2} \exp\{-1 - \sqrt{1 - 2n\kappa d^2}\}. \tag{2.13}$$

PROOF. The proof mimics that of Theorem 7 in [LiY] and we only sketch it here. Let u be an eigenfunction belonging to the eigenvalue λ_1. We may assume that $\sup|u| = \sup u$. Since u is $L^2(w\,dV)$-orthogonal to the constants, the nodal set N of u is not empty. Integrating $|\nabla u|/(\beta \sup|u| - u)$ along the shortest geodesic from N to the point x where u attains its maximum, and using Theorem 2.3, we find

$$\log\left(\frac{\beta}{\beta - 1}\right) \leqslant d\left\{2(n+1)\,\lambda_1 \frac{\beta}{\beta - 1} - 2n\kappa\right\}^{1/2},$$

so that squaring and simplifying we get

$$\lambda_1 \geqslant \frac{1}{2(n+1)} \varrho \left\{\frac{1}{d^2} \log^2 \varrho + 2n\kappa\right\}, \tag{2.14}$$

for every $\varrho = (\beta - 1)/\beta \in (0, 1)$. Under the assumption that $2n\kappa \leqslant d^{-2}$, the right hand side of (2.17) is maximized for $\log \varrho = -1 - \sqrt{1 - 2n\kappa d^2}$, and, since,

$$(1+\sqrt{1-2n\kappa d^2})^2 \geqslant 2-2n\kappa d^2,$$

(2.13) follows. ■

REMARK. If $2n\kappa > d^{-2}$, then the right hand side of (2.14) is maximized for $\varrho = 1$ and this gives the estimate $\lambda_1 \geqslant (n+1)^{-1} n\kappa$ which is worse than the estimate provided by Theorem 2.2. For R_w non-negative, the following theorem gives a better bound for λ_1.

THEOREM 2.6. *Let M be a compact Riemannian manifold with weight function w and assume that $S_w \geqslant \kappa$. Then*

$$\lambda_1 \geqslant \frac{\pi^2}{2d^2} - \max\{-\kappa, 0\}, \tag{2.15}$$

PROOF. Again let u be an eigenfunction belonging to the eigenvalue λ_1. By changing the sign of u if necessary, we may assume that $\sup u \geqslant \geqslant |\inf u|$. Therefore, letting $\alpha = 0$ and $\beta = \sup u$ in Theorem 2.4, we have

$$\frac{|\nabla u|}{\sqrt{(\sup u)^2 - u^2}} \leqslant (\lambda_1 + \max\{-\kappa, 0\})^{1/2}. \tag{2.16}$$

The proof now proceeds as in [Li], Theorem 3: Integrating (2.19) along the minimizing geodesic γ joining the point where u achieves its supremum to the point where it attains its infimum, and taking into account the fact that the length of γ is at most d, we get

$$\frac{\pi}{2} + \sin^{-1}\left(\frac{|\inf u|}{\sup u}\right) \leqslant d(\lambda_1 + \max\{-\kappa, 0\})^{1/2}.$$

If the multiplicity of λ_1 is $\geqslant 2$, Li's argument shows that there is an eigenfunction u for which $\sup u = |\inf u|$ and therefore (2.15) holds with $\pi^2/(2d^2)$ replaced by π^2/d^2. In the general case one considers the product manifold $\widehat{M} = M \times M$ with the weight function $\widehat{w}(x, y) = w(x) \cdot w(y)$. It is easy to see that the operator $\widehat{L}$ induced on $\widehat{M}$ is simply $L_x + L_y$, where $L_x = -\Delta_x - \nabla_x(\log w)(x)$, so that $\widehat{L}f(x, y) = L_x f(x, y) + L_y f(x, y)$. This implies that the first non-zero eigenvalue $\widehat{\lambda}_1$ of $\widehat{L}$ is λ_1

with multiplicity $\geqslant 2$. Moreover, for $\widehat{X}, \widehat{Y} \in T(\widehat{M})$,

$$\widehat{S}_{\widehat{w}}(\widehat{X}, \widehat{Y}) = \widehat{\mathrm{Ric}}(\widehat{X}, \widehat{Y}) - \widehat{\mathrm{Hess}}(\log \widehat{w})(\widehat{X}, \widehat{Y}) = S_w(X_1, Y_1) + S_w(X_2, Y_2),$$

where X_1, X_2 (resp. Y_1, Y_2) are the projections of $\widehat{X}$ (resp. $\widehat{Y}$) on $TM \times \{0\}$ and $\{0\} \times TM$. Thus the hypothesis $\widehat{S}_{\widehat{w}} \geqslant \kappa$ holds and we can conclude as above that

$$\widehat{\lambda}_1 \geqslant \frac{\pi^2}{\widehat{d}^2} + \max\{-\kappa, 0\},$$

where $\widehat{d}$ is the diameter of $\widehat{M} = M \times M$. Since $\widehat{\lambda}_1 = \lambda_1$ and $\widehat{d}^2 = 2d^2$, (2.15) follows. ■

