Rendiconti del Seminario Matematico della Università di Padova

Michele Balzano

Lino Notarantonio

On the asymptotic behavior of Dirichlet problems in a riemannian manifold less small random holes

Rendiconti del Seminario Matematico della Università di Padova, tome 100 (1998), p. 249-282

<http://www.numdam.org/item?id=RSMUP_1998__100__249_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 100 (1998)

On the Asymptotic Behavior of Dirichlet Problems in a Riemannian Manifold Less Small Random Holes.

MICHELE BALZANO (*) - LINO NOTARANTONIO (**)

ABSTRACT - In a Riemannian manifold-with-boundary, $\overline{M} = M \cup \partial M$, we study sequences of Dirichlet Problems of type

$$\begin{cases} -\Delta u_h = f, & \text{in } M \backslash E_h, \\ u_h \in H_0^1(M \backslash E_h), \end{cases}$$

where Δ is the Laplace-Beltrami operator, and E_h is the union of closed geodesic balls, $E_h := \bigcup_{i=1}^{h} \overline{B_{r_h}(x_h^i)}$, $r_h > 0$, $h \in \mathbb{N}$; the family $\{x_h^i : i = 1, \ldots, h\}$ consists of independent, identically distributed random variables whose distribution is given by a Radon measure β with finite energy. By means of a capacitary method and under a suitable assumption on the asymptotic behavior of the sequence of radii $(r_h)_h$, the limit problem (in the sense of the strong convergence in probability of the resolvent operators) has the form

$$\begin{cases} -\Delta u + \nu u = f, & \text{in } M, \\ u \in H_0^1(M); \end{cases}$$

ν is a Radon measure which depends on β. The measure ν is explicitly determined. The proof rests on estimates of the harmonic capacity of concentric geodesic balls.

(*) Indirizzo dell'A.: Università degli Studi di Cassino, Dipartimento di Ingegneria Industriale, Via G. Di Biasio 43, I-03043 Cassino (FR).
E-mail: balzano@ing.unicas.it
(**) Indirizzo dell'A.: School of Mathematics, University of Minnesota, Vincent Hall 127, 206 Church St. S.E., Minneapolis, MN 55455, USA.
E-mail: notaran@math.umn.edu
1991 Mathematics Subject Classification. 31C12, 31C15, 53C21, 58G18, 58G20, 58G30.
Work supported by a fellowship of the Italian Consiglio Nazionale delle Ricerche.

Introduction.

Let $h \in \mathbb{N}$, $I_h := \{1, \dots, h\}$, and let

$$x_h^i \colon \Omega \to M, \qquad i \in I_h$$

be random variables defined on a probability space (Ω, Σ, P) with values in a compact Riemannian manifold-with-boundary $\overline{M} := M \cup \partial M$, $\partial M \neq \emptyset$, $d := \dim M \geq 2$; let $B_r(x)$ denote the geodesic ball centered at x with radius $r > 0$. In this paper we study the asymptotic behavior of sequences of Dirichlet Problems

$$\begin{cases} -\Delta u_h = f, & \text{in } M \setminus E_h, \\ u_h \in H_0^1(M \setminus E_h), \end{cases} \tag{1}$$

where Δ is the Laplace-Beltrami operator, $E_h := \bigcup_{i \in I_h} \overline{B_{r_h}(x_h^i)}$, $r_h > 0$, and the bar denotes the topological closure. In our main result (Theorem 4.2) we prove that if $l \in]0, +\infty[$ is defined as

$$l := \begin{cases} \lim\limits_{h \to +\infty} h r_h^{d-2}, & \text{if } d \geq 3, \\ \lim\limits_{h \to +\infty} h(-\log r_h)^{-1}, & \text{if } d = 2, \end{cases}$$

and if $(x_h^i)_{i \in I_h}$ is a family of independent, identically distributed random variables with distribution

$$P(x_h^i \in B) = \beta(B), \qquad i \in I_h,$$

for every Borel set $B \subset M$, where $\beta(\cdot)$ is a Radon measure of finite energy (cf. Definition 4.4), then the sequence of resolvent operators associated with (1) converges strongly in probability to the resolvent operator associated with the following Relaxed Dirichlet Problem

$$\begin{cases} -\Delta u + \nu u = f, & \text{in } M, \\ u \in H_0^1(M); \end{cases} \tag{2}$$

where ν is the Radon measure defined by

$$\nu := \begin{cases} (d-2)\omega_d l \beta, & \text{if } d \geq 3, \\ 2\pi l \beta, & \text{if } d = 2, \end{cases}$$

and the constant ω_d is the $(d-1)$-dimensional Hausdorff measure of the euclidean sphere $S^{d-1} := \left\{ y \in \mathbb{R}^d : \sum_{i=1}^{d} (y^i)^2 = 1 \right\}$ of radius 1, $d \geqslant 2$.

To prove Theorem 4.2 we adapt to our Riemannian framework a variational method introduced by M. Balzano ([2]) for the study of a similar problem in the euclidean space $\mathbb{R}^d$. This method allows us the «reconstruction» of the measure ν, appearing in (2), from the asymptotic behavior of the harmonic capacity of concentric geodesic balls $\mathrm{cap}\,(B_{r_h}(x_h^i), B_{R_h}(x_h^i))$, for suitably defined R_h with $R_h > r_h > 0$. In this respect the main tools are general estimates of $\mathrm{cap}\,(B_r(\cdot), B_R(\cdot))$ in terms of the harmonic capacity of concentric euclidean balls in $\mathbb{R}^d$ of the same radii (Proposition 2.3), and an asymptotic super-additive result for the harmonic capacity (Lemma 5.1-(iii)). We notice that the harmonic capacity is a sub-additive set function defined on the σ-algebra of Borel sets of M (cf. Remark 2.1), but in general is not a measure.

Even when M is a bounded open set in $\mathbb{R}^d$ our result is more general than the corresponding result by M. Balzano in [2, Theorem 4.3].

Similar problems in a Riemannian manifold have been studied by I. Chavel and E.A. Feldman in [6, 8], using probabilistic methods, with a particular regard to the convergence of the spectrum of the Laplace-Beltrami operator.

Still in a Riemannian framework, G. Dal Maso, R. Gulliver and U. Mosco in [13] studied similar problems, also when an increasing number of handles is attached to the manifold. We mention also the papers of P. Bérard, G. Besson, S. Gallot, I. Chavel, G. Courtois, E.A. Feldman [4, 3, 10, 7, 9], in which these authors studied (with different methods) the case of a Riemannian manifold with a submanifold of codimension greater than or equal to 2 excised, with a marked attention to the convergence of the eigenvalues.

The asymptotic behavior of sequences of Dirichlet problems as (1) in $\mathbb{R}^d$ has been studied by many authors since the '70's, when the articles by E. Ya. Hruslov [15, 16], M. Kac [17], and J. Rauch and M. Taylor [21] appeared in the literature. Both the probabilistic and the analytical approach have been used in dealing with this kind of problems; we refer for a quite complete bibliography to the recent book by G. Dal Maso [12].

This paper is divided into five sections. In the first one, besides some general notation, we introduce the geometric assumptions on the Riemannian manifold M. In the second section we introduce the harmonic capacity, the Green's function associated with the Laplace-Beltrami operator Δ and give a representation result for the capacitary potential. We conclude this section by proving Proposition 2.3. The third section contains some technical results that are needed in the fourth section, where our main result (Theorem 4.2) is stated. The fifth section is completely devoted to the proof of Theorem 4.2.

Acknowledgment. Both the authors are pleased to thank Gianni Dal Maso for his advice and continuous help during the preparation of this article.

The second author is also glad to thank Robert Gulliver for useful discussions and suggestions, especially regarding Proposition 2.3, and for a critical reading of part of the paper.

1. – Notation and Preliminaries.

We consider a *smooth, compact, oriented, connected* Riemannian manifold-with-boundary $\overline{M} = M \cup \partial M$, where M is the interior part of $\overline{M}$, and $\partial M \neq \emptyset$ denotes its boundary; the dimension $\dim M := d \geq 2$ and g is its metric tensor, $g = (g_{ij})_{i,j=1}^d$. Associated to g there is the *Laplace-Beltrami operator* Δ, acting on real valued functions defined on M.

Let $x \in M$, let M_x be the tangent space at x, and let Riem: $M_x \times M_x \times \times M_x \rightarrow M_x$ be the curvature tensor. If ξ, $\eta \in M_x$ are two linearly independent vectors then

$$K(\xi, \eta) := \frac{\langle \operatorname{Riem}(\xi, \eta)\, \xi, \eta \rangle}{[\langle \xi, \xi \rangle \langle \eta, \eta \rangle - (\langle \xi, \eta \rangle)^2]}$$

is the *sectional curvature* of the 2-dimensional plane determined by ξ and η; $\langle \cdot, \cdot \rangle$ denotes the scalar product in M_x induced by the metric tensor g.

The following condition is a consequence of the regularity assumption on M.

PROPERTY 1.1. For every relatively compact open set $A \subset M$ there exists $\kappa > 0$ such that

$$\kappa^{-1} \sum_{i=1}^{d} (\xi^i)^2 \leq \sum_{i,j=1}^{d} g_{ij}(x)\, \xi^j \xi^i \leq \kappa \sum_{i=1}^{d} (\xi^i)^2$$

for all $x \in A$, and for all $\xi \in \mathbb{R}^d$.

We shall need the following assumption of geometrical nature.

ASSUMPTION 1.1. Let $a, b \in \mathbb{R} \cup i\mathbb{R}$, $i = \sqrt{-1}$. The sectional curvature is bounded above by b^2 and below by a^2.

We recall that M is a metric space and the *distance* $\operatorname{dist}(x, y)$ between any two points $x, y \in M$ is given by the infimum of all piecewise smooth curves joining x and y; the *diameter* of any set $E \subset M$ is $\operatorname{diam} E := \sup\{\operatorname{dist}(x, y) : x, y \in E\}$; we say that E is bounded if $\operatorname{diam} E < +\infty$.

By $B_\varrho(x)$ we denote the *open geodesic ball* of center $x \in M$ and radius ϱ.

At a certain places in the following we shall also consider the euclidean metric $(\delta_{ij})_{i,j=1}^d$ defined by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \qquad i, j = 1, \dots, d.$$

The Lebesgue integral of a measurable function $f: M \to \mathbb{R}$ can be expressed locally as ([1, pp. 29-30])

$$\int_M f\, dV = \int_{\phi(U)} (f \sqrt{\det(g)}) \circ \phi^{-1}\, dx,$$

where (U, ϕ) is a local chart of M.

