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Subgaussian Random Variables in Hilbert Spaces (*).

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0. - Introduction.

In the paper [1] the following definition is given:

(0.1) DEFINITION. A real r.v. X is said to be subgaussian if there exists a number $a \geqslant 0$ such that

$$E[e^{tX}] \leqslant \exp\left(\frac{1}{2}a^2t^2\right), \qquad \forall t \in \mathbb{R},$$

If this is the case, the number

$$\tau_{\text{cl}}(X) = \inf\left\{a \geqslant 0 \colon E[e^{tX}] \leqslant \exp\left(\frac{1}{2}a^2t^2\right), \forall t \in \mathbb{R}\right\}$$

is called the *gaussian standard* of X.

Denote by $\mathcal{S}\mathcal{G}(\Omega)$ the set of real subgaussian varialbes. In [1] it is proved that $\mathcal{S}\mathcal{G}(\Omega)$ is a vector space and τ_{cl} is a norm in it. Moreover $\mathcal{S}\mathcal{G}(\Omega)$, endowed with the norm τ_{cl}, is a Banach space.

In this paper we consider random variables taking their values in a separable Hilbert space H, and we give three different definitions of subgaussianity (the first of them is subgaussianity with respect to a linear trace class operator R, symmetric and positive definite; the second is subgaussianity with respect to a complete orthonormal system E in

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H; the last one is subgaussianity *tout-court* (i.e., our definition will not depend on any R or E)). We investigate the relations between these concepts; moreover, we show that the set of E-subgaussian variables, endowed with a suitable norm, is a Banach space; for subgaussian variables in the other two senses, we prove the same thing when H is finite dimensional.

1. – Subgaussianity with respect to an operator.

Let H be a Hilbert space (finite or infinite dimensional), and denote by $\langle \cdot, \cdot \rangle$ its inner product. Let X be an H-valued random variable and R a linear operator on H. Suppose that R is a trace class operator, symmetric and positive definite. We shall denote by $\mathcal{L}_1$ the set of such operators.

We give the following

(1.1) DEFINITION. We say that X is *subgaussian with respect to* $R \in \mathcal{L}_1$ (or *R-subgaussian*) if there exists $a \geq 0$ such that

$$E[e^{\langle x, X \rangle}] \leq \exp\left(\frac{1}{2} a^2 \langle Rx, x \rangle\right) \quad \text{for every } x \in H.$$

If this is the case, we put

$$\sigma_R(X) = \inf\left\{a \geq 0 \colon E[e^{\langle x, X \rangle}] \leq \exp\left(\frac{1}{2} a^2 \langle Rx, x \rangle\right) \text{ for every } x \in H\right\}.$$

(1.3) REMARK. It is clear that

(i) $$\sigma_R(X) = \sup_{x \neq 0} \frac{\tau_{\text{cl}}(\langle x, X \rangle)}{(\langle Rx, x \rangle)^{1/2}},$$

(ii) $$E[e^{\langle x, X \rangle}] \leq \exp\left(\frac{1}{2} \sigma_R(X)^2 \langle Rx, x \rangle\right) \quad \text{for every } x \text{ in } H.$$

(1.4) REMARK. In [4] the following definition is given: X is a subgaussian variable if there exists an H-valued gaussian vector G such that, for every x in H, we have

$$E[e^{\langle x, X \rangle}] \leq E[e^{\langle x, G \rangle}];$$

now, according to the results of [2], we have

$$E[e^{\langle x, G\rangle}] = \exp\left(\frac{1}{2}E[\langle x, G\rangle^2]\right) = \exp\left(\frac{1}{2}\langle S_G x, x\rangle\right),$$

where S_G is the covariance operator of G.

Since G is gaussian, S_G is in $\mathcal{L}_1$ (see [2]); hence X is S_G-subgaussian.

Conversely, if X is R-subgaussian, the operator $\sigma_R^2(X)R$ is in $\mathcal{L}_1$, hence it is the covariance operator of some gaussian vector G, and we have

$$E[\langle x, G\rangle^2] = \langle Sx, x\rangle .$$

Then

$$E[e^{\langle x, X\rangle}] \leq \exp\left(\frac{1}{2}\langle Sx, x\rangle\right) = \exp\left(\frac{1}{2}E[\langle x, G\rangle^2]\right) = E[e^{\langle x, G\rangle}],$$

and X is subgaussian in the sense of [4].