REMARK. From the variational characterisation of λ_1 and the identity

$$\int (f - \langle f \rangle_\mu)^2 \, d\mu = \inf_{t \in \boldsymbol{R}} \int (f - t)^2 \, d\mu ,$$

that holds for every finite measure μ and $f \in L^2(d\mu)$, one concludes that

$$\lambda_1 \geqslant \frac{\min w}{\max w} \lambda_1^{\Delta},$$

where λ_1^{Δ} is the first non-zero eigenvalue of the standard Laplacian of M. $\min w / \max w$ can be estimated by integrating $|\nabla(\log w)|$ along a minimising geodesic joining the points where w attains respectively its minimum and maximum value, thus yielding

$$\lambda_1 \geqslant \lambda_1^{\Delta} \exp\left\{ -\left(\sup_M |\nabla(\log w)| \right) d \right\}.$$

Together with known estimates for λ_1^{Δ} this gives bounds for λ_1 in terms of Ricci curvature, diameter and of $\left(\sup_M |\nabla(\log w)| \right)$. The dependency on the last quantity reflects the perturbative nature of the method. The bounds provided by Theorems 2.2, 2.5, and 2.6, by contrast, are expressed in terms of the tensor R_w which takes into account the mutual, and possibly competing, interaction of the curvature of M and of the behaviour of w.

We conclude this section showing how the techniques introduced above can be used to give lower bounds for the first eigenvalue of L with Dirichlet boundary conditions. In what follows Ω will be an open relatively compact domain in M with smooth boundary $\partial\Omega$. We will denote by $\partial/\partial\nu$ the outward unit normal to $\partial\Omega$, and, for $x \in \partial\Omega$, $H(x)$ will be the mean curvature of $\partial\Omega$ with respect to $\partial/\partial\nu$, defined as

$$\frac{1}{n-1}\operatorname{trace}\left\{X \in T_x(\partial\Omega) \to \nabla_X \frac{\partial}{\partial\nu} \in T_x(\partial\Omega)\right\}.$$

THEOREM 2.7. *Let M be a complete Riemannian manifold with weight w and let Ω be a relatively compact open domain in M with smooth boundary $\partial\Omega$. Assume that $R_w \geqslant \kappa$ in Ω, with $\kappa \leqslant 0$, and that H_o is a lower bound for the mean curvature H of $\partial\Omega$. If u is a positive solution of*

$$Lu = \lambda u \quad \text{in } \Omega \quad (\lambda > 0)$$

$$u \,|\, \partial\Omega = 0\,,$$

then, $\forall \beta \geqslant 1$, either

$$|\nabla u|^2 \leqslant \left\{\frac{2(n+1)\,\lambda}{\beta - 1}\beta - 2n\kappa\right\}\left(\beta \sup_\Omega |u| - u\right)^2 \tag{2.17}$$

or

$$|\nabla u| \leqslant \left\{\sup_{\partial\Omega} |\nabla(\log w)| - (n-1)\,H_o\right\}\left(\beta \sup_\Omega |u| - u\right). \tag{2.18}$$

PROOF. Assuming without loss of generality that $\sup_\Omega |u| = 1$ and $\beta > 1$, we consider again the smooth function on $\overline{\Omega}$

$$F(x) = \frac{|\nabla u|^2}{(\beta - u)^2}\,,$$

and let x_o be the point where F attains its maximum.

If $x_o \in \partial\Omega$, then $\partial F/\partial\nu(x_o) \geqslant 0$ and a computation as in the proof of Theorem 2 in [LiY] shows that

$$-(n-1)\,H(x_o)\left(\frac{\partial u}{\partial\nu}\right)^2 + \Delta u\left(\frac{\partial u}{\partial\nu}\right) + \frac{|\nabla u|^2}{\beta - u}\,\frac{\partial u}{\partial\nu} \geqslant 0\,.$$

Since $u \mid \partial\Omega = 0$, $\nabla u(x_o) = (\partial u/\partial \nu)\, \partial/\partial \nu$ so that

$$\Delta u(x_o) = -\lambda u(x_o) - \langle \nabla(\log w), \nabla u \rangle = -w^{-1}\left(\frac{\partial w}{\partial \nu}\right)\left(\frac{\partial u}{\partial \nu}\right).$$

Moreover, since $u \geqslant 0$ in Ω and $u(x_o) = 0$, we have $\partial u/\partial \nu(x_o) = -|\nabla u(x_o)|$, and the inequality above becomes

$$-(n-1) H_o |\nabla u|^2 + \sup_{\partial\Omega} \left| w^{-1} \frac{\partial w}{\partial \nu} \right| |\nabla u|^2 - \frac{|\nabla u|^3}{\beta - u} \geqslant 0 \quad \text{(at } x_o\text{)},$$

and (2.18) follows. If $x_o \in \Omega$, (2.17) follows from Theorem 2.3. Notice that since now we cannot guarantee that $\lambda_1 \geqslant \kappa$, the positivity of C in the proof of Theorem 2.3 follows from the assumption $\kappa \leqslant 0$. ■

Now let u be the eigenfunction belonging to the first Dirichlet eigenvalue λ_1 of L on Ω. The proof that one gives in the case of the Laplacian (cf. [Bd], ch. IV, Lemma 3) can be used to show that u has constant sign in Ω. Assuming $u > 0$, we can apply the gradient estimate obtained above and deduce the following theorems.

