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Annalisa Malusa

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Harmonic Measures of Perforated Domains.

ANNALISA MALUSA (*)

ABSTRACT - Let L be a linear elliptic operator of the second order with bounded measurable coefficients on a bounded open subset Ω of $\boldsymbol{R}^N$ with smooth boundary, and let $\{\Omega_n\}$ be an arbitrary sequence of open subsets of Ω. For every $n \in N$ and for every $x \in \Omega_n$, let $\mathcal{H}_n(x, \cdot)$ be the harmonic measure of Ω_n at the point x. We consider the extension of the family $\{\mathcal{H}_n(x, \cdot)\}_{x \in \Omega_n}$ obtained by setting $\mathcal{H}_n(x, \cdot) = \delta_x$ for every $x \in \Omega \backslash \Omega_n$, where δ_x is the Dirac mass at the point x. We prove that there exist a subsequence, still denoted by $\{\Omega_n\}$, and a positive Borel measure μ not charging polar sets, such that for almost every $x \in \Omega$ the sequence $\{\mathcal{H}_n(x, \cdot)\}_{n \in N}$ converges in the weak* topology of measures in $\overline{\Omega}$ to a measure $\mathcal{H}_\mu(x, \cdot)$ which is characterized as the unique probability measure in $\overline{\Omega}$ such that for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ the function

$$u(x) = \int_{\overline{\Omega}} g(y) \mathcal{H}_\mu(x, dy)$$

coincides almost everywhere in Ω with the solution to the problem

$$\begin{cases} u - g \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu, \varphi \rangle + \displaystyle\int_\Omega (u - g)\varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

(*) Indirizzo dell'A: Ist. di Matematica, Fac. di Architettura, Via Monteoliveto 3, 80134 Napoli.

1. - Introduction.

Let $Lu = -\sum_{i,j=1}^{N} D_i(a_{ij}D_j u)$ be a uniformly elliptic operator with bounded coefficients on a bounded open subset Ω of $\mathbf{R}^N$, $N \geq 2$, with smooth boundary. We recall that for every $g \in H^1(\Omega)$ there exists a unique solution u to the problem

$$\begin{cases} u - g \in H_0^1(\Omega), \\ \langle Lu, \varphi \rangle = 0, \qquad \forall \varphi \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

and, by De Giorgi-Nash Theorem, the solution u is also locally Hölder continuous in Ω. The notion of solution of a Dirichlet problem with a boundary datum $g \in C(\partial\Omega)$ is usually given in terms of harmonic measures in the following way. One introduces the linear functionals $\{H(\cdot)(x)\}_{x \in \Omega}$ defined in $H^1(\Omega) \cap C(\partial\Omega)$ and with values in $\mathbf{R}$, which associate to every $g \in H^1(\Omega) \cap C(\partial\Omega)$ the value $H(g)(x) = u(x)$ of the solution of (1.1) at x. By the maximum principle, each $H(\cdot)(x)$ turns out to be a bounded functional defined on a dense subspace of $C(\partial\Omega)$. Thus we can extend $H(\cdot)(x)$ to a functional, still denoted by $H(\cdot)(x)$, which is linear and bounded in $C(\partial\Omega)$ endowed with the uniform norm. Then, by the Riesz representation theorem, there exists a family of nonnegative Borel measures $\{\mathcal{H}(x, \cdot)\}_{x \in \Omega}$ carried by $\partial\Omega$, such that

$$H(g)(x) = \int_{\partial\Omega} g(y)\,\mathcal{H}(x, dy),$$

for every $x \in \Omega$ and for every $g \in C(\partial\Omega)$. The measures $\{\mathcal{H}(x, \cdot)\}_{x \in \Omega}$ are called harmonic measures of Ω (associated to the operator L), and $H(g)$ coincides with the Perron-Wiener-Brelot solution of the boundary value problem corresponding to the datum $g \in C(\partial\Omega)$ (see [12], Section 6.3, for more details on this subject, and [8], [13] for applications to the potential theory).

The aim of this paper is to describe the asymptotic behaviour of solutions of Dirichlet problems with inhomogeneous boundary conditions in perforated domains. More precisely, given an arbitrary sequence $\{\Omega_n\}$ of open subsets of Ω, we want to investigate the behaviour of the sequence $\{u_n\}$ of the solutions to the problems

$$\begin{cases} u_n - g \in H_0^1(\Omega_n), \\ \langle Lu_n, \varphi \rangle = 0, \qquad \forall \varphi \in H_0^1(\Omega_n), \end{cases} \tag{1.2}$$

corresponding to a datum $g \in H^1(\Omega)$.

Moreover, we are interested in the asymptotic behaviour of the harmonic measures $\{\mathcal{H}_n(x,\cdot)\}_{x\in\Omega_n}$ carried by $\partial\Omega_n$, as n goes to ∞, in order to describe the limit of the functionals

$$H_n(g)(x) = \int_{\partial\Omega_n} g(y)\,\mathcal{H}_n(x, dy),$$

defined in $C(\overline{\Omega})$.

Among the motivations for this subjects, we mention the application to the study of physical phenomena in domains with a complicated boundary, and the applications in the framework of the shape optimization theory (see, e.g., [4] and the references therein).

We prove that there exist a subsequence, still denoted by $\{\Omega_n\}$, and a positive Borel measure μ not charging sets of capacity zero, such that for every $g \in H^1(\Omega)$ the solutions u_n of (1.2), extended to Ω by setting $u_n = g$ in $\Omega\backslash\Omega_n$, converge to the solution u to the problem

$$(1.3)\quad \begin{cases} u - g \in H_0^1(\Omega) \cap L^2(\Omega,\mu), \\ \langle Lu, \varphi\rangle + \displaystyle\int_\Omega (u-g)\varphi\, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega)\cap L^2(\Omega,\mu), \end{cases}$$

which we call inhomogeneous relaxed Dirichlet problem corresponding to μ and with datum g. The technique used in order to obtain this result relies on the theory of relaxed Dirichlet problems, developed in [6], [7] and [4] for the study of the asymptotic behaviour of Dirichlet problems with homogeneous boundary conditions. Moreover we prove that if we extend the family of the harmonic measures $\{\mathcal{H}_n(x,\cdot)\}_{x\in\Omega_n}$ by

$$\widetilde{\mathcal{H}}_n(x,\cdot) = \begin{cases} \mathcal{H}_n(x,\cdot) & \text{if } x \in \Omega_n, \\ \delta_x & \text{otherwise,} \end{cases}$$

where δ_x is the Dirac mass at x, then for almost every $x \in \Omega$ the sequence $\{\widetilde{\mathcal{H}}_n(x,\cdot)\}_{n\in N}$ converges in the weak* topology of measures in $\overline{\Omega}$ to a probability measure $\mathcal{H}_\mu(x,\cdot)$. Such a measure is called μ-harmonic measure and it is characterized as the unique probability measure such that for every $g \in H^1(\Omega)\cap C(\overline{\Omega})$ the solution u of (1.3) can be written as

$$u(x) = \int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy).$$

Since for every open subset Ω' of Ω there exists a measure μ' not charging sets of capacity zero such that the (unique) solution u of (1.3) corresponding to μ' coincides with the solution to the problem

$$\begin{cases} u - g \in H_0^1(\Omega'), \\ \langle Lu, \varphi \rangle = 0, \qquad \forall \varphi \in H_0^1(\Omega'), \end{cases} \tag{1.4}$$

prolonged to g in $\Omega \backslash \Omega'$, then the class of inhomogeneous relaxed Dirichlet problems contains all the inhomogeneous Dirichlet problems defined in subdomains of Ω. Moreover, if we extend the family $\{\mathcal{H}(x, \cdot)\}_{x \in \Omega'}$ of harmonic measures of Ω' by setting $\mathcal{H}(x, \cdot) = \delta_x$ for every $x \in \Omega \backslash \Omega'$, then there exists a subset N of Ω with capacity zero such that for every $x \in \Omega \backslash N$ the measure $\mathcal{H}_{\mu'}(x, \cdot)$ coincides with $\mathcal{H}(x, \cdot)$.