(1.5) REMARK. Definition (1.1) is a generalization of the one given in [3] for the case $H = \mathbb{R}^n$.

We shall denote by $\mathcal{SG}_R(\Omega)$ the set of H-valued R-subgaussian variables. By recalling that τ_{cl} is a norm in the space of real subgaussian variables (see [1]), Remark (1.3)(i) yields immediately that $\mathcal{SG}_R(\Omega)$ is a vector space and σ_R is a norm in it, i.e. $(\mathcal{SG}_R(\Omega), \sigma_R)$ is a metric space. As we shall see in Section 3, it is a Banach space when H is finite dimensional.

2. – Subgaussianity with respect to a complete orthonormal system.

Let $E = \{e_n\}$ be a complete orthonormal system (C.O.N.S.) in H.

(2.1) DEFINITION. We say that X is *subgaussian with respect to* E (or E-*subgaussian*) if the two following conditions are verified:

(i) For every $x \in H$, the real random variable $\langle x, X\rangle$ is subgaussian;

(ii) We have

$$\tau_E^2(X) \equiv \sum_n \tau_{\mathrm{cl}}^2(\langle e_n, X\rangle) < +\infty .$$

We shall denote by $\mathcal{SG}_E(\Omega)$ the set of H-valued E-subgaussian variables. (This notation is quite similar to the one introduced in Section 1 for the set of variables which are subgaussian with respect to an operator, but this should cause no confusion).

We shall prove the following

(2.2) THEOREM. *$\mathcal{SG}_E(\Omega)$ is a vector space and τ_E is a norm in $\mathcal{SG}_E(\Omega)$; moreover, $(\mathcal{SG}_E(\Omega), \tau_E)$ is a Banach space.*

The proof of (2.2) is a straightforward application of the following general result:

(2.3) THEOREM. *Let (B, ν) be a Banach space, and consider the set*

$$B_2^{\mathrm{N}} = \left\{x = (x_1 . x_2, \ldots) \in B^{\mathrm{N}} \text{ and } \sum_n \nu^2(x_n) < +\infty\right\}.$$

Then B_2^{N} (with sum and product by a scalar defined in the usual way) is a Banach space with norm

$$\varrho(x) = \left(\sum_n \nu^2(x_n)\right)^{1/2}.$$

Theorem (2.3) is standard. Anyway, by the sake of completeness, we sketch the proof in the appendix.

Theorem (2.2) follows from Theorem (2.3) by identifying X with the vector $(\langle X, e_1\rangle, \langle X, e_2\rangle, \ldots)$ and by taking $\nu = \tau_{\mathrm{cl}}$ (recall that, by the results of [1], the set of real subgaussian variables is a Banach space with norm τ_{cl}).

3. – Relation between R-subgaussianity and E-subgaussianity.

(3.1) PROPOSITION. *Let X be subgaussian with respect to an operator $R \in \mathcal{L}_1$, and let E be any C.O.N.S. in H. Then X is subgaussian with respect to E and*

$$\tau_E^2(X) \leq \sigma_R^2(X)\,\mathrm{tr}(R) < +\infty,$$

where $\mathrm{tr}(R)$ *denotes the trace of R.*

PROOF. It is clear by (1.2) and (1.3)(ii) that, for every x in H, $\langle x, X\rangle$ is subgaussian and

$$\tau_{\mathrm{cl}}^2(\langle x, X\rangle) \leq \sigma_R^2(X)\langle Rx, x\rangle.$$

Then

$$\tau_{\mathrm{cl}}^2(\langle e_n, X\rangle) \leqslant \sigma_R^2(X)\langle Re_n, e_n\rangle.$$

We get the conclusion by summing over n and recalling that

$$\mathrm{tr}(R) = \sum_n \langle Re_n, e_n\rangle. \qquad \blacksquare$$

(3.2) REMARK. Proposition (3.1) says that $(\mathcal{SG}_R(\Omega), \sigma_R)$ can be continuously imbedded in $(\mathcal{SG}_E(\Omega), \tau_E)$.