THEOREM 2.8. *Let M and Ω be as above. Suppose that $R_w \geqslant \kappa$, with $\kappa \leqslant 0$. Let H_o be the lower bound for the mean curvature of $\partial\Omega$ and denote by i the inscribed radius of Ω. Then*

$$\lambda_1 \geqslant \frac{1}{2(n+1)\varrho}\left\{\frac{1}{i^2}\log^2 \varrho + 2n\kappa\right\}, \tag{2.19}$$

with

$$\log \varrho = \max\left\{1 + \sqrt{1 - 2n\kappa i^2}, \left(\sup_{\partial\Omega}\left|\frac{\partial(\log w)}{\partial \nu}\right| - (n-1) H_o\right) i\right\}.$$

PROOF. As in [LiY], Theorem 5, one shows that if β in Theorem 2.7 is chosen so that

$$\log \frac{\beta}{\beta - 1} > \left(\sup_{\Omega}\left|\frac{\partial(\log w)}{\partial \nu}\right| - (n-1) H_o\right) i$$

then (2.17) holds. One then proceeds as in the proof of Theorem 2.5 above. ■

3. – The non-compact case.

In this section we study the operator L defined on a non-compact manifold M. Our goal is to find lower bounds for the bottom of the spectrum λ_o of L. Note that by the spectral theorem and the density of $C_c^\infty(M)$ in the domain of L, λ_o admits the following variational characterization:

$$\lambda_o = \inf \frac{\int_M |\nabla f|^2 w\, dV}{\int_M f^2 w\, dV} \tag{3.1}$$

where f varies over $C_c^\infty(M)$, or equivalently, over $H^1(M, w\, dV)$, defined as usual as the space of all functions f such that f, $\nabla f \in L^2(w\, dV)$. Recall that if M is a Riemannian manifold with weight function w and D is a measurable subset of M, the w-volume of D, denoted $\mathrm{vol}_w(D)$, is by definition the measure of D with respect to $w\, dV$. If $\mathrm{vol}_w(M) < \infty$, clearly $\lambda_o = 0$. The proposition below, which extends to L a theorem proven by Brooks ([Br]) for the Laplacian, shows that much more is true.

PROPOSITION 3.1. *Suppose that the w-volume of the geodesic balls of M grows subexponentially, i.e. for some (and therefore all) $p \in M$ and for all $\alpha > 0$,*

$$\lim_{r \to \infty} e^{-\alpha r} \mathrm{vol}_w B(p, r) = 0\,.$$

Then $\lambda_o = 0$.

After replacing the Riemannian volume with the w-volume, the proof is as in Davies [Da2], p. 157, and therefore will be omitted. ■

Turning to positive results,note first that (3.1) immediately implies that

$$\lambda_o \geqslant \left(\inf_M w / \sup_M w\right) \lambda_o^\Delta,$$

where λ_o^Δ is the bottom of the spectrum of the standard Laplacian of M. Of course this is interesting only when it is a priori known that w is bounded above and away from 0. In the general case a more direct approach is needed.

In the following M will be a manifold with a pole, i.e. there exists $p \in \in M$ such that $\exp_p\colon T_pM \to M$ is a diffeomorphism. Let $(r, u) \in \boldsymbol{R}^+ \times$

$\times ST_pM$ be spherical geodesic coordinates at $p \in M$, where $ST_pM \simeq S^{n-1}$ is the unit tangent space at p. We denote by $\sqrt{g}(r, u)$ the area element in the coordinates (r, u), so that the Riemannian volume element of M is given by $dV = \sqrt{g}(r, u)\, dr\, du$, du being the standard measure on ST_pM. Therefore a standard integration by parts argument (cf. [Cl], p. 47) yields:

THEOREM 3.2. *Let M be a manifold with weight function w and let $p \in M$ be a pole. If*

$$(3.2) \qquad (w\sqrt{g})^{-1} \frac{\partial}{\partial r}(w\sqrt{g})(r, u) \geqslant \alpha\,, \qquad \alpha \geqslant 0\,,$$

for all $(r, u) \in \boldsymbol{R}^+ \times ST_pM$, then

$$(3.3) \qquad \lambda_o \geqslant \frac{1}{4}\alpha^2.$$

To apply the result obtained above one needs to estimate $(w\sqrt{g})^{-1}\partial(w\sqrt{g})/\partial r$. The following proposition, of perturbative nature, is a first step in this direction.