On the other hand, every nonnegative Borel measure μ vanishing on sets of capacity zero can appear in the limit problem (1.3) for a suitable choice of the sequence $\{\Omega_n\}$.

For a general measure μ, problem (1.3) is not equivalent to a problem of the form (1.4) and the μ-harmonic measures $\mathcal{H}_\mu(x, \cdot)$ cannot be written in terms of classical harmonic measures. For instance, if μ is the Lebesgue measure, then problem (1.3) reduces to

$$\begin{cases} u - g \in H_0^1(\Omega), \\ \langle Lu, \varphi \rangle + \int_\Omega u\varphi \, dx = \int_\Omega g\varphi \, dx, \qquad \forall \varphi \in H_0^1(\Omega), \end{cases}$$

so that the μ-harmonic measure also charges the interior of Ω, and

$$\int_{\overline{\Omega}} g(y)\, \mathcal{H}_\mu(x, dy) = \int_\Omega G(x, y) g(y)\, dy + \int_{\partial\Omega} g(y)\, \mathcal{H}(x, dy),$$

where G is the Green function associated to the operator $Lu + u$ with homogeneous boundary conditions in Ω, and $\mathcal{H}(x, \cdot)$ is the harmonic measure in Ω relative to the same operator.

Since $\mathcal{H}_\mu(x, \cdot)$ is a probability measure, then for every $g \in C(\overline{\Omega})$ we can consider the function $H_\mu(g)$ defined by

$$H_\mu(g)(x) = \int_{\overline{\Omega}} g(y)\, \mathcal{H}_\mu(x, dy)$$

for every $x \in \Omega$. We prove that, if μ is a finite measure, then $H_\mu(g)$ is a local solution of the inhomogeneous relaxed Dirichlet problem corresponding to μ, and for every $\mu \in \mathcal{M}_0(\Omega)$ we consider $H_\mu(g)$ as a generalized solution of problem (1.3).

As a direct consequence of the compactness result for the harmonic measures, we obtain that for every sequence $\{\Omega_n\}$ of open subsets of Ω, there exists a subsequence, still denoted by $\{\Omega_n\}$, and a measure μ not charging polar sets such that for every $g \in C(\overline{\Omega})$ the generalized solution $u_n(x) = \int_{\overline{\Omega}} g(y)\,\mathcal{H}_n(x, dy)$ of the Dirichlet problem in Ω_n, extended to Ω by $u_n = g$ in $\Omega \backslash \Omega_n$, converges a.e. in Ω to the generalized solution $H_\mu(g)$ to the problem (1.3).

A question strictly related to the study of inhomogeneous Dirichlet problems is whether the solution corresponding to a continuous boundary datum attains its boundary value continuously at a fixed boundary point x_0. The Wiener criterion, proved in [18] for the Laplace operator and generalized in [14] for elliptic operators in divergence form with bounded coefficients, stated that the regularity of a local solution at a certain point $x_0 \in \partial\Omega$ is related to some geometric properties of $\partial\Omega$ near x_0, detected by the so called Wiener modulus. In [6] and [7] a notion of Wiener point with respect to a measure μ was introduced, in order to study the pointwise behaviour of local solutions of relaxed Dirichlet problems near the «irregular boundaries» inside Ω that the presence of the measure μ may produce. Also in this case the regularity at a certain point $x_0 \in \Omega$ is equivalent to the property of vanishing of a Wiener modulus associated to the measure μ.

We prove that $x_0 \in \Omega$ is a Wiener point for the measure μ, in the sense given in [6], if and only if

$$\lim_{x \to x_0} \mathcal{H}_\mu(x, \cdot) = \delta_{x_0},$$

where the limit above is taken in the weak* topology of measures in $\overline{\Omega}$. Thus we obtain a pointwise regularity of all generalized solutions $H_\mu(g)$, at a Wiener point $x_0 \in \Omega$ for the measure μ.

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2. - Preliminaries.

Sobolev spaces and capacity. Throughout this paper Ω will be a bounded open subset of $\boldsymbol{R}^N$, $N \geq 2$, and $B_r(x)$ will be the open ball of center $x \in \boldsymbol{R}^N$ and radius r. For every Borel set B, we shall denote by $\overline{B}$ the closure of B in the Euclidean topology of $\boldsymbol{R}^N$.

We shall denote by $L^q(\Omega, \mu)$ and $L^q_{\mathrm{loc}}(\Omega, \mu)$ $1 \leq q \leq +\infty$, the usual Lebesgue spaces with respect to a Borel measure μ. If $\mu = \mathcal{L}$ is the Lebesgue measure on $\boldsymbol{R}^N$, we shall use the standard notations $L^q(\Omega)$,

$L^q_{\text{loc}}(\Omega)$, and $\mathcal{L}(B) = |B|$ for every Borel set B. We shall denote by $H^1(\Omega)$ and $H^1_0(\Omega)$ the usual Sobolev spaces, and by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$. The duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle$. By $H^1_{\text{loc}}(\Omega)$ we shall denote the set of all functions $u \in L^2_{\text{loc}}(\Omega)$ such that $u \in H^1(\Omega')$ for every open set Ω' compactly contained in Ω ($\Omega' \subset\subset \Omega$).

If Ω is compactly contained in an open set Ω', and $u \in H^1_0(\Omega)$, then we can extend u to Ω' by setting $u = 0$ in $\Omega' \setminus \Omega$. We shall always identify u with this extension, which is an element of $H^1_0(\Omega')$.

For every subset E of Ω, the (harmonic) capacity of E with respect to Ω is defined by

$$\operatorname{cap}(E, \Omega) = \inf \int_\Omega |Du|^2 \, dx$$

where the infimum is taken over all the functions $u \in C^\infty_0(\Omega)$ such that $u \geq 1$ in a neighborhood of E.

It is well known that $\operatorname{cap}(\cdot, \Omega)$ is a monotone nondecreasing, subadditive set function, and that, if Ω' is an open set containing Ω, then for every Borel set $E \subseteq \Omega$ we have that $\operatorname{cap}(E, \Omega) = 0$ if and only if $\operatorname{cap}(E, \Omega') = 0$ (see, e.g., [10]).

We say that a property $\mathcal{P}(x)$ holds quasi everywhere (q.e.) in Ω if there exists a subset E of Ω with capacity zero such that $\mathcal{P}(x)$ holds for every x in $\Omega \setminus E$. The expression «almost everywhere» (a.e.) refers, as usual, to the analogous property for the Lebesgue measure.

A function $u\colon \Omega \to \mathbf{R}$ is said to be quasi continuous if for every $\varepsilon > 0$ there exists a set $E \subseteq \Omega$, with $\operatorname{cap}(E, \Omega) \leq \varepsilon$ such that the restriction of u to $\Omega \setminus E$ is continuous.

We say that a sequence $\{u_n\}$ converges uniformly q.e. in Ω to u, if there exists a set N with $\operatorname{cap}(N, \Omega) = 0$, such that for every $\varepsilon > 0$

$$|u_n(x) - u(x)| \leq \varepsilon,$$

for every $x \in \Omega \setminus N$ and for n large enough. It is easy to see that, if $\{u_n\}$ is a sequence of quasi continuous functions such that there exists a set N with $\operatorname{cap}(N, \Omega) = 0$ and such that for every $\varepsilon > 0$

$$|u_n(x) - u_k(x)| \leq \varepsilon,$$

for every $x \in \Omega \setminus N$ and for n, k large enough, then $\{u_n\}$ converges uniformly q.e. in Ω to a quasi continuous function.

We recall that, if u belongs to $H^1_{\mathrm{loc}}(\Omega)$, the limit of the averages

$$\tilde{u}(x) = \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy\,,$$

exists and is finite for quasi every $x \in \Omega$ (see, e.g., [19]). Moreover $\tilde{u}$ is a quasi continuous function in Ω.

