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Extensions of Unbounded Topological Spaces.

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Introduction.

A method of compactification of locally compact spaces has been proposed in [1]. This method is based on the concept of essential semilattice homomorphism (*ESH* for short). More precisely, let X be a locally compact (non-compact) Hausdorff space and K a compact Hausdorff space. Let $\mathcal{B}$ be an (open) basis of K closed with respect to finite unions, and let $\mathcal{N}_X$ be the family consisting of the empty set and the open subsets of X which are not relatively compact. A map $\pi\colon \mathcal{B} \to \mathcal{N}_X$, with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$, is an *ESH* if the following conditions hold:

*ESH*1) $X - \pi(K) \notin \mathcal{N}_X - \{\emptyset\}$;

*ESH*2) if $U, V \in \mathcal{B}$ then the symmetric difference

$$\pi(U \cup V)\, \Delta (\pi(U) \cup \pi(V)) \notin \mathcal{N}_X - \{\emptyset\};$$

*ESH*3) if $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$ then $\pi(U) \cap \pi(V) \notin \mathcal{N}_X - \{\emptyset\}$.

If T_X is the topology of X and $\mathcal{S} = \{U \cup (\pi(U) \setminus F)\colon U \in \mathcal{B},\, F \subset X,$ F compact$\}$, then $T_X \cup \mathcal{S}$ is a basis for a topology on the disjoint union $X \cup K$. This new space is a Hausdorff compactification of X with remainder K. It is denoted by $X \underset{\pi}{\cup} K$ and is called an *ESH*-compactification of X.

In this paper we present a natural generalization of the construction above. We say that a topological space X is locally bounded with respect to a family (of «bounded» sets) $\mathcal{F}_X \subset 2^X$ (which is closed under finite

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unions and subsets) if every point of X has a bounded neighborhood. We note that, if $\mathcal{F}_X$ is the family of the relatively compact subsets of X (resp. the relatively Lindelöf subsets of a T_3-space X), we have that local boundedness with respect to $\mathcal{F}_X$ is equivalent to local compactness (resp. local Lindelöfness) of X.

We construct dense extensions of unbounded spaces, which we call B-extensions. By adding some requirements, mainly local boundedness of the space, we obtain Hausdorff B-extensions. This construction is obtained with a method similar to the one used to obtain ESH-compactifications. This method can be applied, for instance, to construct Lindelöf extensions of non-Lindelöf locally Lindelöf spaces. As a final remark, we mention that Theorem 2.3 of this paper appears to be a generalization of a Tkachuk's result (see [8], Proposition 1).

1. – B-extensions with respect to a boundedness.

An extension of a topological space will mean a dense extension.

We recall that a non-empty family $\mathcal{F}_X$ of subsets of a space X is said to be a *boundedness* in X if $\mathcal{F}_X$ is closed with respect to finite unions and subsets (see [5]). Elements of $\mathcal{F}_X$ are called *bounded sets* of X. Every subset of X not in $\mathcal{F}_X$ is called *unbounded.*

A space X with boundedness $\mathcal{F}_X$ is said to be *locally bounded* if every point of X has a bounded neighborhood. If X is T_3, this is equivalent to say that the family of the closed bounded neighborhoods of each point of X is a neighborhood base.

We remark that, for a given space X, the family $\mathcal{C}_X = \{A \subset X : \overline{A}$ is compact$\}$ is a boundedness in X, as well as $\mathcal{L}_X = \{A \subset X : \overline{A}$ is Lindelöf$\}$. Clearly, a space X is locally compact iff X is locally bounded with respect to $\mathcal{C}_X$.

A space X is said to be *locally Lindelöf* if every point of X has a Lindelöf neighborhood. If X is T_3, it is equivalent to say that every point of X has a Lindelöf closed neighborhood (or to say that the family of the closed Lindelöf neighborhoods of every point of X is a neighborhood base). Hence, a T_3-space is locally Lindelöf iff X is locally bounded with respect to $\mathcal{L}_X$.

In [8] Tkachuk defines a space X to be locally Lindelöf if every point of X has an open Lindelöf neighborhood. If X is $T_{3\frac{1}{2}}$, the two definitions are equivalent. In fact, if $x \in X$ and U is a Lindelöf neighborhood of x, then there exists $f \in C(X, [0, 1])$ such that $f(x) = 0$ and $f(X \backslash U) = 1$. Hence

$$x \in Z = f^{-1}(0) \subset W = f^{-1}([0, 1)) \subset U$$

and $f^{-1}([0, 1))$ is Lindelöf since it is an F_σ contained in a Lindelöf subspace.

We remark that Z can be chosen to be a zero-set neighborhood of the point x. In fact, it is sufficient to consider the map $g = (2f - 1) \vee 0$.