We now drop for a moment the assumption that $R \in \mathcal{L}_1$, and let $E_R = \{f_n\}$ be the set of normalized eigenvectors of R. We are going to compare σ_R with τ_E. To this extent, we need the following

(3.3) LEMMA. *Let $E = \{e_n\}$ by any C.O.N.S. in H. Then, for every $x \in H$,*

$$\tau_{\mathrm{cl}}(\langle x, X\rangle) \leqslant \sum_n |\langle x, e_n\rangle|\, \tau_{\mathrm{cl}}(\langle e_n, X\rangle).$$

PROOF. For every $\omega \in \Omega$ and every n, put

$$Y_n(\omega) = \sum_{k=1}^{n} \langle x, e_k\rangle \cdot \langle e_k, X(\omega)\rangle.$$

Then $Y_n(\omega)$ converges to $\langle x, X(\omega)\rangle$ for each ω in Ω, as $n \to \infty$.
Moreover, by the triangular inequality for τ_{cl},

$$\tau_{\mathrm{cl}}^2(Y_n) = \tau_{\mathrm{cl}}^2\left(\sum_{k=1}^{n} \langle x, e_k\rangle \cdot \langle e_k, X\rangle\right) \leqslant \left[\sum_{k=1}^{n} |\langle x, e_k\rangle|\, \tau_{\mathrm{cl}}(\langle e_k, X\rangle)\right]^2 \leqslant$$

$$\leqslant \left[\sum_{k=1}^{n} |\langle x, e_k\rangle|^2\right]\cdot\left[\sum_{k=1}^{n} \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle)\right] \leqslant$$

$$\leqslant \left[\sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2\right]\cdot\left[\sum_{k=1}^{\infty} \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle)\right] = \|x\|^2 \tau_E^2(X) < +\infty.$$

Hence, for every $t \in \mathbb{R}$ and every $\varepsilon > 0$,

$$\sup_n E[(e^{tY_n})^{1+\varepsilon}] = \sup_n E[e^{t(1+\varepsilon)Y_n}] \leqslant$$

$$\leqslant \sup_n \exp\left(\frac{1}{2} t^2 (1+\varepsilon)^2 \tau_{\mathrm{cl}}^2(Y_n)\right) < +\infty,$$

so that the r.v. e^{tY_n} are uniformly integrable and, by Lebesgue theorem, we have

$$E[e^{t\langle x, X\rangle}] = E[e^{t \lim_n Y_n}] = E[\lim_n e^{tY_n}] = \lim_n E[e^{tY_n}] \leqslant$$

$$\leqslant \exp\left(\frac{1}{2}t^2 \sup_n \tau_{\text{cl}}^2(Y_n)\right).$$

It follows

$$\tau_{\text{cl}}(\langle x, X\rangle) \leqslant \sup_n \tau_{\text{cl}}(Y_n) = \sup_n \tau_{\text{cl}}\left(\sum_{k=1}^{n} \langle x, e_k\rangle \cdot \langle e_k, X\rangle\right) \leqslant$$

$$\leqslant \sup_n \sum_{k=1}^{n} |\langle x, e_k\rangle| \, \tau_{\text{cl}}(\langle e_k, X\rangle) = \sum_{k=1}^{\infty} |\langle x, e_k\rangle| \tau_{\text{cl}}(\langle e_k, X\rangle). \qquad \blacksquare$$

(3.4) PROPOSITION. *Let* $0 < \alpha_1 \leqslant \alpha_2 \leqslant \ldots$ *be the eigenvalues of* R, *and assume that* X *is* E_R*-subgaussian. Then*

$$\tau_{\text{cl}}^2(\langle x, X\rangle) \leqslant \frac{1}{\alpha_1}\tau_{E_R}^2(X)\langle Rx, x\rangle.$$

PROOF. By Lemma (3.3), we have

$$\tau_{\text{cl}}^2(\langle x, X\rangle) \leqslant \left[\sum_{k=1}^{\infty} |\langle x, f_k\rangle| \, \tau_{\text{cl}}(\langle f_k, X\rangle)\right]^2 =$$

$$= \left[\sum_{k=1}^{\infty} \sqrt{\alpha_k} \, |\langle x, f_k\rangle| \, \frac{1}{\sqrt{\alpha_k}} \tau_{\text{cl}}(\langle f_k, X\rangle)\right]^2 \leqslant$$

$$\leqslant \left[\sum_{k=1}^{\infty} \alpha_k \, |\langle x, f_k\rangle|^2\right]\left[\sum_{k=1}^{\infty} \frac{1}{\alpha_k}\tau_{\text{cl}}^2(\langle f_k, X\rangle)\right] =$$

$$= \langle Rx, x\rangle\left[\sum_{k=1}^{\infty} \frac{1}{\alpha_k}\tau_{\text{cl}}^2(\langle f_k, X\rangle)\right] \leqslant \langle Rx, x\rangle \frac{1}{\alpha_1}\tau_{E_R}^2(X). \qquad \blacksquare$$