We make the following convention about the pointwise values of a function u in $H^1_{\mathrm{loc}}(\Omega)$: for every $x \in \Omega$ we always require that

$$\liminf_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\,dy \leq u(x) \leq \limsup_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)dy\,.$$

With this convention, the quasi continuous representative $\tilde{u}$ defines u up to sets of capacity zero.

Now we give some properties of the quasi continuous representative we shall use in the following.

PROPOSITION 2.1. *Let $\{u_n\}$ be a sequence of functions in $H^1(\Omega)$ that converges strongly to u in the same space. Then there exists a subsequence $\{u_{n_k}\}$ that converges to u q.e. in Ω. Moreover, if u and v are two functions in $H^1(\Omega)$ such that $u \leq v$ a.e. in Ω, then $u \leq v$ q.e. in Ω.*

PROOF. See [10], Theorem 2.1. ■

Measures. By a Borel measure on Ω we mean a nonnegative, countably additive set function defined on the σ-field $\mathcal{B}(\Omega)$ of all Borel subsets of Ω. By a Radon measure on Ω we mean a Borel measure which is finite on every compact subset of Ω. If μ is a Borel measure and $E \in \mathcal{B}(\Omega)$, the Borel measure $\mu \llcorner E$ is defined by $(\mu \llcorner E)(B) = \mu(E \cap B)$ for every set $B \in \mathcal{B}(\Omega)$. We shall denote by $\mathrm{supp}\,(\mu)$ the support of the measure μ, that is the smallest closed set in Ω, whose complement has measure zero.

The Dirac mass at a point $x \in \Omega$ will be denoted by δ_x. Finally, $\mathbf{1}_B$ will be the characteristic function of the set B, which is defined by

$$\mathbf{1}_B(x) = \begin{cases} 1\,, & \text{if } x \in B, \\ 0\,, & \text{if } x \notin B\,. \end{cases}$$

We shall consider the following notions of convergence of measures.

DEFINITION 2.2. *We say that a sequence* $\{\nu_n\}$ *of finite measures in* Ω *converges weakly* to the measure* ν *in* Ω, *if* $\int_\Omega f d\nu_n$ *converges to* $\int_\Omega f d\nu$ *for every continuous function* f *with compact support in* Ω. *If* ν_n *and* ν *are finite measures defined in* $\overline{\Omega}$, *we say that* $\{\nu_n\}$ *converges weakly* in* $\overline{\Omega}$ *to* ν, *if* $\int_{\overline{\Omega}} f d\nu_n$ *converges to* $\int_{\overline{\Omega}} f d\nu$ *for every function* $f \in C(\overline{\Omega})$.

A subset A of Ω is said to be quasi open if for every $\varepsilon > 0$ there exists an open set $U_\varepsilon \subseteq \Omega$, with $\operatorname{cap}(U_\varepsilon, \Omega) \leq \varepsilon$, such that $A \cup U_\varepsilon$ is an open set.

Using the notion of capacity, we can define a class of Borel measures.

DEFINITION 2.3. *We denote by* $\mathcal{M}_0(\Omega)$ *the set of all nonnegative Borel measures* μ *on* Ω *such that*

(i) $\mu(B) = 0$ *for every Borel set* $B \subseteq \Omega$ *with* $\operatorname{cap}(B, \Omega) = 0$;

(ii) $\mu(B) = \inf\{\mu(A)$: *A quasi open,* $B \subseteq A\}$.

For every subset E of Ω we shall denote by ∞_E the measure in $\mathcal{M}_0(\Omega)$ defined by

$$\infty_E(B) = \begin{cases} 0, & \text{if } \operatorname{cap}(B \cap E, \Omega) = 0, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.1}$$

for every Borel set $B \subseteq \Omega$.

It can be easily seen that a function u belongs to $H_0^1(\Omega) \cap \cap L^2(\Omega, \infty_E)$ if and only if $u \in H_0^1(\Omega \backslash E)$ and $u = 0$ q.e. in Ω.

Relaxed Dirichlet problems. Let $(a_{ij})_{i,j=1,\dots,N}$, with $a_{ij}: \boldsymbol{R}^N \to \boldsymbol{R}$, be a matrix with measurable coefficients such that

$$\sum_{i,j=1}^N a_{ij}(x)\xi_i \cdot \xi_j \geq \lambda\, |\xi|^2 \quad \text{for almost every } x \in \boldsymbol{R}^N, \ \forall \xi \in \boldsymbol{R}^N, \tag{2.2}$$

and

$$\|a_{ij}\|_{L^\infty(\boldsymbol{R}^N)} \leq \Lambda \tag{2.3}$$

for some constants $0 < \lambda \leq \Lambda$. Let L: $H_0^1(\Omega) \to H^{-1}(\Omega)$ be the operator defined by $Lu = -\sum_{i,j=1}^N D_i(a_{ij} D_j u)$; thanks to the hypotheses on a_{ij}, L is a uniformly elliptic and bounded operator.

We give the definition of relaxed Dirichlet problems for the operator L as it was given in [6].

DEFINITION 2.4. *Let μ be a measure in $\mathcal{M}_0(\Omega)$. We say that a function v is a local solution of the relaxed Dirichlet problem associated to μ and with right-hand side $f \in H^{-1}(\Omega)$ if $v \in H^1_{\text{loc}}(\Omega) \cap L^2_{\text{loc}}(\Omega, \mu)$, and*

$$\langle Lv, \varphi \rangle + \int_\Omega v\varphi \, d\mu = \langle f, \varphi \rangle, \tag{2.4}$$

for every $\varphi \in H^1(\Omega) \cap L^2(\Omega, \mu)$ with compact support in Ω.

We say that v is a solution of the relaxed Dirichlet problem associated to μ and with right-hand side $f \in H^{-1}(\Omega)$, if v satisfies

$$\begin{cases} v \in H^1_0(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lv, \varphi \rangle + \int_\Omega v\varphi \, d\mu = \langle f, \varphi \rangle, \end{cases} \tag{2.5}$$

for every $\varphi \in H^1_0(\Omega) \cap L^2(\Omega, \mu)$.

THEOREM 2.5. *Suppose that $f \in H^{-1}(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$. Then there exists a unique solution v of the problem* (2.5), *and we have the estimate*

$$\|v\|_{H^1_0(\Omega)} + \|v\|_{L^2(\Omega,\mu)} \leq c\|f\|_{H^{-1}(\Omega)}, \tag{2.6}$$

for some positive constant c depending only on N, λ, and Λ.

PROOF. See [6], Theorem 2.4. ∎

We recall some results about solutions of relaxed Dirichlet problems we shall use in the following sections.

PROPOSITION 2.6. *Let ν be a nonnegative Radon measure belonging to $H^{-1}(\Omega)$, and let v be a local solution of* (2.4) *with right-hand side ν. Then we have*

$$\langle Lv, \varphi \rangle \leq \langle \nu, \varphi \rangle,$$

for every $\varphi \in H^1_0(\Omega)$ with $\varphi \geq 0$ a.e. in Ω.

PROOF. See Proposition 2.6 in [6]. ∎

THEOREM 2.7. *Let $\mu \in \mathcal{M}_0(\Omega)$, $f \in L^\infty(\Omega)$, and v be the solution to the problem* (2.5). *Then v belongs to $L^\infty(\Omega)$, and there exists a constant* *c, depending only on N, λ, Λ, and Ω, such that*

$$\|v\|_{L^\infty(\Omega)} \leq c\|f\|_{L^\infty(\Omega)}\,.$$

Moreover v admits a pointwise value in Ω, which coincides with the limit of its averages. If $f \geq 0$, then such a representative is given by

$$v(x) = v_0(x) - \int_\Omega G(x, y)d\gamma(y)\,,$$

where v_0 is the (continuous) solution of

$$\begin{cases} v_0 \in H_0^1(\Omega), \\ \langle Lv_0, \varphi\rangle = \displaystyle\int_\Omega f\varphi\, dx\,, \qquad \forall\varphi \in H_0^1(\Omega)\,, \end{cases}$$

γ is a suitable nonnegative measure belonging to $H^{-1}(\Omega)$, and $G(\cdot, \cdot)$ is the Green function associated to the operator L, and with homogeneous boundary conditions on Ω.