Therefore, a locally Lindelöf $T_{3\frac{1}{2}}$-space X is locally bounded with respect to the boundedness

$$\mathcal{Z}\mathcal{L}_X = \{A \subset X : A \subset f^{-1}(0),\ f^{-1}([0, 1]) \text{ is Lindelöf } \&\ f \in C(X, [0, 1])\}.$$

We note that, if X is locally bounded with respect to a boundedness $\mathcal{F}_X$, then $\mathcal{C}_X \subset \mathcal{F}_X$. If $\mathcal{F}_X$ is also closed with respect to countable unions, then $\mathcal{L}_X \subset \mathcal{F}_X$ too.

If aX is an extension of X, then there is a natural boundedness in X associated to aX. In fact, if we define

$$\mathcal{H}_X(aX) = \{A \subset X : Cl_X A = Cl_{aX} A\},$$

then $\mathcal{H}_X(aX)$ is a boundedness in X. We remark that if aX is T_3 and $aX \backslash X$ is closed, or aX is T_2 and $aX \backslash X$ is compact, then X is also locally bounded with respect to $\mathcal{H}_X(aX)$.

Now, let X be an unbounded space with respect to $\mathcal{F}_X$ and let $\mathcal{N}_X$ be the collection consisting of the empty set and the unbounded open subsets of X. Let $\mathcal{B}$ be a basis for the open subsets of a topological space Y, and assume that $Y \in \mathcal{B}$ and $\mathcal{B}$ is closed with respect to finite unions.

We say that $\pi = \pi_{\mathcal{B}, \mathcal{F}_X} : \mathcal{B} \to \mathcal{N}_X$, with $\pi(U) \neq \emptyset$ for every $U \neq \emptyset$, is a *B-map*, if it satisfies the following conditions:

$B1)$ if $\{U_i\}_{i \in A} \subset \mathcal{B}$ is a cover of Y, then $X \backslash \bigcup_{i \in A} \pi(U_i) \in \mathcal{F}_X$;
$B2)$ if $U, V \in \mathcal{B}$ then

$$\overline{\pi(U \cup V)\, \Delta (\pi(U) \cup \pi(V))} \in \mathcal{F}_X;$$

$B3)$ if $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$ then $\overline{\pi(U) \cap \pi(V)} \in \mathcal{F}_X$.

In the following, a B-map $\pi : \mathcal{B} \to \mathcal{N}_X$, with $\mathcal{B}$ closed with respect to unions of cardinality $< \alpha$, will be also called an *α-B map*.

Now, a topological extension of X can be constructed by means of a B-map. If T_X is the topology of X and $\mathcal{S} = \{U \cup (\pi(U) \setminus F)\colon U \in \mathcal{B}, F = \overline{F}, F \in \mathcal{F}_X\}$, then $T_X \cup \mathcal{S}$ is a basis for a topology on the disjoint union $X \cup Y$. To prove this, it is sufficient to imitate the proof given in [1] (see p. 852).

The set $X \cup Y$, endowed with the topology generated by $T_X \cup \mathcal{S}$, will be denoted by $X \bigcup_\pi Y$ and will be called a B-extension of X.

We observe that X is open in $X \bigcup_\pi Y$, and the topologies of the subspaces X, Y coincide with the original topologies. If $\emptyset \neq U \in \mathcal{B}$, then $\pi(U) \notin \mathcal{F}_X$. Hence $\pi(U) \setminus F \neq \emptyset$ for every $F \in \mathcal{F}_X$. It follows that X is dense in $X \bigcup_\pi Y$.

If X is a space with boundedness $\mathcal{F}_X$, we say that a continuous map $f\colon X \to Y$ is *B-singular* (with respect to $\mathcal{F}_X$) if, for every non-empty $U \in T_Y$, $f^{-1}(U)$ is unbounded in X. We note that, if $f\colon X \to Y$ is B-singular, then $\pi = f^{-1}\colon T_Y \to \mathcal{N}_X$ is a B-map.

If $\mathcal{F}_X$ is a boundedness in X, then $\tilde{\mathcal{F}}_X = \{F \in \mathcal{F}_X\colon \overline{F} \in \mathcal{F}_X\}$ is also a boundedness and one has that $F \in \tilde{\mathcal{F}}_X$ iff $\overline{F} \in \mathcal{F}_X$.

A boundedness $\mathcal{G}_X$ with the property that $F \in \mathcal{G}_X$ iff $\overline{F} \in \mathcal{G}_X$ (that is $\mathcal{G}_X = \tilde{\mathcal{G}}_X$) will be called a *closed boundedness*. Clearly, $\mathcal{C}_X$, $\mathcal{L}_X$, $\mathcal{Z}\mathcal{L}_X$ and $\mathcal{H}_X(aX)$ are closed boundednesses.