If H is finite dimensional, Proposition (3.4) yields the following upper bound for $\sigma_R(X)$:

(3.5) PROPOSITION. *We have*

$$\sigma_R^2(X) \leqslant \frac{1}{\alpha_1}\tau_{E_R}^2(X).$$

PROOF. For every $x \in H$ we have, by (3.4),

$$E[e^{\langle x, X\rangle}] \leq \exp\left(\frac{1}{2}\tau_{\mathrm{cl}}^2(\langle x, X\rangle)\right) \leq \exp\left(\frac{1}{2}\frac{1}{\alpha_1}\tau_{E_R}^2(X)\langle Rx, x\rangle\right),$$

so that

$$\sigma_R^2(X) = \inf\left\{b \geq 0\colon\ E[e^{\langle x, X\rangle}] \leq \exp\left(\frac{1}{2}b\langle Rx, x\rangle\right) \text{ for every } x \in H\right\} \leq$$

$$\leq \frac{1}{\alpha_1}\tau_{E_R}^2(X). \qquad \blacksquare$$

Propositions (3.1) and (3.4), together with Theorem (2.2), allows us to state the following result

(3.6) THEOREM. *If H is finite dimensional and R is injective, then $\mathcal{SG}_R(\Omega) = \mathcal{SG}_{E_R}(\Omega)$ (this is a set-theoretical inclusion). Moreover the two norms τ_{E_R} and σ_R are equivalent; hence $(\mathcal{SG}_R(\Omega), \sigma_R)$ is a Banach space.*

4. - The space of subgaussian variables.

The two definitions of subgaussianity we have given in Section 1 and 2 depend strongly on the operator R in $\mathcal{L}_1$ and the C.O.N.S. E respectively. Here we give a definition which will not depend on such objects.

(4.1) DEFINITION. We say that X is *subgaussian* if there exists $R \in \mathcal{L}_1$ such that X is R-subgaussian. We shall denote by $\mathcal{SG}(\Omega)$ the set of such variables, and define the quantity

$$\sigma(X) = \sup\{(\operatorname{tr} R)^{1/2}\sigma_R(X);\ R \in \mathcal{L}_1\}.$$

By virtue of the results of Section 2, σ is obviously a norm in $\mathcal{SG}(\Omega)$. Our aim is now to prove the following

(4.2) THEOREM. *If H is finite dimensional, $(\mathcal{SG}(\Omega); \sigma)$, is a Banach space.*

PROOF. Let $\{X_n\}$ be a Cauchy sequence in $(\mathcal{SG}(\Omega); \sigma)$. The inequality

$$(\operatorname{tr} R)^{1/2}\sigma_R(X_n - X_m) \leq \sigma(X_n - X_m)$$

yields that, for every R in $\mathcal{L}_1$, $\{X_n\}$ is Cauchy in $(\mathcal{SG}_R(\Omega), \sigma_R)$. Since the last space is Banach, for each R there exists $Y^{(R)}$ is $\mathcal{SG}_R(\Omega)$ such that

$$(\operatorname{tr} R)^{1/2}\sigma_R(X_n - Y^{(R)}) \to 0 .$$

Let now $E = \{e_n\}$ be any C.O.N.S. in H. From Proposition (3.1) it follows that $\{X_n\}$ τ_E-converges to $Y^{(R)}$, so that $Y^{(R)}$ cannot depend on R; let's call it Y from now on.

From the triangular inequality we now deduce that

$$(\operatorname{tr} R)^{1/2}\sigma_R(X_n - Y) \leq (\operatorname{tr} R)^{1/2}\sigma_R(X_n - X_m) + (\operatorname{tr} R)^{1/2}\sigma_R(X_m - Y) \leq$$

$$\leq \sigma(X_n - X_m) + (\operatorname{tr} R)^{1/2}\sigma_R(X_m - Y).$$

By interchanging the roles of n and m we get

$$|\operatorname{tr} R)^{1/2}\sigma_R(X_n - Y) - (\operatorname{tr} R)^{1/2}\sigma_R(X_m - Y)| \leq \sigma(X_n - X_m).$$