PROOF. See [16], Theorems 3.3, and 5.2. ■

THEOREM 2.8. *Let μ_1 and μ_2 be in $\mathcal{M}_0(\Omega)$, with $\mu_1 \leq \mu_2$. Let f_1 and f_2 be in $H^{-1}(\Omega)$, with $0 \leq f_2 \leq f_1$. Let v_1 and v_2 be the solutions of*

$$\begin{cases} v_i \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lv_i, \varphi\rangle + \displaystyle\int_\Omega v_i\, \varphi\, d\mu_i = \langle f_i, \varphi\rangle, \end{cases} \qquad i = 1, 2\,,$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$. Then $0 \leq v_2 \leq v_1$ almost everywhere in Ω.

PROOF. See [6], Theorem 2.10. ■

For every $\mu \in \mathcal{M}_0(\Omega)$ we shall denote by w_μ the unique solution of the problem

$$\begin{cases} w_\mu \in H_0^1(\Omega) \cap L^2(\Omega, \mu)\,, \\ \langle Lw_\mu, \varphi\rangle + \displaystyle\int_\Omega w_\mu \varphi\, d\mu = \int_\Omega \varphi\, dx, \end{cases} \tag{2.7}$$

The properties of w_μ that we need in the sequel are listed in the following proposition.

PROPOSITION 2.9. *Let w_μ be the solution of problem* (2.7). *Then the following properties hold:*

(*a*) *w_μ has a pointwise value given by the limit of its averages at each point $x \in \Omega$, and this representative is a upper semicontinuous function;*

(*b*) *there exists a constant $c > 0$, depending only on λ, Λ, N, and Ω, such that $0 \leqslant w_\mu \leqslant c$ in Ω;*

(*c*) *if $f \in L^\infty(\Omega)$, and v is the solution of problem* (2.5), *then $|v| \leqslant \leqslant \|f\|_{L^\infty(\Omega)} w_\mu$ in Ω;*

(*d*) *if μ, μ_0 belong to $\mathcal{M}_0(\Omega)$, then $\mu = \mu_0$ if and only if $w_\mu = w_{\mu_0}$ in Ω.*

PROOF. By Theorem 2.7 there exist a continuous function w_0 and a nonnegative measure $\gamma \in H^{-1}(\Omega)$ such that

$$w_\mu(x) = w_0(x) - \int_\Omega G(x, y)\, d\gamma(y),$$

for every $x \in \Omega$. As an easy consequence of the Fatou lemma and of the fact that the Green function is positive and continuous, we have that $\int_\Omega G(x, y)\, d\gamma(y)$ is a lower semicontinuous function. Thus (*a*) is proved. The fact that w_μ is nonnegative is a direct consequence of Theorem 2.8. Moreover, if we apply again Theorem 2.8 to $f_1 = f_2 = 1$, $\mu_1 = 0$, $\mu_2 = \mu$, and the regularity results for classical Dirichlet problems (see [17], Théorème 4.2), we obtain that there exists a constant c, depending only on λ, Λ, N and Ω such that $w_\mu \leqslant c$ in Ω. Property (*c*) is another consequence of Theorem 2.8, applied to $f_2 = f$, and $f_1 = \|f\|_{L^\infty(\Omega)}$, and of Theorem 2.7. Finally property (*d*) was proved in [4], Proposition 3.4. ■

The main tool for the study of the asymptotic behaviour of Dirichlet problems in perforated domains is the following notion of convergence in $\mathcal{M}_0(\Omega)$.

DEFINITION 2.10. *Let $\{\mu_n\}$ be a sequence of measures of $\mathcal{M}_0(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. We say that $\{\mu_n\}$ γ^L-converges to μ (in Ω) if the sequence $\{v_n\}$ of the solutions to the problems*

$$\begin{cases} v_n \in H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega), \\ \langle L v_n, \varphi\rangle + \int_\Omega v_n \varphi\, d\mu_n = \langle f, \varphi\rangle, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2_{\mu_n}(\Omega), \end{cases}$$

converges weakly in $H_0^1(\Omega)$ *to the solution* v *of the problem*

$$\begin{cases} v \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lv, \varphi \rangle + \int_\Omega v \varphi \, d\mu = \langle f, \varphi \rangle, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

for every $f \in H^{-1}(\Omega)$.

This convergence of measures is the natural extension of the notion of γ^L-convergence introduced in [7], when L is the Laplace operator, and in [1] when L is symmetric.

Then main properties of γ^L-convergence are the following.

PROPOSITION 2.11. *Every sequence of measures of* $\mathcal{M}_0(\Omega)$ *contains a* γ^L*-convergent subsequence.*

PROOF. See [4], Theorem 4.5. ■

THEOREM 2.12. *Let* μ *be a measure in* $\mathcal{M}_0(\Omega)$. *Then there exists a sequence* $\{E_n\}$ *of closed subsets of* Ω *such that the sequence of measures* $\{\infty_{E_n}\}$ γ^L*-converges to* μ.

PROOF. See [4], Proposition 4.7, and [5], Theorem 6.2. ■

3. - Inhomogeneous relaxed Dirichlet problems.

Let $\{\Omega_n\}$ be a sequence of open subsets of Ω. Applying Proposition 2.11 to the sequence of measures $\{\infty_{\Omega \setminus \Omega_n}\}$, we obtain that there exists a subsequence $\{\Omega_{n_k}\}$ and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that for every $f \in H^{-1}(\Omega)$ the solutions v_{n_k} to the problems

$$\begin{cases} v_{n_k} \in H_0^1(\Omega_{n_k}), \\ \langle Lv_{n_k}, \varphi \rangle = \langle f, \varphi \rangle, \qquad \forall \varphi \in H_0^1(\Omega_{n_k}), \end{cases}$$

extended to Ω by setting $v_{n_k} = 0$ in $\Omega \setminus \Omega_{n_k}$, converge in the weak topology of $H_0^1(\Omega)$ to the solution v of the relaxed Dirichlet problem

$$\begin{cases} v \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lv, \varphi \rangle + \int_\Omega v \varphi \, d\mu = \langle f, \varphi \rangle, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases} \tag{3.1}$$

Moreover several examples show that the limit problem may not have the same form of the approximating ones, that is there exists no subset E of Ω such that $\mu = \infty_E$ (see, e.g., [2]). Thus the asymptotic behaviour of solutions of elliptic equations with homogeneous boundary conditions on oscillating domains is described by a relaxed Dirichlet problem.

Now we want to investigate the asymptotic behaviour of the solutions of inhomogeneous Dirichlet problems. More precisely, fixed a function $g \in H^1(\Omega)$, let us consider the solutions u_n to the problems

$$\begin{cases} u_n - g \in H_0^1(\Omega_n), \\ \langle Lu_n, \varphi \rangle = 0, \quad \forall \varphi \in H_0^1(\Omega_n). \end{cases}$$

Hence, the function $v_n = u_n - g$ solves the problem

$$\begin{cases} v_n \in H_0^1(\Omega_n), \\ \langle Lv_n, \varphi \rangle = -\langle Lg, \varphi \rangle, \qquad \forall \varphi \in H_0^1(\Omega_n). \end{cases} \tag{3.2}$$

The previous result about homogeneous Dirichlet problems with right-hand side in $H^{-1}(\Omega)$ implies that the sequence $\{v_{n_k}\}$ converges in the weak topology of $H_0^1(\Omega)$ to the solution v to the relaxed Dirichlet problem associated to μ and with right-hand side $f = -Lg$.

Thus, if we call $u = v + g$, we obtain that the sequence $\{u_{n_k}\}$ converges weakly in $H^1(\Omega)$ to the function u which is a solution to the problem

$$\begin{cases} u - g \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu, \varphi \rangle + \int_\Omega (u - g)\varphi \, d\mu = 0, \quad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases} \tag{3.3}$$

Hence this seems to be the class of problems needed for the study of the asymptotic behaviour of solutions of inhomogeneous Dirichlet problems in wildly oscillating domains.