PROPOSITION 3.3. *Let M be a manifold with a pole p and with weight function w. Suppose that* Sect$(\pi) \leqslant 0$ *for all radial 2-planes π and that in geodesic polar coordinates at p we have*

$$(3.4) \qquad \mathrm{Ric}\left(\frac{\partial}{\partial r}(r, u), \frac{\partial}{\partial r}(r, u)\right) \leqslant$$

$$\leqslant -\left(\beta + \sup_{0<t<\infty}\left| w^{-1}\frac{\partial w}{\partial r}(r, u)\right|\right)^2, \qquad \beta \geqslant 0\,,$$

for all (r, u). Then

$$(3.5) \qquad (w\sqrt{g})^{-1}\frac{\partial(w\sqrt{g})}{\partial r} \geqslant \beta\,.$$

In particular this holds for all $p \in M$ if M is complete simply connected with Sect$(M) \leqslant 0$ *and* Ric $\leqslant -(\beta + \|w^{-1}\nabla w\|_\infty)^2$.

PROOF. Observe first that the conclusion of Lemma 3 in [Se1] holds under the weaker hypothesis that the radial sectional curvature and the radial component of the Ricci curvature satisfy the inequalities stated there. Therefore (3.4) implies that

$$\sqrt{g}^{-1}\,\frac{\partial^2\sqrt{g}}{\partial r^2} \geqslant \left\{\beta + \sup_{0<t<\infty}\left|w^{-1}\,\frac{\partial w}{\partial r}\right|\right\}^2 .$$

Since Lemma 4 in [Se1] holds provided $\sqrt{g}^{-1}\,\partial\sqrt{g}/\partial r \geqslant 0$ and this is again guaranteed by our assumption on the radial sectional curvature, we conclude that

$$\sqrt{g}^{-1}\,\frac{\partial\sqrt{g}}{\partial r} \geqslant \left\{\beta + \sup_{0<t<\infty}\left|w^{-1}\,\frac{\partial w}{\partial r}\right|\right\},$$

and (3.5) follows from

$$(w\sqrt{g})^{-1}\,\frac{\partial(w\sqrt{g})}{\partial r} = w^{-1}\,\frac{\partial w}{\partial r} + \sqrt{g}^{-1}\,\frac{\partial\sqrt{g}}{\partial r} . \qquad \blacksquare$$

REMARKS.

1) Example 1 below shows that even though it is rather crude, Proposition 3.3 provides, via Theorem 2, a sharp lower bound for λ_o.

2) Proposition 3.3 can be applied when the supremum of the sectional curvature is zero and the Ricci curvature is bounded above by a negative constant (and the perturbation introduced by the term $w^{-1}\nabla w$ is suitably small). This is the case, for instance, when M is the product of two or more Riemannian manifold with negative curvature. Even in this simple case the bound for λ_o that one finds is far from being sharp. In the proof of Proposition 3.3 above we have used the inequality

$$\sup_t\left|w^{-1}\,\frac{\partial w}{\partial r}(t, u)\right| + w^{-1}\,\frac{\partial w}{\partial r}(r, u) \geqslant 0 .$$

Therefore the bounds one can derive from it are more interesting when $\partial w/\partial r$ is negative. We consider now the case $\partial w/\partial r \geqslant 0$.

THEOREM 3.4. *Let M be a manifold with a pole p and weight function w and denote by $\partial/\partial r$ the radial unit vector field. Assume that for all $q \in M$*

i) *For all radial 2-planes $\pi \subset T_q M$,* $\mathrm{Sect}(\pi) \leqslant -k \leqslant 0$;

ii) $R_w(\partial/\partial r(q), \partial/\partial r(q)) = (\mathrm{Ric} - w^{-1}\,\mathrm{Hess}\, w)(\partial/\partial r(q), \partial/\partial r(q)) \leqslant$
$\leqslant -\alpha \leqslant 0$;

iii) $\partial w/\partial r \geqslant 0$.

Then

$$(w\sqrt{g})^{-1}\,\frac{\partial(w\sqrt{g})}{\partial r} \geqslant \{\alpha + (n-1)(n-2)\,k\}^{1/2}. \tag{3.6}$$

PROOF. An argument as in the proof of Lemma 3 in [Se1], which, as remarked above, can be carried through under the weaker hypothesis that M has a pole and the radial curvatures are controlled, shows that i) and ii) imply

$$\sqrt{g}^{-1}\,\frac{\partial^2\sqrt{g}}{\partial r^2} \geqslant -\,\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) + (n-1)(n-2)\,k =$$
$$= \alpha + (n-1)(n-2)\,k - w^{-1}\,\frac{\partial^2 w}{\partial r^2},$$

whence

$$\frac{\partial}{\partial r}\left(\sqrt{g}^{-1}\,\frac{\partial\sqrt{g}}{\partial r} + w^{-1}\,\frac{\partial w}{\partial r}\right) + \left(\sqrt{g}^{-1}\,\frac{\partial\sqrt{g}}{\partial r} + w^{-1}\,\frac{\partial w}{\partial r}\right)^2 \geqslant \tag{3.7}$$
$$\geqslant \alpha + (n-1)(n-2)\,k\,.$$