The above inequality yields that $(\operatorname{tr} R)^{1/2}\sigma_R(X_n - Y) \to 0$ uniformly in R, and this in turn implies that $\sigma(X_n - Y) \to 0$. ■

(4.3) REMARK. It is easy to see that X is subgaussian in the sense of Definition (4.1) if and only if there exists a C.O.N.S. E such that X is E-subgaussian (in the sense of (2.1)). In $\mathcal{SG}(\Omega)$ one can then consider the quantity

$$\tau(X) = \sup\{\tau_E(X);\ E \text{ C.O.N.S. } in\ H\}.$$

It is immediate to see that τ is a norm in $\mathcal{SG}(\Omega)$ and, by arguments similar to the previous ones (using (3.4) instead of (3.1)), one can easily show that $(\mathcal{SG}(\Omega); \tau)$ is a Banach space. By Proposition (3.1), $(\mathcal{SG}(\Omega); \sigma)$ can be imbedded continuously in $(\mathcal{SG}(\Omega); \tau)$.

5. - A condition for subgaussianity with respect to an operator.

In this section we are looking for a condition which assures the existence of an operator R such that X is R-subgaussian. We need the following

(5.1) LEMMA. *Let $E = \{e_n\}$ be a C.O.N.S. in H; suppose that X is subgaussian with respect to E and that the following assumption holds*:

(5.2) *for every* $n \in \mathbb{N}$ *and every* $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$, *we have*

$$\tau_{\mathrm{cl}}^2\left(\sum_{k=1}^{n} \lambda_k \langle e_k, X\rangle\right) = \sum_{k=1}^{n} \lambda_k^2 \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle).$$

Then

$$\tau_{\mathrm{cl}}^2(\langle x, X\rangle) = \sum_{k=1}^{\infty} \langle x, e_k\rangle^2 \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle)$$

(*the last series converges since it is not greater than* $\|x\|^2 \tau_E^2(X)$).

PROOF. For every $\omega \in \Omega$ and every n, put, as in (3.3),

$$Y_n(\omega) = \sum_{k=1}^{n} \langle x, e_k\rangle \cdot \langle e_k, X(\omega)\rangle.$$

We have

$$\tau_{\mathrm{cl}}^2(Y_n - \langle x, X\rangle) =$$

$$= \tau_{\mathrm{cl}}^2\left(\sum_{k=n+1}^{\infty} \langle x, e_k\rangle \cdot \langle e_k, X\rangle\right) \leq \left[\sum_{k=n+1}^{\infty} |\langle x, e_k\rangle|^2\right]\left[\sum_{k=n+1}^{\infty} \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle)\right] \to 0,$$

as $n \to \infty$.

By the continuity of the norm τ_{cl}, it follows that

$$\tau_{\mathrm{cl}}(Y_n) \to \tau_{\mathrm{cl}}(\langle x, X\rangle).$$

On the other hand, by assumption (5.2),

$$\tau_{\mathrm{cl}}^2(Y_n) = \tau_{\mathrm{cl}}^2\left(\sum_{k=1}^{n} \langle x, e_k\rangle \cdot \langle e_k, X\rangle\right) =$$

$$= \sum_{k=1}^{n} \langle x, e_k\rangle^2 \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle) \to \sum_{k=1}^{\infty} \langle x, e_k\rangle^2 \tau_{\mathrm{cl}}^2(\langle e_k, X\rangle). \qquad \blacksquare$$

(5.3) REMARK. Recall that the variance of the sum of two independent random variables is the sum of their variance. From this point of view, condition (5.2) may be regarded as a sort of independence among the variables $\langle e_n, X\rangle$, $n \in \mathbb{N}$.

(5.4) PROPOSITION. *Suppose that* $\mathrm{span}(\mathrm{Im}\, X) = H$. *Suppose moreover that there exists a C.O.N.S.* $E = \{e_n\}$ *such that* X *is subgaussian with respect to* E *and* (5.2) *holds. Then* X *is subgaussian with respect to* *the operator R defined by*

$$R e_n = \tau_{\mathrm{cl}}^2(\langle e_n, X\rangle)\, e_n \,.$$

Moreover $\sigma_R(X) = 1$.