DEFINITION 3.1. *Let $\mu \in \mathcal{M}_0(\Omega)$ and let g be a function in $H^1_{\mathrm{loc}}(\Omega)$. A function u is said to be a local solution of the inhomogeneous relaxed Dirichlet problem associated to μ and with datum g if $u - g \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu)$, and*

$$\langle Lu, \varphi \rangle + \int_\Omega (u - g)\varphi \, d\mu = 0\,, \tag{3.4}$$

for all $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ with compact support in Ω.

If $g \in H^1(\Omega)$, then a function u which solves (3.3) *is said to be a solution of the inhomogeneous relaxed Dirichlet problem associated to μ and with datum g.*

THEOREM 3.2. *For every $\mu \in \mathcal{M}_0(\Omega)$ and for every $g \in H^1(\Omega)$ there exists a unique solution u of* (3.3), *and there exists a positive constant c, depending only on Ω, Λ, λ, and N, such that*

$$\|u\|_{H^1(\Omega)} \leq c\|g\|_{H^1(\Omega)} . \tag{3.5}$$

Moreover, let h be a function in $H^1(\Omega)$ such that $g - h \in H_0^1(\Omega)$ and $g - h = 0$ q.e. in $\operatorname{supp}(\mu)$. *If z is the solution to the problem*

$$\begin{cases} z - h \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lz, \varphi\rangle + \displaystyle\int_\Omega (z - h)\varphi \, d\mu = 0 , \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases} \tag{3.6}$$

then $u = z$ q.e. in Ω.

PROOF. By Theorem 2.5 there exists a unique solution v to the problem

$$\begin{cases} v \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lv, \varphi\rangle + \displaystyle\int_\Omega v\varphi \, d\mu = -\langle Lg, \varphi\rangle , \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

and there exists a positive constant c, depending only on Ω, Λ, λ, and n such that

$$\|v\|_{H^1(\Omega)} \leq c\|g\|_{H^1(\Omega)} .$$

Thus $u = v + g$ is the unique solution of (3.3), and satisfies (3.5).

Finally, let h be a function satisfying the assumptions of the theorem, and let z be be the solution to (3.6). Taking the difference between the equations solved by u and z, we obtain

$$\begin{cases} u - z \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle L(u - z), \varphi\rangle + \displaystyle\int_\Omega (u - z)\varphi \, d\mu = 0 , \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

which, by the uniqueness, implies that $u = z$ q.e. in Ω. ∎

The first result about this class of problems is a stability theorem with respect to the γ^L-convergence.

THEOREM 3.3. *Let $\{\mu_n\}$ be a sequence of measures in $\mathcal{M}_0(\Omega)$ γ^L-converging to a measure μ. Let $\{g_n\}$ be a sequence in $H^1(\Omega)$ which converges strongly in $H^1(\Omega)$ to a function g. For every $n \in N$, let u_n be the solution to the problem*

$$(3.7) \quad \begin{cases} u_n - g_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu_n), \\ \langle Lu_n, \varphi\rangle + \int_\Omega (u_n - g_n)\varphi \, d\mu_n = 0, \quad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu_n), \end{cases}$$

and let u be the solution to the problem (3.3). Then $\{u_n\}$ converges to u weakly in $H^1(\Omega)$. Moreover, if $\mu_n = \mu$ for every $n \in N$, then u_n converge to u strongly in $H^1(\Omega)$.

PROOF. If we define $v_n = u_n - g_n$, then v_n solves

$$\begin{cases} v_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu_n), \\ \langle Lv_n, \varphi\rangle + \int_\Omega v_n \varphi \, d\mu_n = -\langle Lg_n, \varphi\rangle, \quad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu_n). \end{cases}$$

where $\{Lg_n\}$ converges strongly to Lg in $H^{-1}(\Omega)$. By Proposition 4.8 in [4], the sequence $\{v_n\}$ converges in the weak topology of $H_0^1(\Omega)$ to the solution v of the problem

$$(3.8) \quad \begin{cases} v \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lv, \varphi\rangle + \int_\Omega v\varphi \, d\mu = -\langle Lg, \varphi\rangle, \quad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

Thus the sequence $\{u_n\}$ converges weakly in $H^1(\Omega)$ to the function u solution of (3.3).

Finally, if $\mu_n = \mu$ for every $n \in N$, then, by the linearity of problem (3.3), $u_n - u$ is the solution of the inhomogeneous relaxed Dirichlet problem associated to μ, and with datum $g_n - g$. Thus, by the continuity estimate in Theorem 3.2, we have

$$\|u_n - u\|_{H^1(\Omega)} \le c\|g_n - g\|_{H^1(\Omega)},$$

which implies the strong convergence of u_n to u. ∎

REMARK 3.4. If $\mu = \infty_E$, E closed subset of Ω, then it can be easily seen that u is the solution of (3.3) if and only if $u = g$ q.e. on E and u is the solution of the classical Dirichlet problem

$$\begin{cases} u - g \in H_0^1(\Omega \backslash E), \\ \langle Lu, \varphi \rangle = 0, \qquad \forall \varphi \in H_0^1(\Omega \backslash E). \end{cases}$$

Then the class of inhomogeneous relaxed Dirichlet problems contains all the classical Dirichlet problems with inhomogeneous boundary conditions on open subsets of Ω. The following proposition, which is a direct consequence of the theory of homogeneous relaxed Dirichlet problems, states that the classical Dirichlet problems on subdomains of Ω are dense in the class of inhomogeneous relaxed Dirichlet problems with respect to the strong convergence in $L^2(\Omega)$ of the solutions.

PROPOSITION 3.5. *For every $\mu \in \mathcal{M}_0(\Omega)$, there exists a sequence $\{E_n\}$ of compact subsets of Ω, such that for every $g \in H^1(\Omega)$ the solutions u_n to the problems*

$$\begin{cases} u_n - g \in H_0^1(\Omega \backslash E_n), \\ \langle Lu_n, \varphi \rangle = 0, \qquad \forall \varphi \in H_0^1(\Omega \backslash E_n), \end{cases}$$

extended to Ω by setting $u_n = g$ q.e. in E_n, converge weakly in $H^1(\Omega)$ to the solution u of problem (3.3).

PROOF. The result follows from Theorem 2.12, once we notice that $v = u - g$ is the solution of (3.8). ■

We recall that the positive part of a function ψ is defined by

$$\psi^+(x) = \begin{cases} \psi(x), & \text{if } \psi(x) > 0, \\ 0, & \text{if } \psi(x) \leq 0, \end{cases}$$

and that for every $\psi \in H^1(\Omega)$, the positive part ψ^+ also belongs to $H^1(\Omega)$. The negative part ψ^- of ψ is defined as $(-\psi)^+$.

DEFINITION 3.6. *Let g and h be functions in $H^1(\Omega)$. We say that $g \leq h$ on $\partial\Omega$ if $(g-h)^+ \in H_0^1(\Omega)$.*

If $g \in H^1(\Omega)$, and $E \subseteq \Omega$, the previous definition allows us to introduce the quantities

$$\operatorname*{ess\,sup}_{E \cup \partial\Omega} g = \inf\{M \in \mathbf{R}\colon\ g(x) \leq M \text{ q.e. in } E,\ (g-M)^+ \in H_0^1(\Omega)\},$$

and

$$\operatorname*{ess\,inf}_{E \cup \partial\Omega} g = \sup \{ m \in \boldsymbol{R} \colon g(x) \geq m \text{ q.e. in } E,\ (m-g)^+ \in H_0^1(\Omega) \}.$$

We are now in a position to state a maximum principle for solutions of inhomogeneous relaxed Dirichlet problems.