The measure of a Borel set $E \subset M$, viz. the Lebesgue integral of the indicator function

$$\mathbf{1}_E(x) := \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E, \end{cases}$$

will be denoted by $V(E)$; sometimes, with a little abuse in language, we refer to $V(E)$ as the Lebesgue measure of E. We say that a property $P(x)$ holds *almost everywhere* (*a.e.* in shorthand notation) if $P(x)$ holds for all $x \in M$ except for a set Z with $V(Z) = 0$.

We define $L^2(M)$ as the Hilbert space of all (equivalence classes of) measurable functions $f: M \to \mathbb{R}$ for which the integral of f^2 is finite; its scalar product is

$$(f, h) := \int_M fh\, dV,$$

and the associated norm is given by

$$\|f\| := \sqrt{\int_M f^2\, dV}\,.$$

The space $H^1(M)$ is defined as the completion of $\{f \in C^1(M) \colon \|f\|_1 < < +\infty\}$ in the the norm induced by

$$\|f\|_1 := \sqrt{\int_M f^2\, dV + \int_M |\nabla f|^2\, dV}\,,$$

where $|\cdot|^2 := \langle \cdot, \cdot \rangle$, and $\nabla f = (D_1 f, \ldots, D_d f)$ denotes the gradient of f. The completion of $C_c^1(M)$ (viz. the space of continuosly differentiable functions on M with compact support) w.r.t. the norm $\|\cdot\|_1$ is denoted by $H_0^1(M)$. Given an open subset A of M, we denote by $H_0^1(A)$ the completion of $C_c^1(A)$ w.r.t. the norm $\|\cdot\|_1$; cf. *e.g.* [1, Ch. 2], [6, I.5].

Throughout this paper *Borel* (resp. *Radon*) measure will mean *positive Borel* (*Radon*) measure. We say that a Borel measure μ is a Radon measure if $\mu(K) < +\infty$, for every compact set $K \subset M$. Given a Borel measure μ on M, we denote by $L^p(M, \mu)$ (resp. by $L^p_{\text{loc}}(M, \mu)$) all (equivalence classes of) real-valued measurable functions defined on M whose p-th power is μ-integrable (resp. locally μ-integrable) on M, for $p \in [1, \infty]$.

Finally we let

$$c(r, R) := \begin{cases} \omega_d (d-2)\, \dfrac{r^{d-2}}{1 - (r/R)^{d-2}}\,, & \text{if } d \geq 3\,, \\ 2\pi (\log R - \log r)^{-1}\,, & \text{if } d = 2\,, \end{cases} \tag{3}$$

for any $0 < r < R$.

2. – The harmonic capacity.

Definition 2.1. Let $A \subset M$ be a bounded open set and let E be a Borel subset of A. The *harmonic capacity* of E w.r.t. A is

$$\operatorname{cap}(E, A) = \inf\left\{ \int_A |\nabla u|^2\, dV \colon u \in \mathcal{K}(A, E) \right\} \tag{4}$$

where $\mathcal{K}(A, E) := \{u \in H_0^1(A) \colon u \geq 1 \text{ } a.e. \text{ on a neighborhood of } E\}$.

REMARK 2.1. 1) Let A be a bounded open set of M; it can be proven that the harmonic capacity is a set function which satisfies the following properties (cf. *e.g.* [11, Proposition 1.4]):

(*a*) if $E_1 \subset E_2 \subset A$ are two Borel sets, then $\operatorname{cap}(E_1, A) \leqslant \leqslant \operatorname{cap}(E_2, A)$;

(*b*) if (E_h) is an increasing sequence of Borel sets of A and $E = = \bigcup_{h \in \mathbb{N}} E_h \subset A$, then $\operatorname{cap}\left(\bigcup_{h \in \mathbb{N}} E_h, A\right) = \sup_{h \in \mathbb{N}} \operatorname{cap}(E_h, A)$;

(*c*) if (K_h) is an decreasing sequence of compact sets contained in A and $K = \bigcap_{h \in \mathbb{N}} K_h$, then $\operatorname{cap}\left(\bigcap_{h \in \mathbb{N}} K_h, A\right) = \inf_{h \in \mathbb{N}} \operatorname{cap}(K_h, A)$;

(*d*) if E_1, E_2 are two Borel sets of A, then $\operatorname{cap}(E_1 \cup E_2, A) + + \operatorname{cap}(E_1 \cap E_2, A) \leqslant \operatorname{cap}(E_1, A) + \operatorname{cap}(E_2, A)$;

(*e*) if $A_1 \subset A_2$ are two bounded open sets of M, then $\operatorname{cap}(\cdot, A_1) \geqslant \geqslant \operatorname{cap}(\cdot, A_2)$.

2) We say that a property $P(x)$ holds for *quasi every* $x \in E$ (*q.e.* in shorthand notation) if $P(x)$ holds for all $x \in M$, except for a set Z with $\operatorname{cap}(Z, M) = 0$. Note that (4) does not depend on local coordinates.

3) Each function in $H^1(\mathbb{R}^d)$ has a representative which is defined up to a set of capacity zero (cf. *e.g.* [22, Chapter 3]); by Property 1.1 this continues to hold for functions in $H^1(M)$.

REMARK 2.2. Using standard variational methods, such as those in [18, Chapter II, §6], it is possible to prove that the infimum in (4) is attained. We will call the (unique) function $u_{E,A} \in H_0^1(A)$ which realizes the minimum in (4) the *capacitary potential* associated with $\operatorname{cap}(E, A)$.

We now give a *representation formula* for the capacitary potential $u_{E,A}$ associated with $\operatorname{cap}(E, A)$ in Proposition 2.2 below by means of the *Green's function* of the Laplace-Beltrami operator Δ. This result can be proven adapting to our case the methods developed in [19, §§ 5, 6] for a similar purpose. Before stating the result, we introduce and give some properties of the Green's function of Δ which are needed in the following.

DEFINITION 2.2 (cf. [1]). Let $\overline{W} = W \cup \partial W$ be a compact manifold-with-boundary, $\partial W \neq \emptyset$. The Green's function $g_W(x, y)$ of the Laplace-Beltrami operator, with Dirichlet boundary condition on ∂W, is the function which satisfies, for $x, y \in W$,

$$\Delta_{(y)} g_W(x, y) = \delta_x \qquad \text{for } x, y \in W$$

in the sense of distribution and which vanishes for $x, y \in \partial W$; here δ_x is the Dirac mass at x. The subscript «(y)» indicates that the Laplace-Beltrami operator acts on the function $y \mapsto g_W(x, y)$.

We list in the following proposition the properties of the Green's function we shall need afterwards.

PROPOSITION 2.1. *Let $\overline{W} = W \cup \partial W$ be an oriented compact manifold-with-boundary. There exists $g_W(x, y)$, the Green's function of the Laplace-Beltrami operator, which satisfies the following properties*:

(i) $g_W(x, y) > 0$ *for* $x, y \in W$;

(ii) $g_W(x, y) = g_W(y, x)$;

(iii) $g_W(x, y) < C_0 r^{2-d}$, *if* $d \geq 3$, *while* $g_W(x, y) < C_0(|\log r| + 1)$ *if* $d = 2$, *where* $r := \operatorname{dist}(x, y)$ *and* C_0 *is a constant which depends on the distance of* x *to the boundary of* W.

PROOF. See Theorem 4.17-(a), (e), (c) in [1]. ■

PROPOSITION 2.2. *Let A be a bounded open set in M, let $E \subset A$ be a compact set and let u_E be the capacitary potential associated with* $\operatorname{cap}(E, A)$. *There exists a Radon measure μ_E for which the integral*

$$\int_A g_A(x, y)\, \mu_E(dy)$$

exists, it is finite almost everywhere (w.r.t. the Lebesgue measure on A) and

$$u_E(x) = \int_A g_A(x, y)\, \mu_E(dy).$$

The measure μ_E is called the capacitary distribution of E and μ_E vanishes on all Borel sets having harmonic capacity zero. Moreover the measure μ_E is supported on ∂E.

In the euclidean case the harmonic capacity of a ball of radius $r > 0$ w.r.t. the concentric ball of radius R, with $r < R$, is equal to $c(r, R)$, which has been introduced in (3). Note that in this case the harmonic capacity does not depend on the center of the balls; moreover the capacitary potential $u^e_{r,R} \equiv u^e$ (in shorthand notation) associated with $c(r, R)$ is *radially symmetric*, viz. the value of u^e at a point depends only on the distance of that point from the center of the ball.

In the following result we compare cap $(B_r(x), B_R(x))$ of concentric geodesic balls $B_r(x) \subset B_R(x)$ in M with $c(r, R)$; this result is one of the main tools in the analysis we shall develop in the fourth and fifth sections; it is a quantitative statement of the intuitive fact that cap $(B_r(\cdot\,\cdot), B_R(\cdot))$ should not differ too much from $c(r, R)$ if the radii are sufficiently small. To achieve this proposition we shall rely in an essential way on Assumption 1.1: The sectional curvature is bounded below by a^2, and above by b^2, $a, b \in \mathbb{R} \cup i\mathbb{R}$, $i = \sqrt{-1}$.

PROPOSITION 2.3. *For each $x \in M$ there exists $\overline{R} = \overline{R}(x) > 0$, the injectivity radius of M at x, such that*

$$\left(1 \wedge \frac{aR}{\sin aR}\right)^2 \left(1 \wedge \frac{\sin bR}{bR}\right)^{d-1} c(r, R) \leqslant \tag{5}$$

$$\leqslant \operatorname{cap}(B_r(x), B_R(x)) \leqslant \left(1 \vee \frac{\sin aRa}{R}\right)^{d-1} c(r, R),$$

for any $0 < r < R < \overline{R}$.

NOTATION 2.1. As $a, b \in \mathbb{R} \cup i\mathbb{R}$, we use the following convention: When a^2 or b^2 is less than zero, then we use the following formula

$$\sin(it) = i \sinh(t), \qquad t \in \mathbb{R},$$

while if a^2 or b^2 is equal to zero, then we replace $\sin at/at$ or $\sin bt/bt$ by 1.

PROOF. Let $x \in M$, and consider the exponential map

$$\exp_x \colon M_x \to M .$$

By Gauss's Lemma we write the given metric on M using geodesic polar coordinates as

$$ds^2 = dt^2 + |\mathfrak{a}(t, \vartheta)\, d\vartheta|^2 , \tag{6}$$

where $t > 0$ and $\vartheta = (\vartheta_1, \ldots, \vartheta_{d-1}) \in S^{d-1}$; we denote by $d\vartheta$ the euclidean measure on S^{d-1}. We then identify M_x with $\mathbb{R}^d$ and let the origin of $\mathbb{R}^d$ be the pre-image of x under $\exp_x$. Moreover we let $J(t, \vartheta)$ denote the Jacobian of $\exp_x$.