Now, if $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a B-map, then $\tilde{\pi} = \pi_{\mathcal{B}, \tilde{\mathcal{F}}_X}$, defined by $\tilde{\pi}(U) = \pi(U)$ for every $U \in \mathcal{B}$, is also a B-map and we have that $X \bigcup_\pi Y = X \bigcup_{\tilde{\pi}} Y$. In fact, $\pi(U) \notin \mathcal{F}_X$ implies $\tilde{\pi}(U) \notin \tilde{\mathcal{F}}_X$ and the closed unbounded subsets with respect to $\mathcal{F}_X$ are just the closed unbounded subsets with respect to $\tilde{\mathcal{F}}_X$. Hence, every B-extension of X can be considered as a B-extension with respect to a closed boundedness. Therefore, we can assume, without restriction from the stand point of B-extension, that every boundedness we consider is a closed boundedness.

Now, let aX be a B-extension of X. We show there is a maximal boundedness $\mathcal{M}_X$ and a B-map $\pi' = \pi_{\mathcal{B}, \mathcal{M}_X}$ such that $aX = X \bigcup_{\pi'} Y$.

PROPOSITION 1.1. *Let $aX = X \bigcup_\pi Y$ be a B-extension of X, with B-map $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$. Then $\mathcal{F}_X \subset \mathcal{H}_X(aX)$ and there is a B-map $\pi' = \pi_{\mathcal{B}, \mathcal{H}_X(aX)}$ such that $aX = X \bigcup_{\pi'} Y$. If Y is compact, then $\mathcal{F}_X = \mathcal{H}_X(aX)$.*

PROOF. If $A \in \mathcal{F}_X$, then $Y \cup \pi(Y) \setminus \overline{A}$ is an open neighborhood of Y that does not meet A. Hence $Cl_X A = Cl_{aX} A$ and so $A \in \mathcal{H}_X(aX)$.

Now, we observe that if $\emptyset \neq U \in \mathcal{B}$ then $\pi(U) \notin \mathcal{H}_X(aX)$. Otherwise $U \cup \pi(U) \setminus Cl_{aX}\pi(U) = U \cup \pi(U) \setminus Cl_X \pi(U) = U$ would be a non-empty open subset of aX contained in $aX \setminus X$. Therefore, we can consider the B-map $\pi' = \pi_{\mathcal{B}, \mathcal{H}_X(aX)}$ defined by $\pi'(U) = \pi(U)$ for every $U \in \mathcal{B}$. Since $\mathcal{F}_X \subset \mathcal{H}_X(aX)$, then we have $T_{X \underset{\pi}{\cup} Y} \leq T_{X \underset{\pi'}{\cup} Y}$. On the other hand, every basic open of $X \underset{\pi'}{\cup} Y$ of the form $U \cup \pi'(U) \setminus F = U \cup \pi(U) \setminus F$ is also open in $aX = X \underset{\pi}{\cup} Y$, because $F = Cl_X F = Cl_{aX} F$. Hence $T_{X \underset{\pi}{\cup} Y} = T_{X \underset{\pi'}{\cup} Y}$.

Now, suppose that Y is compact and let $A \in \mathcal{H}_X(aX)$. For every $y \in Y$, there is a basic open $U_y \cup \pi(U_y) \setminus F_y$ containing y such that $(U_y \cup \pi(U_y) \setminus F_y) \cap A = \emptyset$. If $\{U_{y_1}, \ldots, U_{y_n}\}$ is a finite subfamily of $\{U_y\}_{y \in Y}$ that covers Y, then

$$A \subset \left(X \setminus \bigcup_{i=1}^{n} \pi(U_{y_i}) \right) \cup F_{y_1} \cup \ldots \cup F_{y_n} \in \mathcal{F}_X. \quad \blacksquare$$

We note that, if $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a B-map and $X \underset{\pi}{\cup} Y$ is T_2-compact, then $\mathcal{F}_X = \mathcal{H}_X(aX) = \mathcal{C}_X$. Moreover, if X is T_2-locally compact and Y is T_2-compact, then the B-maps (with respect to $\mathcal{C}_X$) are exactly the *ESH*'s as defined in [1]. In fact, let $\pi\colon \mathcal{B} \to \mathcal{N}_X$ be such that $X \setminus \pi(Y) \in \mathcal{C}_X$ and $\pi(U \cup V) \,\Delta\, (\pi(U) \cup \pi(V)) \in \mathcal{C}_X$ for every $U, V \in \mathcal{B}$. If $\mathcal{U} = \{U_i\}_{i \in A} \subset \mathcal{B}$ is a cover of Y and $\{U_{i_1}, \ldots, U_{i_r}\}$ is a finite subcover of Y, then $X \setminus \pi(Y) = X \,\Delta\, \pi(Y) \in \mathcal{C}_X$ and $\pi(Y) \,\Delta \left(\bigcup_{k=1}^{r} \pi(U_{i_k}) \right) \in \mathcal{C}_X$ imply that $X \Delta \left(\bigcup_{k=1}^{r} \pi(U_{i_k}) \right) = X \setminus \left(\bigcup_{k=1}^{r} \pi(U_{i_k}) \right) \in \mathcal{C}_X$. Hence $X \setminus \bigcup_{i \in A} \pi(U_i) \in \mathcal{C}_X$ too.