The rest of the proof is now a simple modification of the proof of Lemma 4 in [Se1]. Since $\sqrt{g}^{-1}\,\partial\sqrt{g}/\partial r \sim (n-1)\,r^{-1}$ as $r \downarrow 0$ and $w^{-1}\,\partial w/\partial r$ is smooth and bounded, we have

$$\frac{\partial}{\partial r}\left(\sqrt{g}^{-1}\,\frac{\partial\sqrt{g}}{\partial r} + w^{-1}\,\frac{\partial w}{\partial r}\right) < 0\,,$$

for r small. Since i) and iii) imply that

$$\left(\sqrt{g}^{-1}\,\frac{\partial\sqrt{g}}{\partial r} + w^{-1}\,\frac{\partial w}{\partial r}\right) > 0\,.$$

The argument in [Se1] shows that (3.6) follows from (3.7).

We conclude the section with two examples.

EXAMPLE 1. Let $M = \boldsymbol{H}^n_{-4}$, the n-dimensional hyperbolic space with constant curvature -4, which we identify with $\boldsymbol{R}^n$ endowed with the metric given in polar coordinates(r, ξ) by $ds^2 = dr^2 + + |1/2 \sinh(2r)\, d\xi|^2$ and let $w(r, \xi) = (\cosh r)^{-n+1}$. Since $(\cosh r)^{-n+1}$ is C^∞ and even on $\boldsymbol{R}$, w is C^∞ on M. Since $w^{-1}\partial w/\partial r = -(n-1)\tan r$ we have $\|w^{-1}\nabla w\|_\infty = (n-1)$ so that we can write

$$\mathrm{Sect}(M) = -4 = -\{\alpha + (n-1)^{-1}\|w^{-1}\nabla w\|_\infty\}^2$$

with $\alpha = 1$. By Proposition 3.3 and Theorem 3.2 it follows that $\lambda_o \geqslant \leqslant 1(n-1)^2/4$. On the other hand $dV = ((1/2)\sinh(2r))^{n-1} dr\, d\xi$ and consequently $w\, dV = (\sinh r)^{n-1} dr\, d\xi$. Therefore we see that if f is a radial function, the Rayleigh quotient for L

$$R_M(f) = \frac{\int_0^\infty \int_{S^{n-1}} |\nabla f|^2_M (\sinh r)^{n-1} dr\, d\xi}{\int_0^\infty \int_{S^{n-1}} f^2 (\sinh r)^{n-1} dr\, d\xi}$$

coincides with the Rayleigh quotient for the standard Laplacian $-\Delta$ on $\boldsymbol{H}^n_{-1}$, the hyperbolic space with constant curvature -1. Since both the metric of M and the weight function w are radially symmetric one can easily see that

$$\inf\{R_M(f) : f \in C_c^\infty(M)\} = \inf\{R_M(f) : f \in C_c^\infty(M) \text{ and radial}\},$$

and since this is also true for the Laplacian $-\Delta$ on $\boldsymbol{H}^n_{-1}$, we conclude that the bottom of the spectrum of L on M coincides with the bottom of the spectrum of $-\Delta$ on $\boldsymbol{H}^n_{-1}$. Thus $\lambda_o = (n-1)^2/4$ ([McK]), showing that the bound obtained above is sharp.

EXAMPLE 2. Let $M = \boldsymbol{R}^2$ with the metric given in polar coordinates (r, θ) by $ds^2 = dr^2 + r^2 d\theta^2$ and let $w(r, \theta) = \cosh r$. Notice that in this case Proposition 3.3 is not applicable. On the other hand we have $\mathrm{Sect}(M) \equiv 0$, $R_w(\partial/\partial r, \partial/\partial r) \equiv -1$ and $\partial w/\partial r = \sinh r > 0$, so Proposition 3.4 implies that $\lambda_o \geqslant 1/4$. We will show that in fact $\lambda_o = 1/4$ which implies that Proposition 3.4 gives a sharp lower bound for λ_o. To see this we adapt a proof by Pinsky [Pk]. Let f be defined by $f(r, \theta) = \phi(r)$, where $\phi(r) = e^{-r/2} \sin[2\pi(r - \delta/2)\,\delta^{-1}]$ if $r \in [\delta/2, \delta]$ and $\phi(r) = 0$ otherwise.