Let $\overline{R} = \overline{R}(x) > 0$ be the injectivity radius of M at x, *i.e.*, $\overline{R}$ is such that for each $R \in]0, \overline{R}[$ the exponential map is a diffeomorphism between the euclidean ball centered at the origin with radius R and $B_R(x)$. Let $0 < r < R < \overline{R}$, consider $B_r(x) \subset B_R(x)$ and let $u_{r,R} \equiv u$ (in shorthand notation) be the capacitary potential associated with $\operatorname{cap}(B_r(x), B_R(x))$. With our choice of coordinates we have

$$\operatorname{cap}(B_r(x), B_R(x)) = \int_{B_R(x)} |\nabla u|^2 \, dV = \int_0^R \int_{S^{d-1}} |\nabla u|^2 J(t, \vartheta) \, t^{d-1} \, dt \, d\vartheta \,. \tag{7}$$

As a consequence of Assumption 1.1 the Jacobian $J(t, \vartheta)$ satisfies the bounds

$$\left(\frac{\sin bt}{bt} \right)^{d-1} \leqslant J(t, \vartheta) \leqslant \left(\frac{\sin at}{at} \right)^{d-1}, \tag{8}$$

for $t \in [0, \overline{R}[$, and $\vartheta \in S^{d-1}$; cf. *e.g.* Theorem 15 in [5, Chapter 11, p. 253].

We point out that the capacitary potential u^e (resp. the capacitary potential u) associated with $c(r, R)$ (resp. with $\operatorname{cap}(B_r(x), B_R(x))$) is an admissible function in the minimum problem associated with $\operatorname{cap}(B_r(x), B_R(x))$ (resp. with $c(r, R)$). This requires the composition with $\exp_x^{-1}$ (resp. $\exp_x$), which we suppress in this computation.

Consider the upper bound of $J(t, \vartheta)$ in (8), recall that u^e is radially symmetric and get $\left(\text{below we use the notation } |\nabla \cdot|_e^2 := \sum_{i,j=1}^{d} \delta^{ij} D_i \cdot D_j \cdot \right)$

$$\operatorname{cap}(B_r(x), B_R(x)) \leqslant \int_0^R \int_{S^{d-1}} |\nabla u^e|^2 J(t, \vartheta) \, t^{d-1} \, dt \, d\vartheta =$$

$$= \int_0^R \int_{S^{d-1}} \left(\frac{\partial u^e}{\partial t} \right)^2 J(t, \vartheta) \, t^{d-1} \, dt \, d\vartheta = \int_0^R \int_{S^{d-1}} |\nabla u^e|_e^2 J(t, \vartheta) \, t^{d-1} \, dt \, d\vartheta \leqslant$$

$$\leqslant \int_0^R \left(\frac{\sin at}{at} \right)^{d-1} \int_{S^{d-1}} |\nabla u^e|_e^2 \, t^{d-1} \, dt \, d\vartheta \leqslant$$

$$\leqslant \left(1 \vee \frac{\sin aR}{aR} \right)^{d-1} \int_0^R \int_{S^{d-1}} |\nabla u^e|_e^2 \, t^{d-1} \, dt \, d\vartheta = \left(1 \vee \frac{\sin aR}{aR} \right)^{d-1} c(r, R);$$

this inequality gives the upper bound in the statement of the proposition.

As for the lower bound, we write the relation (6) in an equivalent fashion as

$$\sum_{i,j=1}^{d} g_{ij}\xi^i\xi^j = g_{11}\xi^1\xi^1 + \sum_{\alpha,\beta=2}^{d} g_{\alpha\beta}\xi^\alpha\xi^\beta ,$$

where g_{11} and $(g_{\alpha\beta})_{\alpha,\beta}$ denote respectively the radial and spherical part of the metric $(g_{ij})_{i,j}$; notice that $g_{11}=1$, and $g_{1\alpha}=0$, $\alpha=2, \ldots, d$. By Assumption 1.1 we have the following lower bound on the sectional curvature $K \geqslant a^2$, $a \in \mathbb{R} \cup i\mathbb{R}$; then the Rauch's Comparison Theorem ([5, Theorem 14, Chapter 11, p. 250-251]) gives

$$\sum_{\alpha,\beta=2}^{d} g_{\alpha\beta}\xi^\alpha\xi^\beta \leqslant \left(\frac{\sin at}{at}\right)^2 \sum_{\alpha,\beta=2}^{d} \delta_{\alpha\beta}\xi^\alpha\xi^\beta ,$$

hence

$$\sum_{\alpha,\beta=2}^{d} g^{\alpha\beta}\xi_\alpha\xi_\beta \geqslant \left(\frac{at}{\sin at}\right)^2 \sum_{\alpha,\beta=2}^{d} \delta^{\alpha\beta}\xi_\alpha\xi_\beta ,$$

for $0 \leqslant t \leqslant R$, *i.e.*, on $B_R(x)$. Therefore

$$\sum_{i,j=1}^{d} g^{ij}\xi_i\xi_j \geqslant \left(1 \wedge \frac{aR}{\sin aR}\right)^2 \sum_{i,j=1}^{d} \delta^{ij}\xi_i\xi_j .$$

Now let u be the capacitary potential associated with $\operatorname{cap}(B_r(x), B_R(x))$. We have

$$\operatorname{cap}(B_r(x), B_R(x)) = \int_{B_R(x)} |\nabla u|^2\, dV =$$

$$= \int_0^R \int_{S^{d-1}} \sum_{i,j} g^{ij} D_i u D_j u J(t,\vartheta)\, t^{d-1}\, d\vartheta\, dt \geqslant$$

$$\geqslant \left(1 \wedge \frac{aR}{\sin aR}\right)^2 \left(1 \wedge \frac{\sin bR}{bR}\right)^{d-1} \int_0^R \int_{S^{d-1}} \sum_{i,j} \delta^{ij} D_i u D_j u t^{d-1}\, d\vartheta\, dt \geqslant$$

$$\geqslant \left(1 \wedge \frac{aR}{\sin aR}\right)^2 \left(1 \wedge \frac{\sin bR}{bR}\right)^{d-1} c(r,R).$$

So this proves the lower bound in (5), hence the proof of the proposition is accomplished. ■

3. – The μ-capacity.

In this section we introduce a family of Borel measures on M which we denote by $\mathcal{M}_0^*$. Using some properties of the μ-capacity (see below for its definition) we are able to give $\mathcal{M}_0^*$ the structure of a measurable space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*))$, which we shall use in the fourth section.

Let us indicate by $\mathcal{B}$ the σ-algebra of all Borel subsets contained in M, by $\mathcal{U}$ the family of relatively compact open subsets of M and by $\mathcal{K}$ the family of all compact subset of M.

DEFINITION 3.1. We indicate by $\mathcal{M}_0^*$ the class of all Borel measures μ on M such that:

— $\mu(B) = 0$ whenever $\operatorname{cap}(B, M) = 0$;

— $\mu(B) = \inf\{\mu(A): A$ quasi open, $B \subset A\}$, for every $B \in \mathcal{B}$.

We say that a Borel set $B \subset M$ is *quasi open* (resp. *quasi closed*) if for every $\varepsilon > 0$ there exists an open set (resp. a closed set) U such that

$$\operatorname{cap}(B \Delta U, M) < \varepsilon ,$$

where here Δ denotes the symmetric difference between two sets. Moreover B is quasi open if and only if its complement B^c is quasi closed; the countable union or the finite intersection of quasi open sets is still quasi open. Notice that an open set (resp. a closed set) is quasi-open (resp. quasi-closed).

For example the measure $\mu(B) = \int_B f\, dV$, for $f \in L^1(M)$, belongs to $\mathcal{M}_0^*$, as well the singular measure

$$\infty_E(B) := \begin{cases} 0, & \text{if } \operatorname{cap}(E \cap B, M) = 0, \\ +\infty, & \text{if } \operatorname{cap}(E \cap B, M) > 0, \end{cases} \tag{9}$$

where E is a (quasi) closed subset of A.

DEFINITION 3.2. Let $\mu \in \mathcal{M}_0^*$. For every $B \in \mathcal{B}$, we define the *μ-capacity* of B as

$$C(\mu, B) = \inf \left\{ \int_M |\nabla u|^2 dV + \int_B (u-1)^2 \, d\mu : u \in H_0^1(M) \right\}.$$

REMARK 3.1. If $\mu = \infty_F$, for a quasi closed set F, then

$$C(\infty_F, B) = \operatorname{cap}(B \cap F, M).$$

PROPOSITION 3.1. *For every $\mu \in \mathcal{M}_0^*$ the set function $C(\mu, \cdot)$ satisfies the following properties*:

(*a*) $C(\mu, \emptyset) = 0$;

(*b*) *if* $B_1, B_2 \in \mathcal{B}$, $B_1 \subset B_2$, *then* $C(\mu, B_1) \leq C(\mu, B_2)$;

(*c*) *if* (B_n) *is an increasing sequence in* $\mathcal{B}$, $B = \bigcup_n B_n$, *then* $C(\mu, B) = \sup_n C(\mu, B_n)$;

(*d*) *if* (B_n) *is a sequence in* $\mathcal{B}$, $B \subset \bigcup_n B_n$, *then* $C(\mu, B) \leq \leq \sum_n C(\mu, B_n)$;

(*e*) $C(\mu, B_1 \cup B_2) + C(\mu, B_1 \cap B_2) \leq C(\mu, B_1) + C(\mu, B_2)$, for all $B_1, B_2 \in \mathcal{B}$;

(*f*) $C(\mu, B) \leq \operatorname{cap}(B, M)$, *for every* $B \in \mathcal{B}$;

(*g*) $C(\mu, B) \leq \mu(B)$, *for every* $B \in \mathcal{B}$;

(*h*) $C(\mu, K) = \inf\{C(\mu, U): K \subset U, U \in \mathcal{U}\}$, *for every* $K \in \mathcal{K}$;

(*i*) $C(\mu, B) = \sup\{C(\mu, K): K \subset B, K \in \mathcal{K}\}$, *for every* $B \in \mathcal{B}$.