If X is a space with boundedness $\mathcal{F}_X$, then the relation defined by $A \sim B$ iff $A \Delta B \in \mathcal{F}_X$ is an equivalence relation in 2^X. Finite unions and intersections are compatible with it. Moreover, if $\mathcal{F}_X$ is closed under unions of cardinality γ, then one has that also unions of cardinality γ are compatible with $\sim$.

PROPOSITION 1.2 (see Prop. 1.1 in [2]). *Let X be a locally bounded space with respect to $\mathcal{F}_X$. If $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}\colon \mathcal{B} \to \mathcal{N}_X$ is a B-map, then every map $\pi'\colon \mathcal{B} \to \mathcal{N}_X$ such that $\pi(U) \,\Delta\, \pi'(U) \in \mathcal{F}_X$, for every $U \in \mathcal{B}$, is also a B-map such that $X \underset{\pi}{\cup} Y \cong X \underset{\pi'}{\cup} Y$.*

PROOF. It is easily seen that π' is a B-map such that $T_{X \underset{\pi}{\cup} Y} = T_{X \underset{\pi'}{\cup} Y}$. $\blacksquare$

Now, we will see that the Hausdorff property of $X \bigcup_{\pi} Y$ is ensured under the assumption that X is locally bounded, namely that X is locally bounded with respect to a closed boundedness.

The proof of the following statement is straightforward (see the proof of Prop. 1 in [1]).

PROPOSITION 1.3. *Let X and Y be Hausdorff spaces. If X is locally bounded with respect to a (closed) boundedness $\mathcal{F}_X$ and $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$ is a B-map, then $X \bigcup_{\pi} Y$ is a Hausdorff space containing X as a dense subspace.*

With an argument suggested by the last part of the proof of Prop. 1.1, one can also prove the following proposition.

PROPOSITION 1.4. *If $aX = X \bigcup_{\pi} Y$ is Hausdorff, with $\pi = \pi_{\mathcal{B}, \mathcal{F}_X}$, and Y is compact, then X is locally bounded with respect to $\mathcal{F}_X$.*

THEOREM 1.5. *Let aX be a T_4-extension of X such that $aX \setminus X$ is closed and 0-dimensional. Then aX is a B-extension of X.*

PROOF. We denote by $\mathcal{B}$ the basis of $Y = aX \setminus X$ consisting of the clopen subsets of Y. If $U \in \mathcal{B}$, by the normality of aX, we have that there is an open subset A of aX such that $A \cap (aX \setminus X) = U$ and $aX \setminus A$ is a neighborhood of $Y \setminus A$. In fact, U and $Y \setminus U$ are closed (and disjoint) in aX. For every $U \in \mathcal{B}$, choose such a set A. Observe that, if B is another subset of aX that satisfies the same conditions, then $(A \cap X) \Delta (B \cap X) \in \mathcal{K}_X(aX)$. Moreover, $(Cl_{aX}(A \cap X)) \setminus X = U$ implies that if $U \neq \emptyset$ then $A \cap X \notin \mathcal{K}_X(aX)$. Hence, we can define $\pi = \pi_{\mathcal{B}, \mathcal{K}_X(aX)}$ by setting $\pi(U) = A \cap X$. We note that X is locally bounded with respect to $\mathcal{K}_X(aX)$.

Now, we check the B-properties of π. First, suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is a cover of Y of members of $\mathcal{B}$. Since every $U_i \cup \pi(U_i)$ is open in aX, then $X \setminus \bigcup_{i \in I} \pi(U_i) = aX \setminus \left(\bigcup_{i \in I} U_i \cup \pi(U_i) \right)$ is closed in aX and so it belongs to $\mathcal{K}_X(aX)$.

Let $U, V \in \mathcal{B}$. If $O = (U \cup \pi(U)) \cup (V \cup \pi(V))$, then $O \cap (aX \setminus X) = U \cup V$ and $aX \setminus O$ is a neighborhood of $Y \setminus (U \cup V)$. Therefore,

$$(O \cap X) \Delta \pi(U \cup V) = (\pi(U) \cup \pi(V)) \Delta \pi(U \cup V) \in \mathcal{K}_X(aX).$$

Now, suppose $U, V \in \mathcal{B}$ and $\overline{U} \cap \overline{V} = \emptyset$, that is $U \cap V = \emptyset$. Then

$$Cl_{aX}(\pi(U) \cap \pi(V)) \setminus X \subset (Cl_{aX}\pi(U)) \cap (Cl_{aX}\pi(V)) \setminus X =$$

$$= (Cl_{aX}\pi(U)) \setminus X \cap (Cl_{aX}\pi(V)) \setminus X = U \cap V = \emptyset$$

implies $Cl_{aX}(\pi(U) \cap \pi(V)) \subset X$ and so $\pi(U) \cap \pi(V) \in \mathcal{K}_X(aX)$.