Then $f \in H^1(M, w\,dv)$, and on $[\delta/2, \delta]$, ϕ satisfies the differential equation

$$\phi'' + \phi' + \left[\frac{1}{4} + \frac{4\pi^2}{\delta^2}\right]\phi = 0 .$$

Integrating by parts, using the differential equation and the fact that $r^{-1} + \tanh r - 1$ is decreasing in $(0, \infty)$, we find:

$$\int_{\delta/2}^{\delta} (\phi')^2 r \cosh r\, dr \leqslant$$

$$\leqslant \left(\frac{2}{\delta} + \tanh\frac{\delta}{2} - 1\right)\int_{\delta/2}^{\delta} |\phi|\,|\phi'|\, r\cosh r\, dr + \left(\frac{1}{4} + \frac{4\pi^2}{\delta^2}\right)\int_{\delta/2}^{\delta} \phi^2 r \cosh r\, dr .$$

Thus, integrating over S^1, dividing through by $\|f\|^2$, and applying the quadratic formula yields

$$\lambda_o^{1/2} \leqslant \frac{\|\nabla f\|}{\|f\|} \leqslant \frac{1}{2}\left(\frac{2}{\delta} + \tanh\frac{\delta}{2} - 1\right) +$$

$$+\left\{\frac{1}{4} + \frac{4\pi^2}{\delta^2} + \frac{1}{4}\left(\frac{2}{\delta} + \tanh\frac{\delta}{2} - 1\right)^2\right\}^{1/2}$$

whence, letting $\delta \to +\infty$ gives $\lambda_o \leqslant 1/4$, and our claim follows. thus proving our claim.

4. – Estimates from above.

In this section, essentially extending ideas of S. Y. Cheng presented in [Cg], we study bounds from above for the eigenvalues of L, in the compact case, and for the bottom of the spectrum, for M non-compact.

For $p \in M$ and $u \in ST_pM$, let $c(u)$ be the distance along the geodesic $\gamma_u = \exp_p(tu)$ from p to the cut locus of p. Note that $\{(r, u) \in \mathbf{R}^+ \times ST_pM : 0 < r < c(u)\}$ is the domain of the polar geodesic coordinates at p. Keeping the notation introduced in the previous section, we have $\sqrt{g}(r, u) > 0\ \forall r \in (0, c(u))$, $\forall u \in ST_pM$ and therefore for every weight function w we have $(w\sqrt{g})(r, u) > 0$ in the same domain. To simplify the notation we let also

$$(4.1)\qquad \begin{cases} S_\kappa(r) = \begin{cases} (\sqrt{-\kappa})^{-1}\sinh(\sqrt{-\kappa}r) & \text{for } \kappa < 0\,, \\ r & \text{for } \kappa = 0\,, \\ (\sqrt{\kappa})^{-1}\sin(\sqrt{\kappa}r) & \text{for } \kappa > 0\,, \end{cases} \\ C_\kappa(r) = \begin{cases} \cosh(\sqrt{-\kappa}r) & \text{for } \kappa < 0\,, \\ 1 & \text{for } \kappa = 0\,, \\ \cos(\sqrt{\kappa}r) & \text{for } \kappa > 0\,. \end{cases} \end{cases}$$

The following theorem generalizes Bishop's comparison theorem and will allow us to extend to L some of Cheng's results.

THEOREM 4.1. *Let M be a Riemannian manifold with weight function w. Assume that in polar geodesic coordinates at p we have*

$$(4.2)\qquad R_w\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \geqslant \alpha\,.$$

Then, $\forall u \in ST_pM$ and $\forall r \in (0, c(u))$,

$$(4.3)\qquad (w\sqrt{g})^{-1}\,\frac{\partial(w\sqrt{g})}{\partial r}(r, u) \leqslant n\frac{C_\beta(r)}{S_\beta(r)}\,,$$

where $\beta = \alpha/n$ and C_β and S_β are defined in (4.1). *Moreover, if $\alpha > 0$,*

$$(4.4)\qquad c(u) \leqslant \frac{\pi}{\sqrt{\beta}}\,.$$

PROOF. Adopting Chavel's notation ([Cl], p. 65 ff.), for $u \in T_pM$, let $A(r, u)$ be the self adjoint isomorphism of $\{u\}^\perp \subset T_pM$ defined by $A(r, u)\,\eta = \tau_r^{-1}(T_{ru}\exp_p)(r\eta)$, where τ_r is the parallel translation along the geodesic $\gamma_u(r) = \exp_p(ru)$. Then $\det A(r, u) = \sqrt{g}(r, u)$ and if we define $U(r, u) = A'(r, u)\,A^{-1}(r, u)$ then U is self adjoint, $\operatorname{tr} U(r, u) = \sqrt{g}^{-1}\,\partial\sqrt{g}/\partial r(r, u)$, and the following differential equation holds:

$$\operatorname{tr} U' + \operatorname{tr} U^2 + \operatorname{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = 0 .$$