THEOREM 3.7. *Let $g \in H^1(\Omega)$ and let $\mu \in \mathcal{M}_0(\Omega)$. If u is the solution to the problem* (3.3), *then*

$$\operatorname*{ess\,inf}_{\operatorname{supp}(\mu) \cup \partial\Omega} g \leq u(x) \leq \operatorname*{ess\,sup}_{\operatorname{supp}(\mu) \cup \partial\Omega} g$$

for quasi every $x \in \Omega$.

PROOF. Let $M = \operatorname*{ess\,sup}_{\operatorname{supp}(\mu) \cup \partial\Omega} g$. It is not restrictive to assume that $M < +\infty$, so that we can introduce the function $\varphi = (u - M)^+$. Since $u - g \in H_0^1(\Omega)$, then $\varphi \in H_0^1(\Omega)$. Moreover

$$\int_\Omega \varphi^2 \, d\mu = \int_{\operatorname{supp}(\mu) \cap \{u \geq M\}} (u - M)^2 \, d\mu \leq \int_{\operatorname{supp}(\mu) \cap \{u \geq M\}} (u - g)^2 \, d\mu, \tag{3.9}$$

so that $\varphi \in L^2(\Omega, \mu)$, and we can choose it as test function in (3.3), obtaining

$$\int_{\{u \geq M\}} \sum_{i,j=1}^N a_{ij} D_j u \cdot D_i u \, dx + \int_{\operatorname{supp}(\mu) \cap \{u \geq M\}} (u - g)(u - M) \, d\mu = 0.$$

By (2.2) and the definition of M, we get

$$\lambda \int_{\{u \geq M\}} |Du|^2 \, dx + \int_{\{u \geq M\}} (u - M)^2 d\mu \leq 0$$

which implies that $u \leq M$ a.e. in Ω, and hence, by Proposition 2.1, q.e. in Ω.

Similarly, setting $m = \operatorname*{ess\,inf}_{\operatorname{supp}(\mu) \cup \partial\Omega} g$ and $\varphi = (m - u)^+$, we obtain that $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$, and

$$\lambda \int_{\{u \leq m\}} |Du|^2 dx + \int_{\operatorname{supp}(\mu) \cap \{u \leq m\}} (m - u)^2 d\mu \leq 0,$$

which implies $m \leq u$ q.e. in Ω. ∎

As a consequence of the maximum principle, we obtain a comparison principle between solutions of inhomogeneous relaxed Dirichlet problems.

COROLLARY 3.8. *Let g, $h \in H^1(\Omega)$, and let u, z be respectively the solution to* (3.3) *and to the problem*

$$\begin{cases} z - h \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lz, \varphi \rangle + \int\limits_\Omega (z-h)\varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

If $g \leq h$ a.e. in $\operatorname{supp}(\mu)$, *and $(g-h)^+ \in H_0^1(\Omega)$, then $u \leq z$ q.e. in Ω.*

PROOF. Thanks to the linearity of the problem, we have that $u - z$ is the solution of the inhomogeneous relaxed Dirichlet problem corresponding to μ and with datum $g - h$. Then, by the maximum principle,

$$u(x) - z(x) \leq \operatorname*{ess\,sup}_{\operatorname{supp}(\mu) \cup \partial\Omega} (g-h) \leq 0,$$

for quasi every $x \in \Omega$, which conclude the proof. ∎

Another consequence of the maximum principle which will be useful is the following stability result with respect to the uniform convergence of the data.

COROLLARY 3.9. *Let $\mu \in \mathcal{M}_0(\Omega)$, and let $\{g_n\}$ be a sequence of functions belonging to $H^1(\Omega) \cap C(\overline{\Omega})$. If $\{g_n\}$ converges uniformly to a function g in $\overline{\Omega}$, then the solutions u_n to the problems*

$$\begin{cases} u_n - g_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu_n, \varphi \rangle + \int\limits_\Omega (u_n - g_n)\varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

converge uniformly q.e. in Ω. If in addition $g \in H^1(\Omega)$, then the limit u of $\{u_n\}$ is the solution of the problem (3.3).

PROOF. By Theorem 3.7, for every $n, k \in N$ we have

$$|u_n(x) - u_k(x)| \leq \|g_n - g_k\|_{C(\overline{\Omega})}$$

for every $x \notin N(n,k)$, with $\operatorname{cap}(N(n,k), \Omega) = 0$. Thus $\{u_n\}$ converges uniformly q.e. in Ω to a quasi continuous function. If in addition g belongs to $H^1(\Omega)$, then, replacing g_k by g in the previous computation, we obtain that

$$|u_n(x) - u(x)| \leq \|g_n - g\|_{C(\overline{\Omega})}$$

for quasi every $x \in \Omega$, where u is the solution of problem (3.3), so that the limit of u_n coincides with u q.e. in Ω. ∎

REMARK 3.10. In the sequel it will be useful to assume that Ω has a smooth boundary. This is not restrictive to our purposes, since if Ω is not regular, we can consider an open set Ω' with smooth boundary such that $\Omega \subset\subset \Omega'$, and we can associate to every measure $\mu \in \mathcal{M}_0(\Omega)$ the measure $\mu' \in \mathcal{M}_0(\Omega')$ defined as $\mu' = \mu + \infty_{\Omega' \setminus \Omega}$. As a direct consequence of the fact that a function φ belongs to $H_0^1(\Omega') \cap L^2(\Omega', \mu')$ if and only if $\varphi = 0$ q.e. in $\Omega' \setminus \Omega$ and $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$, we obtain that for every $g \in H^1(\Omega')$ a function u is the solution to the problem

$$\begin{cases} u - g \in H_0^1(\Omega') \cap L^2_{\mu'}(\Omega'), \\ \langle Lu, \varphi \rangle + \int\limits_{\Omega'} (u-g)\varphi \, d\mu' = 0, \qquad \forall \varphi \in H_0^1(\Omega') \cap L^2_{\mu'}(\Omega'), \end{cases} \tag{3.10}$$

if and only if $u = g$ q.e. in $\Omega' \setminus \Omega$, and u is the solution to the problem (3.3).

Thus from now on we shall always assume that Ω has a smooth boundary, eventually making the previous reduction.

4. – Pointwise value of the solution and μ-harmonic measures.

The aim of this section is to introduce a notion of μ-harmonic measure which generalizes the classical harmonic measure of the potential theory.

We recall that the harmonic measure of a bounded open set Ω is the unique probability measure $\mathcal{H}(x, \cdot)$ such that for every $g \in C(\partial\Omega)$, the Perron-Wiener-Brelot solution $H(g)$ of the Dirichlet problem in Ω with boundary datum g can be represented as

$$H(g)(x) = \int\limits_{\partial\Omega} g(y)\, \mathcal{H}(x, dy) \tag{4.1}$$

for every $x \in \Omega$. We notice that each term in (4.1) is well defined, because the Perron-Wiener-Brelot solution $H(g)$ is, by construction, an harmonic (and thus continuous) function in Ω (see e.g., [8] and [13] for more details on the classical framework). The approach in the case of inhomogeneous relaxed Dirichlet problems will be quite different from the classical one, due to the lack of continuity of the solutions. We shall overcame this difficulty by proving that the solution u of an inhomogeneous relaxed Dirichlet problem corresponding to a datum $g \in H^1(\Omega) \cap C(\overline{\Omega})$ can be defined pointwise as the limit of its averages at each point $x \in \Omega$. To this aim we need the following density result.