PROOF. All these properties can be proven as in [11, §§ 2, 3], so we refer to that paper for the proof. ■

We may associate to every $\mu \in \mathcal{M}_0^*$ the functional

$$F_\mu : L^2(M) \to [0, +\infty]$$

defined by

$$F_\mu(v) = \begin{cases} \int_M |\nabla v|^2 dV + \int_M v^2 \, d\mu - 2 \int_M fv \, dV, & \text{if } u \in H_0^1(M), \\ +\infty, & \text{otherwise in } L^2(M). \end{cases}$$

We recall that each function $u \in H^1(M)$ is defined up to a set of capacity zero; cf. Remark 2.1-(3). As the measure μ does not charge (Borel) sets of capacity zero, it follows that the functional $F_\mu(\cdot)$ is well defined and $F_\mu(\cdot)$ is lower semicontinuous w.r.t. the strong topology of $L^2(M)$.

DEFINITION 3.3. Let (μ_h) be a sequence in $\mathcal{M}_0^*$ and let $\mu \in \mathcal{M}_0^*$. We say that (μ_h) γ-converges to μ if the following conditions are satisfied:

(a) for every $u \in H_0^1(M)$ and for every sequence (u_h) in $H_0^1(M)$ converging to u in $L^2(M)$ we have

$$F_\mu(u) \leq \liminf_{h \to +\infty} F_{\mu_h}(u_h);$$

(b) for every $u \in H_0^1(M)$ there exists a sequence (u_h) in $H_0^1(M)$ such that u_h converges to u in $L^2(M)$ and

$$F_\mu(u) \geq \limsup_{h \to +\infty} F_{\mu_h}(u_h).$$

REMARK 3.2. It can be proven that there exists a unique metrizable topology τ_γ on $\mathcal{M}_0^*$ which induces the γ-convergence. All topological notions we shall consider are relative to τ_γ w.r.t. which $\mathcal{M}_0^*$ is also *metrizable and compact*; cf. [14, § 4] for more details.

The following result establishes a connection between γ-convergence and the convergence of the μ-capacities in this Riemannian framework; cf. [20, Proposition 3.8].

PROPOSITION 3.2. *Let (μ_h) be a sequence in $\mathcal{M}_0^*$ and let $\mu \in \mathcal{M}_0^*$. Then μ_h γ-converges to μ if and only if:*

(a) $C(\mu, U) \leq \liminf_{h \to +\infty} C(\mu_h, U)$;

(b) $C(\mu, K) \geq \limsup_{h \to +\infty} C(\mu_h, K)$;

are satisfied for every $K \in \mathcal{K}$ and for every $U \in \mathcal{U}$.

REMARK 3.3. By the above proposition it follows that a sub-base for the topology τ_γ is given by $\{\mu \in \mathcal{M}_0^*: C(\mu, U) > t\}$, $\{\mu \in \mathcal{M}_0^*: C(\mu, K) < s\}$, for $t, s > 0$, $K \in \mathcal{K}$ and $U \in \mathcal{U}$. We therefore may speak about open and closed sets in $\mathcal{M}_0^*$, hence about Borel sets whose family we denote by $\mathcal{B}(\mathcal{M}_0^*)$.

From the next proposition we get some useful measurability properties of the μ-capacity.

PROPOSITION 3.3. *The family $\mathcal{B}(\mathcal{M}_0^*)$ is the smallest σ-algebra for which the function $C(\cdot, U)$: $\mathcal{M}_0^* \to [0, +\infty]$ is measurable for every $U \in \mathcal{U}$ (resp. the function $C(\cdot, K)$: $\mathcal{M}_0^* \to [0, +\infty]$ is measurable for every $K \in \mathcal{K}$).*

PROOF. It can be obtained adapting the proofs of Proposition 2.3 and Proposition 2.4 in [2]. ∎

From the previous proposition we have the following consequence.

COROLLARY 3.1. *Let (Λ, Σ, P) be a measure space and let $m: \Lambda \to \mathcal{M}_0^*$ be a function. The following statements are equivalent:*

(i) *m is $\Sigma/\mathcal{B}(\mathcal{M}_0^*)$-measurable;*

(ii) *$C(m(\cdot), U)$ is Σ-measurable, for every $U \in \mathcal{U}$;*

(iii) *$C(m(\cdot), K)$ is Σ-measurable, for every $K \in \mathcal{K}$.*

LEMMA 3.1. *Let A be an open, bounded set; for every compact set $K \subset A \subset M$, and for every $R > 0$, the real-valued function, defined on $M \times \ldots \times M$ (p-times) by*

$$(x_1, \ldots, x_p) \mapsto \operatorname{cap}\left(\bigcup_{i=1}^{p} \overline{B_R(x_i)} \cap K, A\right)$$

is upper semicontinuous in $M \times \ldots \times M$.

PROOF. Adapt the proof of Lemma 3.1 in [2]. ∎

4. - The main result.

Let (Ω, Σ, P) be a probability space. We shall denote by $\mathbb{E}$ and

Cov respectively the *expectation* and the *covariance* of a random variable w.r.t. the measure P.

DEFINITION 4.1. A measurable function $m\colon \Omega \to \mathcal{M}_0^*$ will be called a *random measure*.

We recall that necessary and sufficient conditions for the measurability of the function $m\colon \Omega \to \mathcal{M}_0^*$ are given in Corollary 3.1.

Let $\mu \in \mathcal{M}_0^*$ and let $\lambda > 0$ be a parameter. Let us consider the following Dirichlet problem, formally written for every $f \in L^2(M)$ as

$$\text{(10)} \qquad \begin{cases} -\Delta u + \lambda u + \mu u = f & \text{in } M\,, \\ u \in H_0^1(M)\,. \end{cases}$$

We say that $u \in H_0^1(M) \cap L^2(M, \mu)$ is a *weak solution* of (10) if

$$\int_M \left(\sum_{i,j=1}^{d} g^{ij} D_i u D_j v + \lambda uv \right) dV + \int_M uv\, d\mu = \int_M fv\, dV$$

for any $v \in H_0^1(M) \cap L^2(M, \mu)$.

Let $\mu \in \mathcal{M}_0^*$; the *resolvent operator* for the Dirichlet problem (10)

$$R_\lambda^\mu \colon L^2(M) \to L^2(M)$$

is defined as the operator that associates to every $f \in L^2(M)$ the unique solution u to (10). Observe that R_λ^μ is a *positive* and *linear* operator.

In the sequel we are interested in sequences of Dirichlet problems such as

$$\text{(11)} \qquad \begin{cases} -\Delta u + \lambda u + m_h u = f & \text{in } M\,, \\ u \in H_0^1(M)\,. \end{cases}$$

where (m_h) is a sequence of random measures. In particular we want also to study the asymptotic behavior as $h \to +\infty$ of the resolvent operators $R_\lambda^{m_h}$ associated to the random measures m_h. The following result, Theorem 4.1 gives an answer in this sense.

We recall that $\mathcal{U}$ denotes the family of all relatively compact open subsets of M.

DEFINITION 4.2. Let us define the following set functions:

$$\alpha'(U) = \liminf_{h \to +\infty} \mathbb{E}[C(m_h(\cdot), U)],$$

$$\alpha''(U) = \limsup_{h \to +\infty} \mathbb{E}[C(m_h(\cdot), U)],$$

for every $U \in \mathcal{U}$. Next consider the *inner regularization* of α' and α'' defined for every $U \in \mathcal{U}$ by

$$\overline{\alpha'}(U) = \sup\{\alpha'(V) \colon V \in \mathcal{U},\ \overline{U} \subset V\},$$

$$\overline{\alpha''}(U) = \sup\{\alpha''(V) \colon V \in \mathcal{U},\ \overline{U} \subset V\}.$$

Then extend $\overline{\alpha'}$ and $\overline{\alpha''}$ to arbitrary Borel sets $B \in \mathcal{B}$ by

$$\overline{\alpha'}(B) = \inf\{\overline{\alpha'}(U) \colon V \in \mathcal{U},\ B \subset U\}, \tag{12}$$

$$\overline{\alpha''}(B) = \inf\{\overline{\alpha''}(U) \colon V \in \mathcal{U},\ B \subset U\}. \tag{13}$$

Finally denote by ν', ν'' the least superadditive set functions defined on $\mathcal{B}$ greater than or equal to $\overline{\alpha'}$ and $\overline{\alpha''}$ respectively.

The following theorem can be obtained adapting the arguments used in [2, Theorem 4.1].

THEOREM 4.1. *Let (m_h) be a sequence of random measures. Let α' and α'' be defined as in Definition* 4.2 *above; let ν' and ν'' be the least superadditive set functions on $\mathcal{B}$ greater that or equal to $\overline{\alpha'}$ and $\overline{\alpha''}$ respectively. Assume that:*

(i) $\nu'(B) = \nu''(B)$, *for every $B \in \mathcal{B}$, and denote by $\nu(B)$ their common value;*

(ii) *there exist $\varepsilon > 0$, a continuous function $\xi \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with* $\xi(0, 0) = 0$ *and a Radon measure β on $\mathcal{B}$ such that*

$$\limsup_{h \to +\infty} |\mathrm{Cov}(C(m_h(\cdot), U), C(m_h(\cdot), V))| \leq \xi(\mathrm{diam}(U), \mathrm{diam}(V))\, \beta(U)\, \beta(V)$$

for every pair U, V of relatively compact open sets with $\overline{U} \cap \overline{V} = \emptyset$ and $\mathrm{diam}(U) < \varepsilon$, $\mathrm{diam}(V) < \varepsilon$.

Then the set function ν is a measure and for every $\lambda > 0$, the se-

quence $(R_\lambda^{m_h})_h$ *converges strongly in probability to* R_λ^ν *in* $L^2(M)$, *i.e.,*

$$\lim_{h\to+\infty} P\{\omega\in\Omega\colon \|R_\lambda^{m_h(\omega)}(f)-R_\lambda^\nu(f)\|_{L^2(M)}>\eta\}=0$$

for every $\eta>0$ *and for any* $f\in L^2(M)$.

From now on, we shall consider a particular class of random measures, which are related to Dirichlet problems with random holes.

Let us denote by $\mathcal{C}$ the family of closed sets contained in A.

DEFINITION 4.3. A function $F\colon \Omega\to\mathcal{C}$ is called a *random set* if the function $m\colon \Omega\to\mathcal{M}_0^*$ defined by $m(\omega)=\infty_{F(\omega)}$ for each $\omega\in\Omega$ is Σ-measurable, where $\infty_{F(\omega)}$ is the singular measure defined in (9).

Let $F\colon \Omega\to\mathcal{C}$ be a function; using the notation introduced in § 2, it follows from Corollary 3.1 that the following statements are equivalent:

a) F is a random set;

b) $C(F(\cdot), U)$ is Σ-measurable for every $U\in\mathcal{U}$;

c) $C(F(\cdot), K)$ is Σ-measurable for every $K\in\mathcal{K}$.