Using the inequalities

i) $\operatorname{tr} U^2 \geqslant (\operatorname{tr} U)^2/(n-1)$, which follows from the self adjointness of U,

ii) $\operatorname{Ric}(\partial/\partial r, \partial/\partial r) \geqslant \alpha + (w^{-1}\partial w/\partial r)^2 + (w^{-1}\partial w/\partial r)'$,

iii) and $A^2/(n-1) + B^2 \geqslant (A+B)^2/n$ $\forall A, B$, with equality iff $A = (n-1)B$, it follows that

$$0 \geqslant (\operatorname{tr} U)' + \frac{1}{n-1}(\operatorname{tr} U)^2 + \alpha + w^{-1}\operatorname{Hess} w\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) \geqslant$$

$$\geqslant \left(\sqrt{g}^{-1}\frac{\partial\sqrt{g}}{\partial r} + w^{-1}\frac{\partial w}{\partial r}\right)' + \frac{1}{n}\left(\sqrt{g}^{-1}\frac{\partial\sqrt{g}}{\partial r} + w^{-1}\frac{\partial w}{\partial r}\right)^2 + \alpha ,$$

Thus the function $\phi(r)$ defined by

$$\phi(r) = (w\sqrt{g})^{-1}\frac{\partial}{\partial r}(w\sqrt{g})(r, u)$$

for $r \in (0, c(u))$, satisfies the differential inequality

$$\phi' + \frac{1}{n}\phi^2 + n\beta \leqslant 0 , \qquad \text{in } (0, c(u)) ,$$

where $\beta = \alpha/n$, and Setting $\psi(r) = nC_\beta/S_\beta(r)$, one verifies that ψ satisfies the differential equation

$$\psi' + \frac{1}{n}\psi^2 + n\beta = 0 ,$$

and therefore, as in the proof of Bishop's comparison theorem ([Cl], p. 73), one concludes that $\phi(r) \leqslant \psi(r)$ in $(0, \pi/\sqrt{\beta}) \cap (0, c(u))$, where $\pi/\sqrt{\beta} = +\infty$ if $\beta \leqslant 0$. If $\beta \leqslant 0$, the proof is complete. If $\beta \geqslant 0$, take $r_o \in (0, c(u)) \cap (0, \pi/\sqrt{\beta})$, and note that, since

$$(d/dr)\log\{(w\sqrt{g})/(\sqrt{\beta}^{-1}\sin\sqrt{\beta}\, r)^n\} \leqslant 0 ,$$

we have

$$w\sqrt{g}(r, u) \leqslant e^{b_u}(\sqrt{\beta}^{-1}\sin\sqrt{\beta}\, r)^n \quad \text{in} \quad (r_o, c(u)) \cap \left(r_o, \frac{\pi}{\sqrt{\beta}}\right),$$

where $\beta_u = \log\{(w\sqrt{g})(r_o, u)/(\sqrt{\beta}^{-1}\sin\sqrt{\beta}\, r_o)^n\}$. Since $(w\sqrt{g})(r, u) > 0$ in $(0, c(u))$, while $(\sqrt{\beta}^{-1}\sin\sqrt{\beta}\, r)^n = 0$ at $r = \pi/\sqrt{\beta}$, (4.4) follows. ■

REMARK. Let M^n and w be as in the statement of the Theorem, with $\alpha > 0$. As in the proof of the Bonnet-Myers Theorem, since M is complete and the geodesic γ_u is not minimizing after $c(u)$, it follows that $\operatorname{dist}(q, p) \leqslant \pi/\sqrt{\beta}$ for all $q \in M$, and M^n is compact by the Hopf-Rinow theorem. If we assume that $R_w \geqslant \alpha$, with $\alpha > 0$, we can conclude that, in fact, $\operatorname{diam}(M) \leqslant \pi/\sqrt{\beta}$. In connection with this, observe that Bakry ([Bk1]) has shown that $S_w \geqslant \alpha > 0$ implies that $\operatorname{vol}_w(M) < \infty$.

THEOREM 4.2. *Let $p \in M$ and $R > 0$, and consider the generalized Dirichlet problem for L on the geodesic ball in M centered at p with radius R, $B_p(R)$. If*

$$R_w\left(\frac{\partial}{\partial r}(q), \frac{\partial}{\partial r}(q)\right) \geqslant \alpha, \tag{4.5}$$

in $B_p(R)$, then

$$\lambda_o(B_p(R)) \leqslant \lambda^{-\Delta}(B_\beta^{n+1}(R)), \tag{4.6}$$

where $\lambda_o(B_p(R)$ is the bottom of the spectrum of L on $B_p(R)$ and $\lambda^{-\Delta}(B_\beta^{n+1}))$ is the smallest Dirichlet eigenvalue for $-\Delta$ on the disk of radius R in $\boldsymbol{M}_\beta^{n+1}$, the $(n+1)$-dimensional space with constant curvature $\beta = \alpha/n$.