LEMMA 4.1. *The class of all functions $g \in H^1(\Omega) \cap C(\overline{\Omega})$ with $Lg \in L^\infty(\Omega)$ is dense in $C(\overline{\Omega})$ with respect to the uniform norm.*

PROOF. It is enough to prove that the class of all functions $g \in H^1(\Omega) \cap C(\overline{\Omega})$ with $Lg \in L^\infty(\Omega)$ is dense in $C^1(\overline{\Omega})$ with respect to the uniform norm. Let us fix f belonging to $C^1(\overline{\Omega})$. Since $Lf \in H^{-1,p}(\Omega)$, with $p > N$, there exists a sequence $\{\psi_n\}$ of functions in $C_0^\infty(\Omega)$ which converges to Lf in the strong topology of $H^{-1,p}(\Omega)$. Let g_n be the solution of the classical Dirichlet problem

$$\text{(4.2)} \qquad \begin{cases} g_n - f \in H_0^1(\Omega), \\ \langle Lg_n, \varphi \rangle = \int_\Omega \psi_n \varphi \, dx, \qquad \forall \varphi \in H_0^1(\Omega). \end{cases}$$

Since Ω has a smooth boundary, by Theorem 7.3 in [17], the sequence $\{g_n\}$ is uniformly Hölder continuous in $\overline{\Omega}$. Thus, by Ascoli-Arzelà compactness theorem, there exists a subsequence, still denoted by $\{g_n\}$, and a continuous function g such that g_n converge uniformly to g in $\overline{\Omega}$. On the other hand, standard arguments applied to (4.2) assure that $\{g_n\}$ is also equibounded in $H^1(\Omega)$, so that, taking a further subsequence, we obtain that $\{g_n\}$ converges to g in the weak topology of $H^1(\Omega)$. Finally, taking the limit in (4.2) we obtain that $g = f$ q.e. in Ω. ■

THEOREM 4.2. *Let g be a function in $H^1(\Omega) \cap C(\overline{\Omega})$. For every $\mu \in \mathcal{M}_0(\Omega)$, let u be the solution of* (3.3). *Then there exists the limit*

$$\text{(4.3)} \qquad \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y) \, dy$$

for every $x \in \Omega$.

PROOF. If we assume, in addition, that $Lg \in L^\infty(\Omega)$, then the result follows from Theorem 2.7 applied to $v = u - g$.

Let us fix now $g \in H^1(\Omega) \cap C(\overline{\Omega})$. By Lemma 4.1, for every $\varepsilon > 0$ there exists $g_\varepsilon \in H^1(\Omega) \cap C(\overline{\Omega})$ with $Lg_\varepsilon \in L^\infty(\Omega)$ such that $\|g - g_\varepsilon\|_{C(\overline{\Omega})} \leq \varepsilon$. Let u_ε be the solution of problem (3.3) corresponding to the datum g_ε. Then, by Corollary 3.9, $\|u - u_\varepsilon\|_{L^\infty(\Omega)} \leq \varepsilon$ and, by by the previous step, for every $x \in \Omega$ there exists ϱ_0 such that

$$\left| \frac{1}{|B_\varrho(x)|} \int_{B_\varrho(x)} u_\varepsilon(y)\,dy - \frac{1}{|B_{\varrho'}(x)|} \int_{B_{\varrho'}(x)} u_\varepsilon(y)\,dy \right| \leqslant \varepsilon,$$

for every ϱ, $\varrho' < \varrho_0$. Thus we have

$$\left| \frac{1}{|B_\varrho(x)|} \int_{B_\varrho(x)} u(y)\,dy - \frac{1}{|B_{\varrho'}(x)|} \int_{B_{\varrho'}(x)} u(y)\,dy \right| \leqslant$$

$$\leqslant \left| \frac{1}{|B_\varrho(x)|} \int_{B_\varrho(x)} u(y)\,dy - \frac{1}{|B_\varrho(x)|} \int_{B_\varrho(x)} u_\varepsilon(y)\,dy \right| +$$

$$\leqslant \left| \frac{1}{|B_{\varrho'}(x)|} \int_{B_{\varrho'}(x)} u(y)\,dy - \frac{1}{|B_{\varrho'}(x)|} \int_{B_{\varrho'}(x)} u_\varepsilon(y)\,dy \right| + \varepsilon \leqslant 3\varepsilon$$

so that the sequence of the averages of u at a point x is a Cauchy sequence with respect to ϱ, for every $x \in \Omega$, and this implies the result. ■

COROLLARY 4.3. *For every $g \in H^1(\Omega) \cap C(\overline{\Omega})$, if we define pointwise the solution u of the problem* (3.3) *as the limit of its averages* (4.3), *then*

$$|u(x)| \leqslant \sup_{y \in \overline{\Omega}} |g(y)|\,, \tag{4.4}$$

for every $x \in \Omega$.

PROOF. By the maximum principle, (4.4) holds q.e. in Ω. Thus the result follows taking the limit of the averages for u, and from the fact that for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$, $\operatorname{ess\,sup}_{\Omega} |g| = \sup_{\Omega} |g|$. ■

For every $x \in \Omega$ we can consider the map $H_\mu(\cdot)(x)$: $H^1(\Omega) \cap \cap C(\overline{\Omega}) \to \boldsymbol{R}$, which associates to every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ the pointwise value in $x \in \Omega$ of the solution u to the problem (3.3). By Lemma 4.1 and Corollary 4.3, $H_\mu(\cdot)(x)$ turns out to be a linear and bounded functional defined in a dense subset of $C(\overline{\Omega})$. Thus there exists a unique extension, still denoted by $H_\mu(\cdot)(x)$, linear and bounded in $C(\overline{\Omega})$ endowed with the uniform norm. By the Riesz representation theorem, for every $x \in \Omega$

there exists a Radon measure $\mathcal{H}_\mu(x,\cdot)$ such that

$$H_\mu(g)(x) = \int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy). \tag{4.5}$$

DEFINITION 4.4. The measures $\{\mathcal{H}_\mu(x,\cdot)\}_{x\in\Omega}$ are the μ-harmonic measures associated to the operator L.

5. – Properties of the μ-harmonic measures.

The following propositions single out some properties of the μ-harmonic measures.

LEMMA 5.1. *For every $x \in \Omega$ the measure $\mathcal{H}_\mu(x,\cdot)$ is a positive Radon measure with $\mathcal{H}_\mu(x,\overline{\Omega}) = 1$, and such that* $\operatorname{supp}(\mathcal{H}_\mu(x,\cdot)) \subseteq \operatorname{supp}(\mu) \cup \partial\Omega$.

PROOF. By the maximum principle

$$\int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy) = u(x) \geqslant 0, \tag{5.1}$$

for every nonnegative $g \in H^1(\Omega) \cap C(\overline{\Omega})$. Moreover every nonnegative $f \in C(\overline{\Omega})$ can be approximate by meaning of nonnegative functions $g_n \in H^1(\Omega) \cap C(\overline{\Omega})$ in the uniform norm, then (5.1) remains valid for every $g \in C(\overline{\Omega})$, and it implies that $\mathcal{H}_\mu(x,\cdot)$ is a positive measure.

Since $u = 1$ is the solution to the problem (3.3) corresponding to $g = 1$, then $\mathcal{H}_\mu(x,\overline{\Omega}) = u(x) = 1$ for every $x \in \Omega$.

Finally, let U be an open subset of Ω such that $U \cap \operatorname{supp}(\mu) = \emptyset$. By the second part of Theorem 3.2, for every $g \in C_0^\infty(U)$ we have that the solution to the problem (3.3) corresponding to g is identically zero. Since, in particular, $g \in H^1(\Omega) \cap C(\overline{\Omega})$, we get

$$\int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy) = 0$$

for every $x \in \Omega$, so that $U \cap supp(\mathcal{H}_\mu(x,\cdot)) = \emptyset$ for every $x \in \Omega$. ■

Now we partially describe the connection between the μ-harmonic measures and the capacity. A complete description of the mass of the μ-harmonic measures inside sets of capacity zero needs some additional tools, and will be given in Theorem 7.6.