Let (F_h) be a sequence of random sets and let $(m_h(\omega))_h$ be the sequence of random measures so defined

$$m_h(\omega)=\infty_{F_h(\omega)} \qquad \text{for each } \omega\in\Omega\,.$$

Let $f\in L^2(M)$ and $\lambda>0$ be a parameter. We shall consider the weak solution u_h of the following Dirichlet problem on random domains

$$\begin{cases} -\Delta u_h+\lambda u=f & \text{in } M\setminus F_h\,,\\ u\in H_0^1(M\setminus F_h)\,. \end{cases} \tag{14}$$

As in [14], it can be shown that the above Dirichlet problem can be written using the measures ∞_{F_h} as

$$\begin{cases} -\Delta u_h+\lambda u+\infty_{F_h}u_h=f & \text{in } M\,,\\ u\in H_0^1(M)\,; \end{cases}$$

the resolvent operator is defined as

$$R_\lambda^{\infty_{F_h}}(f) = \begin{cases} R_\lambda^h(f), & \text{on } M \setminus F_h, \\ 0, & \text{on } F_h, \end{cases}$$

where R_λ^h is the resolvent operator associated to (14).

DEFINITION 4.4. Let $\mathcal{R}$ be the class of Radon measures on M. For $\beta \in \mathcal{R}$, we define

$$(15) \qquad \mathcal{E}(\beta, A) := \begin{cases} \displaystyle\int_{A \times A}\!\!\int \frac{\beta(dx)\,\beta(dy)}{(\operatorname{dist}(x, y))^{d-2}}, & \text{if } d \geq 3, \\ \displaystyle\int_{A \times A}\!\!\int \log \frac{e^2}{\operatorname{dist}(x, y)}\,\beta(dx)\,\beta(dy), & \text{if } d = 2, \end{cases}$$

for each relatively compact open set $A \subset M$. In analogy with the euclidean case, we call $\mathcal{E}(\beta, A)$ the *energy* of β on A.

We say that $\beta \in \mathcal{R}$ has *finite energy* if

$$\sup_{A \in \mathcal{U}} \mathcal{E}(\beta, A) < +\infty ,$$

where $\mathcal{U}$ is the class of all relatively compact open subsets of M.

ASSUMPTIONS 4.1. Let us assume the following hypotheses:

(i_1) let β be a probability law on M of finite energy;

(i_2) for every $h \in \mathbb{N}$ we set $I_h := \{1, \ldots, h\}$ and we consider h measurable functions $x_h^i \colon \Omega \to M$, $i \in I_h$, such that $(x_h^i)_{i \in I_h}$ is a family of independent, identically distributed random variables with probability distribution β, viz.

$$P(x_h^i \in B) = \beta(B), \qquad i \in I_h,$$

for every Borel set $B \subset M$;

(i_3) let (r_h) be a sequence of strictly positive numbers such that

$$l := \begin{cases} \lim\limits_{h \to +\infty} h r_h^{d-2}, & \text{if } d \geq 3, \\ \lim\limits_{h \to +\infty} h(-\log r_h)^{-1}, & \text{if } d = 2, \end{cases}$$

and $l \in]0, +\infty[$.

From now on, we shall consider the sequence of closed sets (E_h), given by

$$E_h := \bigcup_{i \in I_h} \overline{B_{r_h}(x_h^i)}. \tag{16}$$

By Lemma 3.1 it follows that the sets E_h are actually random sets, according to Definition 4.3.

The next theorem is our main result.

THEOREM 4.2. *Let (E_h) be the sequence of random sets, as defined by* (16). *Assume the general hypotheses* (i_1), (i_2), (i_3). *Then for any $\psi \in L^2(M)$ and for every $\varepsilon > 0$*

$$\lim_{h \to +\infty} P\{\omega \in \Omega : \|R_\lambda^{\infty_{E_h(\omega)}} \psi - R_\lambda \psi\|_{L^2(M)} > \varepsilon\} = 0$$

where R_λ is the resolvent operator associated with the measure

$$\nu = \begin{cases} \omega_d (d-2)\, l\beta\,, & \text{if } d \geq 3\,, \\ 2\pi l\beta\,, & \text{if } d = 2\,. \end{cases} \tag{17}$$

5. - Proof of the main result.

By Theorem 4.1, Theorem 4.2 is an immediate consequence of the following proposition.

PROPOSITION 5.1. *Let (E_h) be the sequence of random sets defined in* (16). *Let α', α'' be the set functions as in Definition* 4.2. *Then if general hypotheses* (i_1), (i_2), (i_3) *are satisfied, we have*:

(t_1) $\nu'(B) = \nu''(B) = \nu(B)$, *for every $B \in \mathcal{B}$*;

(t_2) *there exist a constant $\varepsilon > 0$, a continuous function $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $\xi(0, 0) = 0$ and a Radon measure β_1 such that*

$$\limsup_{h \to +\infty} |\mathrm{Cov}[\mathrm{cap}(E_h(\cdot) \cap U, M), \mathrm{cap}(E_h(\cdot) \cap V, M)]| \leq$$
$$\leq \xi(\mathrm{diam}(U), \mathrm{diam}(V))\, \beta_1(U)\, \beta_1(V)$$

for any pair of relatively compact open sets U, V with $\overline{U} \cap \overline{V} = \emptyset$ and $\mathrm{diam}(U) < \varepsilon$, $\mathrm{diam}(V) < \varepsilon$.

In order to prove the above proposition, we need to introduce some more notation.

Let $\{x_i \colon i \in I\}$ be a finite family of independent, identically distributed random variables, with values in M, and with distribution given by

$$P(x_i \in B) = \beta(B), \qquad \forall B \in \mathcal{B},$$

where β is a probability measure on M.

For $0 < r < R$ and for any subset Z of M let us introduce the following random set of indices

$$N(Z) := \{i \in I \colon x_i \in Z\},$$

$$I(Z) := \{i \in I \colon B(x_i, R) \subset Z, \operatorname{dist}(x_i, x_j) \geqslant 4R, \forall j \in I, j \neq i\},$$

$$J(Z) := \{i \in I \colon B(x_i, R) \subset Z, \exists j \in I, j \neq i, \operatorname{dist}(x_i, x_j) \leqslant 4R\},$$

and for every $\eta > 0$

$$I_\eta(Z) := \begin{cases} \left\{ i \in I(Z) \colon \sum\limits_{\substack{j \in I(Z) \\ j \neq i}} \dfrac{r^{d-2}}{(\operatorname{dist}(x_i, x_j) - 2R)^{d-2}} < \dfrac{\eta}{2} \right\}, & \text{if } d \geqslant 3, \\ \left\{ i \in I(Z) \colon (-\log r)^{-1} \sum\limits_{\substack{j \in I(Z) \\ j \neq i}} \log \dfrac{e}{\operatorname{dist}(x_i, x_j) - 2R} < \dfrac{\eta}{2} \right\}, & \\ & \text{if } d = 2; \end{cases}$$

and finally

$$J_\eta(Z) := I(Z) \setminus I_\eta(Z).$$

Loosely speaking, the (random) set $I(Z)$ gives a «separating condition» among the x_i's ($2R$ could play the role of «separating radius»), while if $i \in I_\eta(Z)$ then we have an upper bound on the potential at x_i.

Let A be a relatively compact open subset of M. Applying Proposition 2.3, and taking into acccount the relation (3), we get in particular that for each $i \in I(A)$ there exists a constant $\overline{R}(x_i)$ (the injectivity radius of M at x_i) such that

$$\text{(18)} \qquad \operatorname{cap}(B_r(x_i), B_R(x_i)) \leqslant \left(1 \vee \frac{\sin aR}{aR}\right)^{d-1} c(r, R), \qquad d \geqslant 2,$$

for any $0 < r < R < \overline{R}(x_i)$, where $c(r, R)$ is as in (3). Let us set

$$\overline{R} := \inf_{i \in I(A)} \overline{R}(x_i).$$

The following lemma will be essential in the proof of Proposition 5.1.

LEMMA 5.1. *Let us consider the notation introduced above. For any* $0 < r < R < \overline{R}$ *let us set*

$$\delta := \begin{cases} 2\left(\dfrac{r}{R}\right)^{d-2}, & \text{if } d \geq 3, \\ 2(-\log r)^{-1} \log \dfrac{e}{R-r}, & \text{if } d = 2. \end{cases}$$

Then:

(i) *The expectation of the random variable* $\#(J_\delta(A))$ *satisfies the inequality*

$$\mathbb{E}[\#(J_\delta(A))] \leq \begin{cases} (2/\delta)(2r)^{d-2}\,\mathcal{E}(\beta, A)(\#(I))^2, & \text{if } d \geq 3, \\ (2/\delta)(-\log r)^{-1}\,\mathcal{E}(\beta, A)(\#(I))^2, & \text{if } d = 2; \end{cases}$$

(ii) *the expectation of the random variable* $\#(J(A))$ *satisfies the inequality*

$$\mathbb{E}[\#(J(A))] \leq \begin{cases} (4R)^{d-2}\,\mathcal{E}(\beta, \tilde{A})(\#(I))^2, & \text{if } d \geq 3, \\ (-\log 4R)^{-1}\,\mathcal{E}(\beta, \tilde{A})(\#(I))^2, & \text{if } d = 2, \end{cases}$$

where $\tilde{A} := \{y \in M : \operatorname{dist}(y, A) < 4R\}$, *and* $\mathcal{E}(\beta, \cdot)$ *is the energy of the measure* β *introduced in Definition* 4.4;

moreover there exists a constant $C = C(r, R, d, A)$ *such that*

(iii) $(1 - C\delta)^2 \sum_{i \in I_\delta(A)} \operatorname{cap}(B_r(x_i), B_R(x_i)) \leq \operatorname{cap}\left(\bigcup_{i \in I_\delta(A)} B_r(x_i), A\right)$
for all δ *sufficiently small so that* $C\delta < 1$.

PROOF. We give the proof for the case $d \geq 3$, for the remaining case $d = 2$ can be proved similarly.