REMARKS.

i) Since we do not assume that $B_p(R)$ is contained in the domain of the normal coordinates at p, ∂B may fail to be smooth and it is necessary to consider the generalized Dirichlet problem on $B_p(R)$. In any case the bottom of the spectrum of L is given by

$$(4.7)\qquad \lambda_o(B_p(R)) = \inf\left\{\frac{\int_{B_p(R)} |\nabla f|^2 w\, dV}{\int_{B_p(R)} f^2 w\, dV}\right\},$$

where f ranges over $C_c^\infty(B_p(R))$ or, equivalently, over $H_o^1(B_p(R), w\, dV)$.

ii) If $\alpha > 0$ so that $\beta > 0$, then M_β^{n+1} is the $n+1$ sphere with curvature β and Riemannian diameter $\pi/\sqrt{\beta}$. According to Theorem 4.1 we also have $d(p, q) \leq \pi/\sqrt{\beta}$, $\forall q \in M$. Consequently the theorem has content only for $R \leq \pi/\sqrt{\beta}$, for if $R > \pi/\sqrt{\beta}$, then $B_p(R) = M$ and $B_\beta^{n+1}(R) = M_\beta^{n+1}$ and the theorem holds trivially with $\lambda_o(M) = 0 = \lambda^{-\Delta}(M_\beta^{n+1})$. Notice that if $R = \pi/\sqrt{\beta}$, then $\lambda^{-\Delta}(B_\beta^{n+1}(R)) = 0$ (cf. [Cl], p. 53), and so again $\lambda_o(B_p(R)) = 0 = \lambda^{-\Delta}(B_\beta^{n+1}(R))$.

Proof. Since the proof is very similar to that of Cheng's theorem ([Cg], Theorem 1.1, or [Cl], pp. 74-77) we will only indicate how to adapt Cheng's proof to our case. Let T be the radial eigenfunction belonging to $\lambda^{-\Delta}(B_\beta^{n+1}(R))$ and define F by $F(q) = T(r(q))$, where $r(q) = d(q, p)$. It is easy to see that $F \in H^1(B_p(R), w\, dV)$. Letting for convenience $b(u) = \min(c(u), R)$, and using Theorem 4.1 instead of Bishop's Comparison Theorem in the integration by parts argument in [Cg], pp. 290-91, yields

$$\int_0^{b(u)} |T'|^2 w\sqrt{g}(r, u)\, dr = \lambda^{-\Delta}(B_\beta^{n+1}(R)) \int_0^{b(u)} T^2 w\sqrt{g}(r, u)\, dr,$$

so that, integrating over $u \in ST_pM$ we find

$$\|\nabla F\|^2_{L^2(w\,dV)} \leq \lambda^{-\Delta}(B_\beta^{n+1}(r))\|F\|^2_{L^2(w\,dV)},$$

and (4.6) follows from (4.7). ■

Corollary 4.3. *Let M be a compact manifold with weight function w and assume that*

$$(4.8)\qquad R_w \geq \alpha\,.$$

Let

$$0 = \lambda_o < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \nearrow \infty$$

be the sequence of the eigenvalues of L, where each eigenvalue is repeated according to its multiplicity. Then for every m we have

$$\lambda_m \leq \lambda^{-\Delta}\left(B_\beta^{n+1}\left(\frac{d}{2m}\right)\right), \tag{4.9}$$

where d is the diameter of M and β and $\lambda^{-\Delta}(B_\beta^{n+1}(R))$ *are defined in Theorem* 4.2.

PROOF. For every m, let ϕ_m be the (normalized) eigenfunction belonging to λ_m. As in the case of the Laplacian it is easy to see that

$$\lambda_m = \inf\left\{\frac{\int_M |\nabla f|^2 w\, dV}{\int_M f^2 w\, dV}\right\},$$

$$f \in H^1(M, w\, dV), \qquad f \perp \text{Span}\{\phi_0, \phi_1, \dots \phi_{m-1}\},$$

with equality iff $f = \phi_m$. Using Theorem 4.2 above the proof follows exactly as in [Cg], Theorem 2.1. ∎

Using the estimate for $\lambda^{-\Delta}(B_\kappa^n(R))$ obtained by Cheng ([CG]), we also have:

COROLLARY 4.4. *Let M be as in Corollary* 4.3, *and assume that* $R_w \geq -\alpha$, *with* $\alpha \geq 0$. *Then*

$$\lambda_m \leq n\frac{\alpha}{4} + C(n)\frac{m^2}{d^2}$$

with C(n) depending only upon n. ∎

We conclude remarking that Cheng's results relative to non-compact manifolds ([Cg], § 4) also extend effortlessly to the operator L and allow us to state the following corollary:

COROLLARY 4.5. *Let M be an n-dimensional complete, non-compact Riemannian manifold with weight function w. If*

$$R_w \geq -\alpha, \qquad \alpha \geq 0,$$

then

$$\lambda_o(M) \leqslant \frac{na}{4}. \quad \blacksquare$$

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