Finally, we show that $aX = X \underset{\pi}{\cup} Y$. Obviously, $T_{X \underset{\pi}{\cup} Y} \leqslant T_{aX}$. Now, let T be an open subset of aX and suppose $y \in T \cap Y$. Choose $U_y \in \mathcal{B}$ such that $y \in U_y$ and $U_y \subset T$. Clearly, $\pi(U_y) \setminus T$ has no adherence points in T (that is open in aX). Also, $\pi(U_y) \setminus T$ has no adherence points in $Y \setminus T$. In fact,

$$(Cl_{aX}(\pi(U_y) \setminus T)) \setminus X \subset (Cl_{aX}\pi(U_y)) \setminus X = U_y \subset T\,.$$

Therefore, $F_y = \pi(U_y) \setminus T \in \mathcal{K}_X(aX)$ and we have $y \in U_y \cup \pi(U_y) \setminus \overline{F}_y \subset T$. Hence,

$$T = \bigcup_{y \in T \cap Y} (U_y \cup \pi(U_y) \setminus \overline{F}_y) \cup (T \cap X)\,,$$

and so $T \in T_{X \underset{\pi}{\cup} Y}$. ■

2. – Lindelöf and other special extensions.

The proof of the next theorem is routine (see Proposition 1 in [1]).

THEOREM 2.1. *Let X be a locally bounded T_2-space with respect to a (closed) boundedness $\mathcal{G}_X \subset \mathcal{L}_X$ (hence X is locally Lindelöf) and let Y be a Lindelöf T_2-space. If $\pi = \pi_{\mathcal{B}, \mathcal{G}_X}$ is a B-map, then $X \underset{\pi}{\cup} Y$ is a Lindelöf T_2-space.*

If X is a Hausdorff space, locally bounded with respect to $\mathcal{C}_X$, and Y is compact and T_2, then $X \underset{\pi}{\cup} Y$ is a T_4-space.

Now, we investigate regularity and normality of $X \underset{\pi}{\cup} Y$ when X is locally Lindelöf and Y is Lindelöf. Of course, we suppose that X and Y are T_3-spaces.

THEOREM 2.2. *Let X be a locally bounded T_3-space with respect to a (closed) boundedness $\mathcal{G}_X \subset \mathcal{L}_X$. Assume that every $F \in \mathcal{G}_X$ is contained in an open subset $A \in \mathcal{G}_X$. If Y is a Lindelöf T_3-space and $\pi = \pi_{\mathcal{B}, \mathcal{G}_X}$ is an ω_1-B map, then $X \underset{\pi}{\cup} Y$ is a T_4-space.*

PROOF. In view of Proposition 1.3 and Theorem 2.1 we have only to show that $X \underset{\pi}{\cup} Y$ is regular.

Let I be a neighborhood of $x \in X$ in $X \underset{\pi}{\cup} Y$. If $U_1 = Cl_X U_1 \subset I \cap X$ and $U_2 = Cl_X U_2 \in \mathcal{G}_X$ are neighborhoods of x in X, then $V = U_1 \cap U_2 \subset I$ is a closed bounded neighborhood of x in X. Hence V is also a closed neighborhood of x in $X \underset{\pi}{\cup} Y$.

Now, let $x \in Y$ with $x \in U \cup \pi(U) \setminus F_1$. By hypotheses, there is an open subset A_1 of X such that $F_1 \subset A_1 \subset \overline{A_1}$ and $G_1 = \overline{A_1} \in \mathcal{G}_X$. By the regularity of Y, there exist W, $W_1 \in \mathcal{B}$ such that $x \in W \subset \overline{W} \subset W_1 \subset \overline{W_1} \subset U$. If $y \in \in Y \setminus U \subset Y \setminus \overline{W_1}$, let $V_y \in \mathcal{B}$ be such that $x \in V_y \subset Y \setminus \overline{W_1}$. Since $Y \setminus U$ is closed and Y is Lindelöf, then the open cover $\{V_y\}_{y \in Y}$ of $Y \setminus U$ has a countable subcover $\{V_n\}_{n=1}^{\infty}$, and one has

$$Y \setminus U \subset V = \bigcup_{n=1}^{\infty} V_n \subset Y \setminus \overline{W_1}.$$

Now U, $V \in \mathcal{B}$ and $U \cup V = Y$ imply $F_2 = X \setminus (\pi(U) \cup \pi(V)) \in \mathcal{G}_X$. Denote by A_2 an open subset of X such that $F_2 \subset A_2 \subset \overline{A_2}$ and $G_2 = \overline{A_2} \in \mathcal{G}_X$.