We start off proving the inequality in (i). First of all we notice that

$$J_\delta(A) = \left\{ i \in I(A) \colon \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(\operatorname{dist}(x_i, x_j) - 2R)^{d-2}} > \frac{\delta}{2} \right\}.$$

Therefore

$$\#(J_\delta(A)) < \frac{2}{\delta} \sum_{i \in J_\delta} \left[\sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(\operatorname{dist}(x_i, x_j) - 2R)^{d-2}} \right],$$

hence

$$\begin{aligned}
\mathbb{E}[\#(J_\delta(A))] &< \frac{2}{\delta} \left(\mathbb{E}\left[\sum_{\substack{i, j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(\operatorname{dist}(x_i, x_j) - 2R)^{d-2}} \right] \right) < \\
&< \frac{2}{\delta} \left((2r)^{d-2} \mathbb{E}\left[\sum_{\substack{i, j \in I(A) \\ j \neq i}} \frac{1}{\operatorname{dist}(x_i, x_j)^{d-2}} \right] \right) = \\
&= \frac{2}{\delta} \left((2r)^{d-2} \sum_{\substack{i, j \in I \\ j \neq i}} \mathbb{E}\left[\frac{\mathbf{1}_A(x_i)\, \mathbf{1}_A(x_j)}{\operatorname{dist}(x_i, x_j)^{d-2}} \right] \right) = \\
&= \frac{2}{\delta} \left((2r)^{d-2} \sum_{\substack{i, j \in I \\ j \neq i}} \int\int_{A \times A} \frac{\beta(dx)\, \beta(dy)}{(\operatorname{dist}(x, y))^{d-2}} \right) \leqslant \\
&\leqslant \frac{2}{\delta} \left((2r)^{d-2} \mathcal{E}(\beta, A) (\#(I))^2 \right).
\end{aligned}$$

Let us prove the inequality in (ii). Define

$$\widehat{A} := \{ y \in A \colon \operatorname{dist}(y, \partial) A > R \},$$

and recall

$$\widetilde{A} := \{ y \in M \colon \operatorname{dist}(y, A) < 4R \}.$$

We notice that

$$J(A) = \{i \in N(\widehat{A})\colon \exists j \in N(\widetilde{A}),\ \operatorname{dist}(x_i, x_j) < 4R\}.$$

If $i \in J(A)$, we have that $4R > \operatorname{dist}(x_i, x_j)$, for some $j \in N(\widetilde{A})$; hence

$$(4R)^{d-2} \sum_{j \in N(\widetilde{A})} \frac{1}{(\operatorname{dist}(x_i, x_j))^{d-2}} > 1,$$

for any $i \in J(A)$. Therefore

$$\#(J(A)) \leq (4R)^{d-2} \sum_{i \in N(\widehat{A})} \sum_{j \in N(\widetilde{A})} \frac{1}{(\operatorname{dist}(x_i, x_j))^{d-2}},$$

and

$$\mathbb{E}[\#(J(A))] \leq (4R)^{d-2} \sum_{i \in I} \sum_{\substack{j \in I \\ j \neq i}} \mathbb{E}\left[\frac{\mathbf{1}_{\widehat{A}}(x_i)\, \mathbf{1}_{\widetilde{A}}(x_j)}{\operatorname{dist}(x_i, x_j)^{d-2}} \right] \leq$$

$$\leq (4R)^{d-2} (\#(I))^2 \int\int_{\widetilde{A} \times \widetilde{A}} \frac{\beta(dx)\, \beta(dy)}{\operatorname{dist}(x, y)^{d-2}} = (4R)^{d-2} (\#(I))^2 \mathcal{E}(\beta, \widetilde{A}).$$

Finally we prove (iii). Let us consider the Green's function $g(\cdot, \cdot)$ with Dirichlet boundary condition on ∂M. By Proposition 2.1-(iii)

$$g(x, y) \leq C_0 (\operatorname{dist}(x, y))^{2-d},$$

where $C_0 > 0$ is uniform for $x, y \in A$.

Let us define

$$C = \left(1 \vee \frac{\sin aR}{aR}\right)^{d-1} \frac{2\omega_d (d-2)\, C_0}{(1 - (r/R)^{d-2})(1 - r/R)^{-2}}, \tag{19}$$

and take $\delta > 0$ sufficiently small so that $C\delta < 1$.

Let u be the capacitary potential associated with

$$\operatorname{cap}\left(\bigcup_{i \in I_\delta(A)} B_r(x_i), A\right).$$

If $u < C\delta$ on $\partial B_R(x_i)$, for each $i \in I_\delta(A)$, we claim that the proof is achieved. Let us introduce indeed $v = (1 - C\delta)^{-1}(u - C\delta)^+$; the function $v \in H_0^1(A)$, $v \geq 1$ q.e. on $\bigcup_{i \in I_\delta(A)} B_r(x_i)$ and $v = 0$ on $\partial B_R(x_i)$, for every $i \in$

$\in I_\delta(A)$. We then have

$$\operatorname{cap}(B_r(x_i), B_R(x_i) \leqslant \int_{B_R(x_i)} |\nabla v|^2 \, dV$$

for every $i \in I_\delta(A)$. Hence

$$\int_A |\nabla v|^2 \, dV \geqslant \sum_{i \in I_\delta(A)} \int_{B_R(x_i)} |\nabla v|^2 \, dV \geqslant \sum_{i \in I_\delta(A)} \operatorname{cap}(B_r(x_i), B_R(x_i)) .$$

On the other hand we have also

$$\int_A |\nabla v|^2 \, dV = \frac{1}{(1 - C\delta)^2} \int_A |\nabla (u - C\delta)^+|^2 \, dV \leqslant \frac{1}{(1 - C\delta)^2} \int_A |\nabla u|^2 \, dV =$$

$$= \frac{1}{(1 - C\delta)^2} \operatorname{cap}\left(\bigcup_{i \in I_\delta(A)} B_r(x_i), A \right);$$

therefore we have

$$\operatorname{cap}\left(\bigcup_{i \in I_\delta(A)} B_r(x_i), A \right) \geqslant (1 - C\delta)^2 \sum_{i \in I_\delta(A)} \operatorname{cap}(B_r(x_i), B_R(x_i)) .$$

Now we verify that $u \leqslant C\delta$, on $\partial B_R(x_i)$, for each $i \in I_\delta(A)$. Let

$$u_i(x) = \int_M g(x, y) \, \mu_i(dy) ,$$

be the capacitary potential associated with $\operatorname{cap}(B_r(x_i), M)$, where μ_i is the corresponding capacitary distribution (cf. Proposition 2.2). Define

$$z(x) := \sum_{i \in I_\delta(A)} u_i(x) , \qquad \forall x \in M .$$

Adapting a classical comparison result ([18, Chapter 6, § 7]) to our case, we find that for each $x \in A$ $u(x) \leqslant z(x)$. Let $y \in \partial B_R(x_i)$, for a fixed $i \in$ $\in I_\delta(A)$; we have

$$z(y) = \sum_{j \in I_\delta(A)} u_j(y) = \int_M g(y, \xi) \, \mu_i(d\xi) + \sum_{\substack{j \in I_\delta(A) \\ j \neq i}} \int_M g(y, \zeta) \, \mu_j(d\zeta) .$$

For $\xi \in \partial B_r(x_i)$ we have $\operatorname{dist}(y, \xi) \geqslant R - r$, while for $\zeta \in \partial B_r(x_j), j \neq i$, we have

$$\operatorname{dist}(x_i, x_j) \leqslant \operatorname{dist}(x_i, y) + \operatorname{dist}(y, \zeta) + \operatorname{dist}(\zeta, x_j) = R + \operatorname{dist}(y, \zeta) + r,$$

hence $\operatorname{dist}(y, \zeta) \geqslant \operatorname{dist}(x_i, x_j) - 2R$. Then

$$\begin{aligned} z(y) \leqslant C_0 \Bigg(\left(\frac{1}{R(1 - r/R)} \right)^{d-2} & \operatorname{cap}(B_r(x_i), M) + \\ & + \sum_{\substack{j \in I_\delta(A) \\ j \neq i}} \frac{\operatorname{cap}(B_r(x_j), M)}{(\operatorname{dist}(x_i, x_j) - 2R)^{d-2}} \Bigg) \leqslant \\ & \leqslant C_0 \Bigg(\left(\frac{1}{R(1 - r/R)} \right)^{d-2} \operatorname{cap}(B_r(x_i), B_R(x_i)) + \\ & + \sum_{\substack{j \in I_\delta(A) \\ j \neq i}} \frac{\operatorname{cap}(B_r(x_j), B_R(x_j))}{(\operatorname{dist}(x_i, x_j) - 2R)^{d-2}} \Bigg) \leqslant \\ & \leqslant \left(1 \vee \frac{\sin aR}{aR} \right)^{d-1} \frac{\omega_d (d-2)\, C_0}{(1 - (r/R)^{d-2})(1 - r/R)^{d-2}} \cdot \\ & \cdot \left(\left(\frac{r}{R} \right)^{d-2} + \sum_{\substack{j \in I(A) \\ j \neq i}} \frac{r^{d-2}}{(\operatorname{dist}(x_j, x_i) - 2R)^{d-2}} \right) \leqslant C\delta . \end{aligned}$$

C is the constant defined in (19), and in passing from the second to the third inequality we have used the relation (19) together with the inequality (5) in Proposition 2.3. ■

REMARK 5.1. Since the measure β has finite energy, it does not charge sets of capacity zero; this implies that the measure σ, defined on the Borel family of $M \times M$ by

$$\sigma(E) := \iint_E \frac{\beta(dx)\, \beta(dy)}{(\operatorname{dist}(x, y))^{d-2}}, \qquad d \geqslant 3,$$

does not charge singletons. This property also holds true in the case $d = 2$.

Now consider a non-atomic, finite measure σ on a separable, metric space X. For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every A with

$\operatorname{diam} A < \delta$ we have $\sigma(A) < \varepsilon$. In fact, suppose for the moment that the measure has support in a compact subset of X and assume, by contradiction, that there exists $\varepsilon_0 > 0$ such that for every $h \in \mathbb{N}$ there exists A_h with $\operatorname{diam}(A_h) < 1/h$ and $\sigma(A_h) \geqslant \varepsilon_0$. Let $x_h \in A_h$; then $x_h \to x \in X$ and for h sufficiently large we have

$$A_h \subset B_r(x).$$

Then $\sigma(A_h) \geqslant \varepsilon_0$ implies $\sigma(B_r(x)) \geqslant \varepsilon_0$. If we let $r \to 0$, we have $B_r(x) \to \{x\}$, hence $\sigma(\{x\}) \geqslant \varepsilon_0 > 0$, but we have a contradiction, since the measure σ is non-atomic. If the finite measure σ is not supported on a compact set, given $\varepsilon > 0$, there exists a compact set K such that $\sigma(X \setminus K) < \varepsilon$; now we repeat the argument above in the compact set K.

