RENDICONTI
del
SEMINARIO MATEMATICO
della
UNIVERSITÀ DI PADOVA

A. MOHAMMADI HASSANABADI

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Rendiconti del Seminario Matematico della Università di Padova, tome 100 (1998), p. 137-142

<http://www.numdam.org/item?id=RSMUP_1998__100__137_0>

REND. SEM. MAT. UNIV. PADOVA, Vol. 100 (1998)

A Property Equivalent to Permutability for Groups.

A. MOHAMMADI HASSANABADI (*)

ABSTRACT - In this note we prove the following: Let m and n be positive integers and G a group such that $X_1 X_2 \ldots X_n \cap \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)} \neq \emptyset$ for all subsets X_i of G where $|X_i| = m$ for all $i = 1, 2, \ldots, n$; then G is finite-by-abelian-by-finite.

1. – Introduction.

Permutable groups have been studied by various people–see [1], [2], [3], [4] and [5]. Recall that a group G is called n-permutable if given any sequence $x_1, x_2, \ldots, x_n$ of elements of G, then $x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)}$ for some permutation $\sigma \neq 1$ of the set $\{1, 2, \ldots, n\}$. Also a group is said to be permutable if it is $n-$permutable for some $n > 1$. The main result for groups in this class was obtained by Curzio, Longobardi, Maj and Robinson in [3] where it was shown that such groups are finite-by-abelian-by-finite.

Let m, n be positive integers. A natural extension of permutable groups, namely (m, n)-permutable groups–groups in which

$$X_1 X_2 \ldots X_n \subseteq \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)}$$

(*) Indirizzo dell'A.: Department of Mathematics, University of Isfahan, Isfahan, Iran.

The author wishes to express his sincere gratitude to Professor Akbar Rhemtulla, of the Department of Mathematical Sciences, University of Alberta for his very warm hospitality, encouragement and advice while this work was done. He also would like to thank the Referee for his valuable suggestions.

for all subsets X_i of G where $|X_i| = m$ for all $i = 1, 2, \ldots, n$–was introduced in [6]. It was proved there that such a group is either n-permutable or it is finite of order bounded by a function of m and n. Here we deal with another extension of (m, n)-permutable groups.

For positive integers m and n call a group G restricted (m, n)-permutable if

$$X_1 X_2 \ldots X_n \cap \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)} \neq \emptyset$$

for all subsets X_i of G where $|X_i| = m$ for all $i = 1, 2, \ldots, n$. Thus restricted $(1, n)$-permutable groups are just $(1, n)$-permutable groups which are n-permutable groups.

The main result of this note is the following

THEOREM. *Suppose that m and n are positive integers, and G a restricted (m, n)-permutable group. Then G is finite-by-abelian-by-finite.*

2. – Proofs.

Throughout we assume G to be a group and denote its centre by $Z(G)$. We first prove that the centre of a restricted (m, n)-permutable-group which is not n-permutable has finite order bounded by a function of m and n.

LEMMA 1. *Suppose that m and n are positive integers and G a restricted (m, n)-permutable group which is not n-permutable. Then* $\exp(Z(G)) \leq ((mn)^{n!+1})!$

PROOF. Let $z \in Z(G)$. Let $x_1, x_2, \ldots, x_n$ be elements in G such that the product $x_1 x_2 \ldots x_n$ cannot be rewritten and consider the sets

$$X_i = x_i \{z, z^2, \ldots, z^m\}, \qquad i = 1, 2, \ldots, n .$$

Then there exists a non-trivial permutation σ such that $x_1 z^{i_1} x_2 z^{i_2} \ldots \ldots x_n z^{i_n} = x_{\sigma(1)} z^{j_1} x_{\sigma(2)} z^{j_2} \ldots x_{\sigma(n)} z^{j_n}$ where $i_s, j_s \in \{1, 2, \ldots, m\}$ for $s = 1, 2, \ldots, n$; so that

$$x_1 x_2 \ldots x_n = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)} z^{\alpha_1},$$

where $-n(m-1) \leq \alpha_1 \leq n(m-1)$; $\alpha_1 \neq 0$.

Now replace z by $z^{(mn)^k}$ and let k run through $0, 1, 2, \ldots, n!$. Then, there exist t and t' in $\{0, 1, \ldots, n!\}$, such that $t \neq t'$, say $t > t'$ and $z^{(mn)^t \alpha} = z^{(mn)^{t'} \beta}$. So $z^{(mn)^t \alpha - (mn)^{t'} \beta} = 1$ where $-n(m-1) \leq \alpha, \beta \leq n(m-1)$; $\alpha, \beta \neq 0$ and thus $(mn)^t \alpha - (mn)^{t'} \beta \neq 0$, since otherwise $(mn)^{t-t'} = \alpha/\beta$ which is not possible. Thus $o(z) \mid (mn)^t \alpha - (mn)^{t'} \beta = \gamma$ and $|\gamma| \leq |(mn)^t \alpha| \leq (mn)^{t+1} \leq (mn)^{n!+1}$. Therefore $\exp(Z(G)) \leq ((mn)^{n!+1})!$

LEMMA 2. *Let m and n be positive integers. If G is a restricted (m, n)-permutable group, then either G is n-permutable or $|Z(G)|$ is finite bounded by* $((mn)^{n!+1})![mn(n!+1)]$.