Since $\overline{V} \cap \overline{W} = \emptyset$ we have that $\pi(V) \cap \pi(W) \in \mathcal{G}_X$. If we set $G_3 = \overline{\pi(V) \cap \pi(W)}$ then one has $\pi(V) \cap (\pi(W) \setminus G_3) = \emptyset$. Now, we claim that

$$\overline{W \cup \pi(W) \setminus (G_1 \cup G_2 \cup G_3)} \subset U \cup \pi(U) \setminus F_1.$$

Suppose $z \notin U \cup \pi(U) \setminus F_1$ and $z \in X$. Then

$$z \in \pi(V) \cup F_1 \cup F_2 \subset \pi(V) \cup A_1 \cup A_2 \subset \pi(V) \cup G_1 \cup G_2,$$

and $\pi(V) \cup A_1 \cup A_2$ is an open neighborhood of z that does not meet $W \cup \pi(W) \setminus (G_1 \cup G_2 \cup G_3)$.

Finally, let $z \notin U \cup \pi(U) \setminus F_1$ and $z \in Y$. Then $z \in V$ and $V \cup \pi(V)$ is an open neighborhood of z that does not meet $W \cup \pi(W) \setminus G_3$. ■

By the way, we note that, in the previous theorem, if $\mathcal{G}_X$ is closed under countable unions, it is sufficient to assume that π is a B-map (in place of ω_1-B map).

Let us see the theory at work in the following example. It also shows that Lindelöf extensions of locally Lindelöf spaces are not always compactifications.

EXAMPLE 2.3. Let E be an uncountable set, viewed as a discrete space, and consider $X = S \times E$, $f : S \times E \to S$ the canonical projection, with S the Sorgenfrey line. Then X is a non-Lindelöf locally Lindelöf space, which has a Lindelöf non-compact B-extension $aX = X \bigcup_{\pi} S$, with respect to the B-map

$$\pi = f^{-1}\colon T_S \to \mathcal{N}_X = \{\emptyset\} \cup \{M \subset X\colon M \text{ is not relatively Lindelöf}\}.$$

In fact, for every $U \in T_S \setminus \{\emptyset\}$, $f^{-1}(U) = U \times E$ has a closure $\overline{U} \times E = \bigcup \{\overline{U} \times q\colon q \in E\}$, namely $\overline{U} \times E$ has a partition in an uncountable family of non-empty open sets. Then $f^{-1}(U)$ is not relatively Lindelöf and f is B-singular. By definition S is closed as a subset of aX, therefore aX is not compact. On the other hand, X is obviously $T_{3\frac{1}{2}}$, therefore has T_2-compactifications, and in each of them the remainder is not closed, since X is not locally compact. From Theorem 2.2, we know that $X \bigcup_{\pi} Y$ is a T_4-space.

If a space X is $T_{3\frac{1}{2}}$, we have already observed that X is locally bounded with respect to $\mathcal{Z}\mathcal{L}_X$. Since for every $F \in \mathcal{Z}\mathcal{L}_X$ there is an open $A \in \mathcal{Z}\mathcal{L}_X$ containing F, we have the following result.

THEOREM 2.3. *Let X be a locally Lindelöf $T_{3\frac{1}{2}}$-space and let Y be a Lindelöf T_3-space. If $\pi = \pi_{\mathcal{B}, \mathcal{Z}\mathcal{L}_X}$ is an ω_1-B map then $X \bigcup_{\pi} Y$ is a Lindelöf T_4-space.*

In the previous theorem, if X is a locally Lindelöf T_4-space, $\mathcal{Z}\mathcal{L}_X$ can be replaced by $\mathcal{L}_X$. In fact, we have the following proposition.

PROPOSITION 2.4. *If X is a locally Lindelöf T_4-space then $\mathcal{Z}\mathcal{L}_X = \mathcal{L}_X$.*

PROOF. Let $F \in \mathcal{L}_X$. For every $y \in \overline{F}$, let U_y be an open neighborhood of y with Lindelöf closure. The open cover $\{U_y\}_{y \in \overline{F}}$ of $\overline{F}$ has a countable subcover $\{U_n\}_{n=1}^{\infty}$. Then the open subset $U = \bigcup_{n=1}^{\infty} U_n$ of X contains $\overline{F}$ and is contained in $U' = \bigcup_{n=1}^{\infty} \overline{U_n}$, that is Lindelöf. Now, by the normality of X, there is $f \in C(X, [0, 1])$ such that

$$\overline{F} \subset f^{-1}(0) \subset f^{-1}([0, 1)) \subset U \subset U'.$$

Then $f^{-1}([0, 1))$ is Lindelöf because it is a cozero-set contained in the Lindelöf set U'. ■

Now, let X be non-Lindelöf locally Lindelöf , and $Y = \{\infty\}$. If X is $T_{3\frac{1}{2}}$, let $\pi = \pi_{T_Y, \mathcal{Z}\mathcal{L}_X}$ be the map defined by $\pi(\infty) = X$. Then $X \bigcup_{\pi} \{\infty\}$, that is T_4, is just the «single-point Lindelöfication» considered by Tkachuk in [8].