PROOF OF THE PROPOSITION 5.1. We shall prove the proposition when $d \geqslant 3$, because the proof of the remaining case $d = 2$ can be adapted in a straightforward way.

Consider the function

$$f(\delta) := 4\omega_d (d-2)\, C_0 \frac{\delta/2}{(1 - (\delta/2)^{1/d-2})^{d-2}(1 - \delta/2)}, \qquad \delta \in [0, 1[.$$

It is not difficult to see that there is $\delta_0 \in]0, 1[$ such that $f(\delta) < 1$, for every $0 < \delta < \delta_0$. For $\delta \in]0, \delta_0[$, and $h \in \mathbb{N}$ let us define $R_h > 0$ so that

$$\left(\frac{r_h}{R_h}\right)^{d-2} = \frac{\delta}{2}.$$

Let $C_h = C_h(r_h, R_h, d, A)$ be the constant defined in (19) with $r = r_h$ and $R = R_h$, viz.

$$C_h := \left(1 \vee \frac{\sin aR_h}{aR_h}\right)^{d-1} \frac{2\omega_d (d-2)\, C_0}{(1 - (r_h/R_h)^{d-2})(1 - r_h/R_h)^{d-2}}.$$

We notice that for h sufficiently large we have $C_h \delta < 1$. Let us introduce the following families of random indices

$$N_h(A) := \{i \in I_h \colon x_i^h \in A\},$$

$$I_h(A) := \{i \in I_h \colon B_R(x_i^h) \subset A,\ \operatorname{dist}(x_h^i, x_h^j) \geqslant 4R_h,\ \forall j \in I_h, j \neq i\},$$

$$I_{\delta,h}(A) := \left\{ i \in I_h(A) \colon \sum_{\substack{j \in I_h(A) \\ j \neq i}} \frac{r_h^{d-2}}{(\operatorname{dist}(x_h^i, x_h^j) - R_h)^{d-2}} \leq \delta \right\},$$

$$J_h(A) := \{ i \in I \colon B_R(x_i^h) \subset A,\ \exists j \in I_h(A), j \neq i,\ \operatorname{dist}(x_h^i, x_h^j) \leq 4R_h \},$$

and set $J_{\delta,h}(A) = I_h(A) \setminus I_{\delta,h}(A)$. Moreover let

$$\widehat{A_h} := \{ y \in A \colon \operatorname{dist}(y, \partial A) > R_h \}.$$

It is not difficult to see that $I_h(A) = N_h(\widehat{A_h}) \setminus J_h(A)$. Denote by E_h' the random set $E_h' := \bigcup_{i \in I_{\delta,h}(A)} \overline{B_{r_h}(x_h^i)}$. Note that $\operatorname{cap}(E_h \cap A, A) \geq \operatorname{cap}(E_h' \cap \cap A, A)$. We apply Lemma 5.1-(iii), the lower bound in Proposition 2.3 and find that

$$\operatorname{cap}(E_h' \cap A, A) \geq (1 - C_h \delta)^2 \sum_{i \in I_{\delta,h}(A)} \operatorname{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i)) \geq \tag{20}$$

$$\geq \omega_d (d-2)(1 - C_h \delta)^2 \Pi_h [\#(N_h(A_h)) - \#(J_{\delta,h}(A)) - \#(J_h(A))] r_h^{d-2},$$

for h sufficiently large, where we have set for ease of notation

$$\Pi_h := \frac{1}{1 - (r_h/R_h)^{d-2}} \left(1 \wedge \frac{aR_h}{\sin aR_h} \right)^2 \left(1 \wedge \frac{\sin bR_h}{bR_h} \right)^{d-1};$$

we recall that a and b are respectively the lower and upper bound on the sectional curvature, as in Assumption 1.1.

We introduce

$$\widetilde{A_h} := \{ x \in M \colon \operatorname{dist}(x, A) < 4R_h \},$$

and we notice that $E_h \cap A \subset \bigcup_{i \in N_h(\widetilde{A_h})} B_{r_h}(x_h^i)$. We have

$$\operatorname{cap}(E_h \cap A, M) \leq \operatorname{cap}\left(\bigcup_{i \in N_h(\widetilde{A_h})} B_{r_h}(x_h^i) \cap \widetilde{A_h}, M \right) \leq \tag{21}$$

$$\leq \sum_{i \in N_h(\widetilde{A_h})} \operatorname{cap}(B_{r_h}(x_h^i), A) \leq \sum_{i \in N_h(\widetilde{A_h})} \operatorname{cap}(B_{r_h}(x_h^i), B_{R_h}(x_h^i)) \leq$$

$$\leq \frac{\omega_d (d-2)}{1 - (r_h/R_h)^{d-2}} \left(1 \vee \frac{\sin aR_h}{aR_h} \right)^{d-1} [\#(N_h(\widetilde{A_h}))] r_h^{d-2},$$

for h sufficiently large. In the second inequality of the first line we have used the subadditive property of the harmonic capacity (cf. Remark 2.1-(d)) and we have used Proposition 2.3 in the last line.

PROOF OF (t_1). We recall that by Definition 4.2 we have

$$\alpha'(A) = \liminf_{h \to +\infty} \mathbb{E}[\operatorname{cap}(E_h \cap A, M)],$$

$$\alpha''(A) = \limsup_{h \to +\infty} \mathbb{E}[\operatorname{cap}(E_h \cap A, M)],$$

Moreover by Lemma 5.1 we deduce that

$$\limsup_{h \to +\infty} \frac{\mathbb{E}[\#(J_{\delta, h}(A))]}{h} \leqslant \frac{4^{d-1} l}{\delta} \mathcal{E}(\beta, A), \tag{22}$$

and

$$\limsup_{h \to +\infty} \frac{\mathbb{E}[\#(J_h(A))]}{h} \leqslant \frac{4^{d-1} l}{\delta} \mathcal{E}(\beta, A). \tag{23}$$

Let

$$C_\delta := \lim_{h \to +\infty} C_h = \frac{\omega_d (d-2) C_0}{(1-\delta/2)(1-(\delta/2))^{d-2}}. \tag{24}$$

Then it is not difficult to check that

$$\beta(A) = \begin{cases} \lim\limits_{h \to +\infty} \dfrac{\mathbb{E}[\#(N_h(\widehat{A_h}))]}{h}, \\ \lim\limits_{h \to +\infty} \dfrac{\mathbb{E}[\#(N_h(\widetilde{A_h}))]]}{h}, \end{cases} \tag{25}$$

for every $A \in \mathcal{U}$ with $\beta(\partial A) = 0$.

Therefore from (21), (25), we get

$$\overline{\alpha''}(B) \leqslant \frac{\omega_d (d-2)\, l}{1-\delta/2} \beta(B), \tag{26}$$

for every $B \in \mathcal{B}$ and from (20), (25), (22), and (23) it follows that

$$\overline{\alpha'}(B') \geqslant (1 - C_\delta \delta)^2 \omega_d (d-2)\, l \frac{1}{1-\delta/2} \left(\beta(B') - \frac{4^d l}{\delta} \mathcal{E}(\beta, B') \right) \tag{27}$$

for every $B' \in \mathcal{B}$. From (26) we have

$$\nu''(B) \leqslant \frac{\omega_d(d-2)\,l}{1-\delta/2}\,\beta(B) \tag{28}$$

for every $B \in \mathcal{B}$. On the other hand we have also

$$\nu'(B) \geqslant (1-C_\delta\,\delta)^2\,\frac{\omega_d(d-2)l}{1-\delta/2}\,\beta(B), \tag{29}$$

for each $B \in \mathcal{B}$. Let us fix indeed $B \in \mathcal{B}$; for arbitrary $0<\eta<1$, take a Borel partition $(B_k)_{k\in K}$ of B with $\operatorname{diam} B_k < \eta$. Since ν' is superadditive, we have

$$\nu'(B) \geqslant \sum_{k\in K}\nu'(B_k) \geqslant$$

$$\geqslant (1-C_\delta\,\delta)^2\,\frac{\omega_d(d-2)\,l}{1-\delta/2}\left(\beta(B) - \frac{4^d l}{\delta}\sum_{k\in K}\mathcal{E}(\beta, B_k)\right) =$$

$$= (1-C_\delta\,\delta)^2\,\frac{\omega_d(d-2)\,l}{1-\delta/2}\left(\beta(B) - \frac{4^d l}{\delta}\sum_{k\in K}\int\!\!\int_{B_k\times B_k}\frac{\beta(dx)\beta(dy)}{(\operatorname{dist}(x,y))^{d-2}}\right) \geqslant$$

$$\geqslant (1-C_\delta\,\delta)^2\,\frac{\omega_d(d-2)\,l}{1-\delta/2}\left(\beta(B) - \frac{4^d l}{\delta}\int\!\!\int_{D_\eta}\frac{\beta(dx)\,\beta(dy)}{(\operatorname{dist}(x,y))^{d-2}}\right),$$

where $D_\eta = \{(x,y)\in M\times M\colon \operatorname{dist}(x,y)<\eta\}$, so that $B_k\times B_k \subset D_\eta$, for every $k\in K$; notice that $\operatorname{diam} D_\eta<\eta$. Since β is a measure of finite energy, and taking into account of Remark 5.1, we find that

$$\lim_{\eta\to 0}\int\!\!\int_{D_\eta}\frac{\beta(dx)\,\beta(dy)}{(\operatorname{dist}(x,y))^{d-2}} = 0\,,$$

and we get (29); letting $\delta\to 0$ we get from (29) and (28)

$$\nu'(B) \geqslant \omega_d(d-2)\,l\beta(B) \geqslant \nu''(B)$$

and (t_1) is proved, because we always have $\nu''(\cdot)\geqslant\nu'(\cdot)$. ■