PROOF. Suppose that G is a counterexample. Then since by Lemma 1 $\exp(Z(G)) \leq ((mn)^{n!+1})!$

$$(*) \qquad Z(G) = \langle z_1 \rangle \times \langle z_2 \rangle \times \ldots$$

is a direct product of cyclic groups of size at most $\exp(Z(G))$. Let $x_1, x_2, \ldots, x_n$ be elements in G such that the product $x_1 x_2 \ldots x_n$ cannot be rewritten and consider

$$X_1 = x_1 \{z_1, \ldots, z_m\},$$

$$X_2 = x_2 \{z_{m+1}, \ldots, z_{2m}\},$$

$$\vdots$$

$$X_n = x_n \{z_{(n-1)m}, \ldots, z_{nm}\}.$$

Then $x_1 x_2 \ldots x_n = x_{\sigma_1(1)} x_{\sigma_1(2)} \ldots x_{\sigma_1(n)} z_1'$ for some $\sigma_1 \in S_n \backslash 1$ and $z_1' \in \langle z_1 \rangle \times \ldots \times \langle z_{nm} \rangle$; $z_1' \neq 1$. Now use the next nm direct factors in $(*)$ to obtain

$$x_1 x_2 \ldots x_n = x_{\sigma_2(1)} x_{\sigma_2(2)} \ldots x_{\sigma_2(n)} z_2'$$

with $\sigma_2 \in S_n \backslash 1$ and $z_2' \in \langle z_{nm+1} \rangle \times \ldots \times \langle z_{2nm} \rangle$; $z_2' \neq 1$.

If $|Z(G)| > ((mn)^{n!+1})![mn(n!+1)]$ then the number of factors in (*) is at least $(n!+1)mn$ and we may continue the above process to obtain

$$x_1 x_2 \ldots x_n = x_{\sigma_3(1)} x_{\sigma_3(2)} \ldots x_{\sigma_3(n)} z_3'$$

for some $\sigma_3 \in S_n \backslash 1$ and $z_3' \in \langle z_{2nm+1} \rangle \times \ldots \times \langle z_{3mn} \rangle$; $z_3' \neq 1$

$$\vdots$$

$$x_1 x_2 \ldots x_n = x_{\sigma_{n!+1}(1)} x_{\sigma_{n!+1}(2)} \ldots x_{\sigma_{n!+1}(n)} z'_{n!+1}$$

for some $\sigma_{n!+1} \in S_n \backslash 1$ and $z'_{n!+1} \in \langle z_{n!(mn)+1} \rangle \times \ldots \times \langle z_{(n!+1)mn} \rangle$, $z'_{n!+1} \neq 1$. Thus there exist i and j, $1 \leqslant i, j \leqslant n! + 1$ such that $i \neq j$ and $z_i' = z_j'$ which is not possible. This completes the proof.

We next want to prove our key lemma that the *FC*-centre of a non-trivial restricted (m, n)-permutable group is not trivial, and we find it easier to show this first for a general version of the restricted (m, n)-permutable groups.

Let $m_1, m_2, \ldots, m_n$ be positive integers and call a group G restricted $(m_1, \ldots, m_n)$-permutable if $X_1 X_2 \ldots X_n \cap \bigcup_{\sigma \in S_n \backslash 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)} \neq \emptyset$ for all subsets X_i of size m_i in G, $i = 1, 2, \ldots, n$. Then we have

LEMMA 3. *Suppose that G is a non-trivial restricted $(m_1, \ldots, m_n)$-permutable group. Then the FC-centre of G is non-trivial.*

PROOF. We use induction on the sum $s = m_1 + \ldots + m_n$. If $s = n$ then $m_i = 1$ for all i and so G is n-permutable and the result follows from [3]. So assume that the result holds for all $s < t$ and suppose that $m_1 + m_2 + \ldots + m_n = t$ and G is restricted $(m_1, m_2, \ldots, m_n)$-permutable. Then by induction there exist subsets $Y_1, Y_2, \ldots, Y_n$ of G such that $|Y_1| + |Y_2| + \ldots + |Y_n| = t - 1$ and

$$(**) \qquad Y_1 Y_2 \ldots Y_n \cap \bigcup_{\sigma \in S_n \backslash 1} Y_{\sigma(1)} Y_{\sigma(2)} \ldots Y_{\sigma(n)} = \emptyset .$$