If X is T_3, but not $T_{3\frac{1}{2}}$, and $\pi = \pi_{T_Y, \mathcal{L}_X}$ is defined by $\pi(\infty) = X$ then $X \bigcup_\pi \{\infty\}$ is a Lindelöf T_2-extension of X, that is not T_4.

We observe that, by Theorem 2.2, it follows that a T_3-space X is locally Lindelöf and $T_{3\frac{1}{2}}$ if and only if X is locally bounded with respect to $\mathcal{Z}\mathcal{L}_X$.

If aX is a T_2-extension of a space X with $|aX \setminus X| = n$, then X is locally bounded with respect to $\mathcal{K}_X(aX)$. If aX is also Lindelöf, then $\mathcal{K}_X(aX) \subset \subset \mathcal{L}_X$ and so X is locally Lindelöf. Now, we show that aX is a B-extension of X. This result could be proved in a way similar to that of Theorem 1.5. Here, a slightly different proof is presented, which will be useful to prove the next theorem.

PROPOSITION 2.5. *Let aX be a T_2-extension of a space X with $|aX \setminus X| = n$. Then X is locally bounded with respect to $\mathcal{K}_X(aX)$ and aX is a B-extension of X.*

PROOF. Let $Y = aX \setminus X = \{y_1, \dots, y_n\}$ and let $U_1, \dots, U_n$ be mutually disjoint open neighborhoods of $\{y_1, \dots, y_n\}$ in aX. Since $U_i \cap X \notin \notin \mathcal{K}_X(aX)$ for every i, we can define $\pi = \pi_{T_Y, \mathcal{K}_X(aX)}$ by setting

$$\pi(\{y_{i_1}, \dots, y_{i_r}\}) = (U_{i_1} \cap X) \cup \dots \cup (U_{i_r} \cap X).$$

It is easily seen that π is a B-map. Now, we show that $X \bigcup_\pi Y = aX$. Clearly $T_{X \bigcup_\pi Y} \leq T_{aX}$. Conversely, let A be an open subset of aX. If $A \subset X$, then A is obviously an open subset of $X \bigcup_\pi Y$. Suppose $y_k \in A$ for some k. Since $y_i \notin Cl_{aX}(U_k \setminus A)$ for every $i = 1, \dots, n$, we get $F_k = Cl_X(U_k \setminus A) = Cl_{aX}(U_k \setminus A) \in \mathcal{K}_X(aX)$. Hence, if $A \cap (aX \setminus X) = = \{y_{k_1}, \dots, y_{k_s}\}$, then we have

$$A = \left(\bigcup_{j=1}^{s} (\{y_{k_j}\} \cup \pi(\{y_{k_j}\}) \setminus F_{k_j}) \right) \cup (X \cap A),$$

and A is open in $X \bigcup_\pi Y$. ∎

Now, we characterize the spaces that have a T_2-extension with finite remainder (compare [6]).

THEOREM 2.6. *A Hausdorff space X has a T_2-extension aX with $|aX\backslash X| = n$ iff X is locally bounded with respect to a boundedness $\mathcal{F}_X$ and there exist n mutually disjoint unbounded open sets $A_1, \dots, A_n \subset X$ and a bounded set $F \subset X$ such that*

$$X = A_1 \cup \dots \cup A_n \cup F .$$

PROOF. The proof of the above proposition shows that the condition is necessary. Conversely, suppose that the condition holds. If $y_1, \dots, y_n$ are n points not in X and $Y = \{y_1, \dots, y_n\}$, we define $\pi = \pi_{\tau_Y, \mathcal{F}_X}$ by $\pi(\{y_{i_1}, \dots, y_{i_r}\}) = A_{i_1} \cup \dots \cup A_{i_r}$. Then π is a B-map and $aX = X \bigcup_{\pi} Y$ is a T_2-extension with $|aX\backslash X| = n$. ■

From Prop. 1.1, it follows that in a Hausdorff space X, the existence of a boundedness $\mathcal{F}_X$ with the properties as in in Theorem 2.6, implies that $\mathcal{F}_X = \mathcal{K}_X(aX)$ for a suitable T_2-extension aX of X.

A consequence of Theorem 2.6 and Theorem 2.1 is the following.

COROLLARY 2.7. *A Hausdorff space X has a Lindelöf T_2-extension aX with $|aX\backslash X| = n$ iff X is locally bounded with respect to a boundedness $\mathcal{G}_X \subset \mathcal{L}_X$ and there exist n mutually disjoint unbounded open sets $A_1, \dots, A_n \subset X$ and a bounded set $F \subset X$ such that*

$$X = A_1 \cup \dots \cup A_n \cup F .$$

Now, we give an example of a Lindelöf T_2-extension of a locally Lindelöf space X that cannot be obtained as a B-extension with respect to $\mathcal{L}_X$.