PROOF OF (T_2). First of all we observe that by the Strong Law of

Large Numbers we have

$$\lim_{h \to +\infty} \frac{\#(N_h(\widehat{U_h}))}{h} = \beta(U), \qquad \text{for a.e. } \omega \in \Omega \tag{30}$$

and

$$\lim_{h \to +\infty} \frac{\#(N_h(\widehat{U_h}))}{h} = \beta(U), \qquad \text{in } L^1(\Omega) \tag{31}$$

for any $U \in \mathcal{U}$. Actually we have, for any $U \in \mathcal{U}$,

$$\lim_{h \to +\infty} \frac{\#(N_h(\widehat{U_h}))}{h} = \beta(U), \qquad \text{in } L^2(\Omega) \tag{32}$$

since $h^{-1}\#(N_h(\widehat{U_h}))$ is an equibounded sequence of random variables. By (30), (31), (32) we have

$$\begin{aligned}(33)\quad & \liminf_{h \to +\infty} \mathbb{E}[\operatorname{cap}(E_h(\cdot) \cap U) \operatorname{cap}(E_h(\cdot) \cap V)] \geqslant \\ & \geqslant \left(\frac{\omega_d (d-2)\, l (1 - C_\delta \delta)^2}{1 - \delta/2} \right)^2 \liminf_{h \to +\infty} \left\{ \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h} \frac{\#(N_h(\widehat{V_h}))}{h} \right] - \right. \\ & - \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h} \frac{\#(J_{\delta,h}(V))}{h} \right] - \mathbb{E}\left[\frac{\#(N_h(\widehat{V_h}))}{h} \frac{\#(J_{\delta,h}(U))}{h} \right] - \\ & \left. - \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h} \frac{\#(J_h(V))}{h} \right] - \mathbb{E}\left[\frac{\#(N_h(\widehat{V_h}))}{h} \frac{\#(J_h(U))}{h} \right] \right\}\end{aligned}$$

for any pair $U, V \in \mathcal{U}$ with $\overline{U} \cap \overline{V} = \emptyset$. From (32) we obtain

$$\lim_{h \to +\infty} \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h} \frac{\#(N_h(\widehat{V_h}))}{h} \right] = \beta(U)\,\beta(V); \tag{34}$$

moreover by Lemma 5.1, (22), (23) and (30) we have

$$\limsup_{h \to +\infty} \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h} \frac{\#(J_{\delta,h}(V))}{h} \right] \leqslant \frac{4^{d-1} l}{\delta}\, \beta(U)\, \mathcal{E}(\beta, V), \tag{35}$$

$$\limsup_{h\to+\infty} \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h}\,\frac{\#(J_{\delta,h}(U))}{h}\right] \leqslant \frac{4^{d-1}l}{\delta}\,\beta(V)\,\mathcal{E}(\beta, U), \tag{36}$$

$$\limsup_{h\to+\infty} \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h}\,\frac{\#(J_h(V))}{h}\right] \leqslant \frac{4^{d-1}l}{\delta}\,\beta(U)\,\mathcal{E}(\beta, V), \tag{37}$$

$$\limsup_{h\to+\infty} \mathbb{E}\left[\frac{\#(N_h(\widehat{U_h}))}{h}\,\frac{\#(J_h(U))}{h}\right] \leqslant \frac{4^{d-1}l}{\delta}\,\beta(V)\,\mathcal{E}(\beta, U), \tag{38}$$

for any $U, V \in \mathcal{U}$. Then by (33), (34), (35), (36), (37), (38)

$$\begin{aligned} \liminf_{h\to+\infty} \mathbb{E}\,[\operatorname{cap}(E_h(\cdot)\cap U, M)\operatorname{cap}(E_h(\cdot)\cap V, M)] &\geqslant \\ \geqslant \left(\frac{\omega_d(d-2)(1-C_\delta\delta)^2 l}{1-\delta/2}\right)^2 &\times \\ \times\left[\beta(U)\,\beta(V) - \frac{4^d l}{\delta}\,\beta(U)\,\mathcal{E}(\beta, V) - \frac{4^d l}{\delta}\,\beta(V)\,\mathcal{E}(\beta, U)\right]& \end{aligned} \tag{39}$$

for every $U, V \in \mathcal{U}$ with $\overline{U}\cap\overline{V} = \emptyset$.

By (21) and (32), we also deduce

$$\begin{aligned} \limsup_{h\to+\infty} \mathbb{E}\,[\operatorname{cap}(E_h(\cdot)\cap U, M)\operatorname{cap}(E_h(\cdot)\cap V, M)] &\leqslant \\ \leqslant \left(\frac{\omega_d(d-2)\,l}{1-\delta/2}\right)^2 \beta(U)\,\beta(V)& \end{aligned} \tag{40}$$

for any $U, V \in \mathcal{U}$ with $\beta(\partial U) = \beta(\partial V) = 0$. Estimates similar to (39) and (40) for the upper and lower limit of the sequence $\mathbb{E}\,[\operatorname{cap}(E_h(\cdot)\cap \cap U, M)]\,\mathbb{E}\,[\operatorname{cap}(E_h(\cdot)\cap V, M)]$ can be obtained in the same way. Therefore we get for every $U, V \in \mathcal{U}$ with $\overline{U}\cap\overline{V} = \emptyset$,

$$\begin{aligned} &\limsup_{h\to+\infty} |\operatorname{Cov}[\operatorname{cap}(E_h(\cdot)\cap U, M), \operatorname{cap}(E_h(\cdot)\cap V, M)]| \leqslant \\ &\leqslant \left(\frac{\omega_d(d-2)\,l}{1-\delta/2}\right)^2 \beta(U)\,\beta(V) - \left(\frac{\omega_d(d-2)\,l(1-C_\delta\delta)^2}{1-\delta/2}\right)^2 \times \\ &\times\left[\beta(U)\,\beta(V) - \frac{4^d l}{\delta}\,\beta(U)\,\mathcal{E}(\beta, V) - \frac{4^d l}{\delta}\,\beta(V)\,\mathcal{E}(\beta, U)\right] \leqslant \end{aligned}$$

$$\leqslant \left(\frac{\omega_d(d-2)\,l}{1-\delta/2}\right)^2 \left[\beta(U)\,\beta(V) + \frac{4^d l}{\delta}\,\beta(U)\,\mathcal{E}(\beta, V) + \frac{4^d l}{\delta}\,\beta(V)\,\mathcal{E}(\beta, U)\right] \leqslant$$

$$\leqslant c(\omega_d(d-2)\,l)^2 \left(\frac{\delta}{1-\delta/2}\right)^2 [\tilde{\beta}(U)\,\mathcal{E}(\tilde{\beta}, V) + \tilde{\beta}(V)\,\mathcal{E}(\tilde{\beta}, U) + \tilde{\beta}(U)\,\tilde{\beta}(V)],$$

where $\tilde{\beta}(\cdot)$ is the Radon measure defined by $\tilde{\beta}(\cdot) := (1/\delta)\,\beta(\cdot)$ and $c = \max\{1, 4^d l\}$. Finally defining

$$\beta_1(U) := \tilde{\beta}(U) + \mathcal{E}(\tilde{\beta}, U),$$

for every $U \in \mathcal{U}$, we have

$$\limsup_{h \to +\infty} |\mathrm{Cov}[\mathrm{cap}(E_h(\cdot) \cap U, M), \mathrm{cap}(E_h(\cdot) \cap V, M)]| \leqslant$$

$$\leqslant c(\omega_d(d-2)\,l)^2 \left(\frac{\delta}{1-\delta/2}\right)^2 \beta_1(U)\,\beta_1(V).$$

We note that $\beta_1(\cdot)$ is a Radon measure and we take $\delta = \max\{\mathrm{diam}\, U, \mathrm{diam}\, V\}$

$$\xi(x, y) := \left(\frac{\max\{x, y\}}{1-(1/2)\max\{x, y\}}\right)^2,$$

and $\varepsilon = 2\delta$. The proof of (t_2) follows, and hence the proof of the proposition is accomplished. ■

REFERENCES

[1] T. Aubin, *Nonlinear Analysis on Manifold, Monge-Ampére Equations*, Grundlehren der Matematischen Wiessenschaften, Springer-Verlag, New York, Berlin, Tokio (1982).

[2] M. Balzano, *Randon relaxed dirichlet problems*, Ann. Mat. Pura Appl., **53** (1988), pp. 133-174.

[3] P. Bérard - S. Gallot, *Remarques sur quelques estimées géometriques explicites*, C. R. Acad. Sci. Paris, **297** (1983), pp. 185-188.

[4] G. Besson, *Comportement asymptotique des valeurs propres du Laplacien dans un trou*, Bull. Soc. Math. France, **113** (1985), pp. 211-230.

[5] R. Bishop - R. Crittenden, *Geometry of Manifolds*, Academic Press, New York, London (1964).

[6] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, New York (1984).

[7] I. CHAVEL - E. FELDMAN, *Spectra of domains in compact manifolds*, J. Funct. Anal., **30** (1978), pp. 198-222.
[8] I. CHAVEL - E. FELDMAN, *The Lenz shift and Wiener sausage in Riemannian manifolds, Compositio Math.*, **60** (1986), pp. 65-84.
[9] I. CHAVEL - E. FELDMAN, *Spectra of manifolds less a small domain*, Duke Math., J., **56** (1988), pp. 399-414.
[10] G. COURTOIS, *Comportement du spectre d'une variété riemanniene compacte sous perturbation topologique par excision d'une domaine*, Ph.D. thesis, Inst. Fourier (1987).
[11] G. DAL MASO, *γ-convergence and μ-capacities*, Ann. Scuola Norm. Sup. Cl. Sci. (4) (1987), pp. 432-464.
[12] G. DAL MASO, *An Introduction to Γ-convergence*, Birkhäuser (1993).
[13] G. DAL MASO - R. GULLIVER - U. MOSCO, *Asymptotic spectrum of manifolds of increasing topological type*, to appear.
[14] G. DAL MASO - U. MOSCO, *Wiener criterion and Γ-convergence*, Appl. Math. Opt., **15** (1987), pp. 15-63.
[15] E. HRUSLOV, *The method of orthogonal projection and the Dirichlet problem in domains with a fine-grained boundary*, Math. U.S.S.R. Sb. (1972).
[16] E. HRUSLOV, *The first boundary problem in domains with a complicated boundary for higher order equation*, Math. U.S.S.R. Sb., **32** (1977), pp. 535-549.
[17] M. KAC, *Probabilistic method in some problems of scattering theory*, Rocky Mountain J. Math., **4** (1974), pp. 511-537.
[18] D. KINDERLEHRER - G. STAMPACCHIA, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York (1980).
[19] W. LITTMAN - G. STAMPACCHIA - H. WEINBERGER, *Regular points for elliptic equation with discontinuous coefficients*, Ann. Scuola Norm. Pisa Cl. Sci., **17** (1963), pp. 45-79.
[20] L. NOTARANTONIO, *Asymptotic behavior of Dirichlet problems on a Riemannian manifold*, Ricerche di Mat., **41** (1992), pp. 327-367.
[21] J. RAUCH - M. TAYLOR, *Potential scattering theory on wildly perturbed domains*, J. Funct. Anal., **18** (1975), pp. 27-59.
[22] W. ZIEMER, *Weakly Differentiable Functions*, Springer-Verlag, Berlin, Heidelberg, New York (1989).

Manoscritto pervenuto in redazione il 7 aprile 1997.