PROOF. It is easily seen that, by Lemma (5.1),

$$\langle Rx, x\rangle = \tau_{\mathrm{cl}}^2(\langle x, X\rangle)\,.$$

Since $\mathrm{span}(\mathrm{Im}\, X) = H$, the operator R is definite positive. It is also a trace class operator since

$$\mathrm{tr}(R) = \sum_n \langle Re_n, e_n\rangle = \tau_R^2(X) < +\infty \,.$$

Then

$$E[e^{\langle x, X\rangle}] \leqslant \exp\left(\frac{1}{2}\tau_{\mathrm{cl}}^2(\langle x, X\rangle)\right) = \exp\left(\frac{1}{2}\langle Rx, x\rangle\right).$$

The infimum property of $\tau_{\mathrm{cl}}(\langle x, X\rangle)$ gives the last statement of the proposition. ■

Appendix. Proof of Theorem (2.3).

If x and y are two elements of B_2^{N}, then

$$\sum_n \nu^2(x_n + y_n) \leqslant \sum_n (\nu(x_n) + \nu(y_n))^2 \leqslant 2\left[\sum_n \nu^2(x_n) + \sum_n \nu^2(y_n)\right],$$

so that $x + y \in B_2^{\mathrm{N}}$.

It is immediate to see that, for every $\lambda \in \mathbb{R}$, $\lambda x \in B_2^{\mathrm{N}}$ if $x \in B_2^{\mathrm{N}}$.

Let's now see that ϱ is a norm in B_2^{N}. The only non trivial thing to check is the triangular inequality. We have

$$\begin{aligned}\varrho^2(x+y) &= \sum_n \nu^2(x_n + y_n) \leqslant \sum_n (\nu(x_n) + \nu(y_n))\,\nu(x_n + y_n) = \\ &= \sum_n \nu(x_n)\,\nu(x_n + y_n) + \sum_n \nu(y_n)\,\nu(x_n + y_n) \leqslant \\ &\leqslant \left(\sum_n \nu^2(x_n)\right)^{1/2}\left(\sum_n \nu^2(x_n + y_n)\right)^{1/2} + \\ &+ \left(\sum_n \nu^2(y_n)\right)^{1/2}\left(\sum_n \nu^2(x_n + y_n)\right)^{1/2} = \varrho(x)\,\varrho(x+y) + \varrho(y)\,\varrho(x+y),\end{aligned}$$

(where the first $\leqslant$ is due to the triangular inequality for ν and the second $\leqslant$ to the Schwartz inequality).

We now prove that ϱ is a Banach norm.

Let $(x^{(p)})_p$ be a Cauchy sequence in B_2^{N}. This means that, for every $\varepsilon > 0$, there exists p_0 such that, for every $p, q > p_0$, we have

$$\varrho^2(x^{(p)} - x^{(q)}) = \sum_n \nu(x_n^{(p)} - x_n^{(q)})^2 < \varepsilon, \tag{A.1}$$

and it is easy to see that the series $\sum_n \nu(x_n^{(p)} - x_n^{(q)})^2$ converges uniformly in p, q.

The inequality

$$\nu(x_n^{(p)} - x_n^{(q)})^2 \leqslant \varrho^2(x^{(p)} - x^{(q)}) < \varepsilon,$$

valid for $p, q > p_0$, implies that, for each n, $(x_n^{(p)})$ is a Cauchy sequence in B, hence converges in B (since B is Banach). Let

$$y_n = \lim_p x_n^{(p)}, \qquad y = (y_1, y_2, \ldots).$$

Passing to the limit in (A.1) with respect to q, we get, for $p > p_0$,

$$\lim_q \sum_n \nu(x_n^{(p)} - x_n^{(q)})^2 = \sum_n \lim_q \nu(x_n^{(p)} - x_n^{(q)})^2 = \sum_n \nu(x_n^{(p)} - y_n)^2 \leqslant \varepsilon \tag{A.2}$$

(where the first equality is due to the uniform convergence of the series with respect to q, and the second to the continuity of the norm ν).

Hence, for $p > p_0$,

$$\sum_n \nu^2(y_n) = \sum_n \nu^2(y_n - x_n^{(p)} + x_n^{(p)}) \leqslant$$

$$\leqslant 2\left[\sum_n \nu(x_n^{(p)} - y_n)^2 + \sum_n \nu(x_n^{(p)})^2\right] \leqslant 2\varepsilon + 2\sum_n \nu(x_n^{(p)})^2 < +\infty,$$

that is, $y \in B_2^{\mathrm{N}}$. Finally, relation (A.2) may be rephrased as «$(x^{(p)})_p$ converges to y in norm as $p \to \infty$». ■

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