Let

$$S = (Y_{n-1}^{-1} \ldots Y_1^{-1}) \left(\bigcup_{\sigma \in S_n \backslash 1} Y_{\sigma(1)} \ldots Y_{\sigma(n)} \right) \bigcup_{\substack{\sigma \in S_n \backslash 1 \\ 1 \leqslant i \leqslant n}},$$

$$Y_{\sigma(i-1)}^{-1} \ldots Y_{\sigma(1)}^{-1} Y_1 \ldots Y_n Y_{\sigma(n)}^{-1} \ldots Y_{\sigma(i+1)}^{-1},$$

with the convention that $Y_{\sigma(i-1)}^{-1} \ldots Y_{\sigma(1)}^{-1} = 1$ if $i = 1$ and $Y_{\sigma(n)}^{-1} \ldots Y_{\sigma(i+1)}^{-1} = 1$ if $i = n$. Then S is a finite subset of G.

Now for any $a \in G \backslash S$, define $X_i = Y_i$; $i = 1, 2, \ldots, n-1$ and $X_n = Y_n \cup \{a\}$. Then $X_1 X_2 \ldots X_n \cap \bigcup_{\sigma \in S_n \backslash 1} X_{\sigma(1)} X_{\sigma(2)} \ldots X_{\sigma(n)} \neq \emptyset$. This together with $(**)$ and the choice of a imply that there exist $x_i \in X_i$; $i =$

$=1, 2, \dots, n-1$ such that $x_1 x_2 \dots x_{n-1} a = x'_{\sigma(1)} \dots x'_{\sigma(i-1)} a x'_{\sigma(i+1)} \dots x'_{\sigma(n)}$ for some $\sigma \in S_n \backslash 1$ and $x_j' \in X_j$. This gives $a g_\sigma a^{-1} = f_\sigma^{-1} c$ where $f_\sigma = x'_{\sigma(1)} \dots x'_{\sigma(i-1)}$, $g_\sigma = x'_{\sigma(i+1)} \dots x'_{\sigma(n)}$ and $c = x_1 x_2 \dots x_{n-1}$. Now there exist only finitely many choices for f_σ and g_σ, and so G is the union of finitely many cosets of centralizers of g_τ's ($\tau \in S_n \backslash \{1\}$). Therefore, by a famous theorem of B. H. Neumann [7], one of the centralizers is of finite index and the proof is complete.

As an immediate corollary to Lemma 3 we have

LEMMA 4. *Let m and n be positive integers. Then a non-trivial restricted (m, n)-permutable group has non-trivial FC-centre.*

We are now able to give the proof of the main result.

PROOF OF THE THEOREM. By Lemma 4 there exists an element $x_1 \in G \backslash \{1\}$ such that $[G: C_G(x_1)]$ is finite. If $G_1 := C_G(x_1)$ is n-permutable then G is finite-by- abelian-by-finite. Thus we may assume that G_1 is not n-permutable. So $Z(G_1)$ is finite by Lemma 2, and $\langle x_1 \rangle \leqslant Z(G_1)$ is finite. Therefore $G_1/\langle x_1 \rangle \neq 1$ and, again by Lemma 4, there exists $x_2 \in G_1 \backslash \langle x_1 \rangle$ such that $[G_1/\langle x_1 \rangle: C_{G_1/\langle x_1 \rangle}(x_2)]$ is finite.

Write $V/\langle x_1 \rangle := C_{G_1/\langle x_1 \rangle}(x_2)$. Then $[V: C_{G_1}(x_2)]$ is finite, since $\langle x_1 \rangle$ is finite, and $[G_1: C_{G_1}(x_2)]$ is finite. If $G_2 := C_{G_1}(x_2)$ is n-permutable, we are done. So suppose that G_2 is not n-permutable. Continuing the above process we obtain sequences $x_1, x_2, \dots$ of distinct elements of G and $G_1, G_2, \dots$ of subgroups of G such that for each $i = 1, 2, \dots$; $\langle x_1, x_2, \dots, x_i \rangle \leqslant Z(G_i)$. Now if G_j is n-permutable for some j then by [3] G_j and therefore G is finite-by-abelian-by-finite. Otherwise, by Lemma 2, $Z(G_i)$ is boundedly finite for all i and so the process must stop after a bounded number of times. This means that there exists some positive integer l such that G_l is an n-permutable group. This completes the proof.

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Manoscritto pervenuto in redazione il 2 agosto 1996
e, in forma revisionata, il 2 gennaio 1997.