EXAMPLE 2.8. Let E be a discrete non countable space, and let $E = S \cup T$ be a partition with S countably infinite. Consider further a two points set $Y = \{a, b\}$ disjoint from E, $\tilde{S} = S \cup \{a\}$ the one-point compactification of S, $\tilde{T} = T \cup \{b\}$ the Lindelöf one-point B-extension of T, with respect to $\pi = \pi_{\{b\}, \mathcal{L}_T}$ defined by $\pi(\{b\}) = T$. Then, the topological sum $L = \tilde{S} + \tilde{T}$ is T_2 Lindelöf and E is dense in L. But L, that is a B-extension of E with respect to the boundedness $\mathcal{K}_E(L)$, cannot be obtained as a B-extension of E with respect to the «Lindelöf» boundedness $\mathcal{L}_E$. In fact, $\mathcal{K}_E(L) \neq \mathcal{L}_E$, and, from Prop. 1.1, we have that if $\pi' = \pi_{\mathcal{B}, \mathcal{F}_E}$ is a B-map such that $L = E \bigcup_{\pi'} Y$ then $\mathcal{F}_E = \mathcal{K}_E(L)$.

We recall that a space X is said to be $[\theta, \kappa]$-compact if every open cover of X of cardinality $\leq \kappa$ has a subcover of cardinality $< \theta$. If $\theta = \omega$, then X is said to be initially κ-compact, and if $|X| \leq \kappa$, then X is said to be finally θ-compact. Lindelöf spaces are exactly the finally ω_1-compact spaces.

For a space X, $\mathcal{C}_X(\theta, \kappa) = \{A \subset X : \overline{A} \text{ is } [\theta, \kappa]\text{-compact}\}$ is a boundedness in X. We say that a space X is locally $[\theta, \kappa]$-compact if every point of X has a $[\theta, \kappa]$-compact neighborhood. As in the Lindelöf case, if X is T_3, then X is locally $[\theta, \kappa]$-compact if and only if X is locally bounded with respect to $\mathcal{C}_X(\theta, \kappa)$.

Proposition 2.1 can be easily generalized to $[\theta, \kappa]$-compact case.

PROPOSITION 2.9. *Let X be a locally bounded T_2-space with respect to a (closed) boundedness $\mathcal{G}_X \subset \mathcal{C}_X(\theta, \kappa)$ (hence X is locally $[\theta, \kappa]$-compact) and let Y be a $[\theta, \kappa]$-compact T_2-space. If $\pi = \pi_{\mathcal{B}, \mathcal{G}_X}$ is a B-map, then $X \bigcup_\pi Y$ is a $[\theta, \kappa]$-compact T_2-space.*

Theorem 2.2 can be generalized to finally θ-compact case in the following way.

THEOREM 2.10. *Let X be a locally bounded T_3-space with respect to a (closed) boundedness $\mathcal{G}_X \subset \mathcal{C}_X(\theta, \kappa)$, with $|X| \leq \kappa$. Assume that every $F \in \mathcal{G}_X$ is contained in an open subset $A \in \mathcal{G}_X$. If Y is T_3 and $\pi = \pi_{\mathcal{B}, \mathcal{G}_X}$ is a θ-B map, then $X \bigcup_\pi Y$ is a T_3-space.*

Finally, Theorem 2.3 and Corollary 2.7 have similar generalizations to the finally θ-compact case.

REFERENCES

[1] A. CATERINO - G. D. FAULKNER - M. C. VIPERA, *Construction of compactifications using essential semilattice homomorphisms*, Proc. Amer. Math. Soc., **116** (1992), pp. 851-860.

[2] A. CATERINO, *ESH which induce βN*, Rend. Mat. Univ. Trieste, **25** (1993), pp. 57-66.

[3] R. E. CHANDLER, *Hausdorff Compactifications*, Marcel Dekker, New York (1976).

[4] U. N. B. Dissanayake - K. P. R. Sastry, *Locally Lindelöf spaces*, Indian J. Pure Appl. Math., **18** (1987), pp. 876-881.
[5] S. T. Hu, *Boundedness in a topological space*, J. Math. Pures Appl., **28** (1949), pp. 287-320.
[6] K. D. Magill Jr., *N-points compactifications*, Amer. Math. Monthly, **72** (1965), pp. 1075-1081.
[7] S. A. Naimpally - B. D. Warrack, *Proximity Spaces*, Cambridge University Press, Cambridge (1970).
[8] V. V. Tkachuk, *Almost Lindelöf and locally Lindelöf spaces*, Izvestiya VUZ. Matematika, **32** (1988), pp. 84-88.

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