LEMMA 5.2. *Fixed a bounded open set Ω' such that $\Omega \subset\subset \Omega'$, let B be a Borel subset of $\overline{\Omega}$ with* $\operatorname{cap}(B, \Omega') = 0$. *Then $\mathcal{H}_\mu(x, B) = 0$ for every $x \in \Omega \setminus B$.*

PROOF. As a first step we consider the case where $B = K$ is compact subset of Ω. Let x_0 be a point outside K, and let us choose $r > 0$ be such that $B_r(x_0) \cap U = \emptyset$ for a suitable neighborhood U of K in Ω'. Since $\operatorname{cap}(K, U) = 0$, there exists a sequence $\{g_n\}$ of nonnegative functions belonging to $C_0^\infty(U)$ with $g_n \geq 1$ on K and such that

$$\lim_{n \to \infty} \int_U |Dg_n|^2 \, dx = 0.$$

Let us consider the solution u_n to the problems

$$\begin{cases} u_n - g_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu_n, \varphi \rangle + \int_\Omega (u_n - g_n) \varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

Since $g_n \in H^1(\Omega) \cap C(\overline{\Omega})$ we have that

$$u_n(x) = \int_{\overline{\Omega}} g_n(y) \, \mathcal{H}_\mu(x, dy) \geq \mathcal{H}_\mu(x, K) \tag{5.2}$$

for every $x \in \Omega$. It remains to prove that $\lim_{n \to \infty} u_n(x_0) = 0$. Since $g_n = 0$ in $\overline{B}_r(x_0)$, then the function u_n is a local solution to the relaxed Dirichlet problem (2.4) in $B_r(x_0)$ with right-hand side $f = 0$. Let us consider the solution z_n to the problem

$$\begin{cases} z_n - u_n \in H_0^1(B_r(x_0)), \\ \langle Lz_n, \psi \rangle = 0, \qquad \forall \psi \in H_0^1(B_r(x_0)). \end{cases}$$

By (5.2), the functions u_n are nonnegative, and then, by the classical maximum principle, we have $z_n \geq 0$ a.e. in $B_r(x_0)$ (see [12], Theorem 8.1). Moreover Corollaire 5.2 of [17] implies that there exists a constant $\alpha > 0$ such that

$$\max_{B_{r/4}(x_0)} z_n \leq \frac{\alpha}{r^{N/2}} \left(\int_{B_{r/2}(x_0)} |z_n|^2 \, dx \right)^{1/2}.$$

Finally, by Proposition 2.6, $Lu_n \leq 0$ in the sense of distributions in

$B_r(x_0)$, so that $L(z_n - u_n) \geq 0$ in the sense of distributions in $B_r(x_0)$, and $z_n - u_n \in H_0^1(B_r(x_0))$. Then, by Theorem 8.1 in [12], we have that $z_n \geq u_n$ in $B_r(x_0)$. Thus we obtain

$$\max_{B_{r/4}(x_0)} u_n \leq \max_{B_{r/4}(x_0)} z_n \leq \frac{a}{r^{N/2}} \left(\int_{B_{r/2}(x_0)} |z_n|^2 dx \right)^{1/2}. \tag{5.3}$$

By Theorem 3.3, and the fact that $\{g_n\}$ converges to zero strongly in $H^1(\Omega)$, we have that $\{u_n\}$ also converges strongly to zero in $H^1(\Omega)$. This implies that also $\{z_n\}$ converges to zero strongly in $H^1(\Omega)$, and a passage to the limit in (5.3) gives $\lim_{n \to \infty} u_n(x_0) = 0$, so that the result is proved for the compact sets.

If B is a Borel subset of $\overline{\Omega}$, then for every compact set $K \subseteq B$ we have $\operatorname{cap}(K, \Omega') = 0$ so that $\mathcal{H}_\mu(x, K) = 0$ for every $x \in \Omega \backslash B$. Finally, since $\mathcal{H}_\mu(x, \cdot)$ is a Radon measure, then $\mathcal{H}_\mu(x, B) = \sup\{\mathcal{H}_\mu(x, K)\colon K$ compact, $K \subseteq B\} = 0$. ■

Fixed $x_0 \in \Omega$, for every $\varepsilon > 0$, we consider the measure

$$\mathcal{H}_\mu^\varepsilon(x_0, \cdot) = \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} \mathcal{H}_\mu(x, \cdot)\, dx.$$

It is well known that, if $\mu = 0$ and $L = -\Delta$, then the harmonic measure $\mathcal{H}(x, B)$ of a Borel set B is an harmonic function, and thus $\mathcal{H}(x_0, B) = 1/|B_\varepsilon(x_0)| \int_{B_\varepsilon(x_0)} \mathcal{H}(x, B)$ for every $\varepsilon > 0$.

The behaviour of the sequence $\{\mathcal{H}_\mu^\varepsilon(x_0, \cdot)\}_{\varepsilon > 0}$ in the general case is described in the following lemma.

LEMMA 5.3. *The sequence $\{\mathcal{H}_\mu^\varepsilon(x_0, \cdot)\}_{\varepsilon > 0}$ converges to $\mathcal{H}_\mu(x_0, \cdot)$ in the weak* topology of measures in $\overline{\Omega}$.*

PROOF. It is enough to prove that for every $x_0 \in \Omega$ the following two properties hold:

(i) $\mathcal{H}_\mu(x_0, B) \leq \liminf_{\varepsilon \to 0^+} \mathcal{H}_\mu^\varepsilon(x_0, B)$, for every open subset B of $\overline{\Omega}$;

(ii) $\limsup_{\varepsilon \to 0^+} \mathcal{H}_\mu^\varepsilon(x_0, K) \leq \mathcal{H}_\mu(x_0, K)$, for every compact subset K of $\overline{\Omega}$.

In order to prove (i), we consider two open subsets A and B of $\overline{\Omega}$ such that $A \subset\subset B$, and a function $g \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\mathbf{1}_A \leq g \leq$

$\leq 1_B$ q.e. in $\overline{\Omega}$. Then, by Theorem 4.2, denoting by u the solution of problem (3.3) corresponding to g, we have

$$\mathcal{H}_\mu(x_0, A) \leq \int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x_0, dy) = u(x_0) = \lim_{\varepsilon\to 0^+} \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} u(x)\,dx =$$

$$= \lim_{\varepsilon\to 0^+} \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} \left(\int_{\overline{\Omega}} g(z)\,\mathcal{H}_\mu(x, dz) \right) dx \leq$$

$$\leq \liminf_{\varepsilon\to 0^+} \frac{1}{|B_\varepsilon(x_0)|} \int_{B_\varepsilon(x_0)} \mathcal{H}_\mu(x, B)\,dx = \liminf_{\varepsilon\to 0^+} \mathcal{H}_\mu^\varepsilon(x_0, B).$$

If we take a sequence $\{A_n\}$ of open sets, such that $A_n \subset\subset B$ for every $n \in N$, and with A_n increasing to B, then we obtain

$$\mathcal{H}_\mu(x_0, B) \leq \liminf_{\varepsilon\to 0^+} \mathcal{H}_\mu^\varepsilon(x_0, B),$$

for every $x_0 \in \Omega$.

Similarly one can prove that (ii) holds. ∎

The following result shows how the measures $\{\mathcal{H}_\mu(x, \cdot)\}_{x\in\Omega}$ depend on μ.

THEOREM 5.4. *Let $\mu, \mu_0 \in \mathcal{M}_0(\Omega)$. If we assume that $\mu = \mu_0$, then $\mathcal{H}_\mu(x, \cdot) = \mathcal{H}_{\mu_0}(x, \cdot)$ for every $x \in \Omega$. Conversely, if $\mathcal{H}_\mu(x, \cdot) \llcorner \Omega = \mathcal{H}_{\mu_0}(x, \cdot) \llcorner \Omega$ for almost every x in Ω, then $\mu = \mu_0$.*

PROOF. If $\mu = \mu_0$, then

$$\int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy) = \int_{\overline{\Omega}} g(y)\,\mathcal{H}_{\mu_0}(x, dy) \tag{5.4}$$

for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ and for every $x \in \Omega$. As $H^1(\Omega) \cap C(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ with respect to the uniform norm, then (5.4) extends to $g \in C(\overline{\Omega})$, that is $\mathcal{H}_\mu(x, \cdot) = \mathcal{H}_{\mu_0}(x, \cdot)$ for every $x \in \Omega$.

Conversely, let us suppose that $\mathcal{H}_\mu(x, \cdot) \llcorner \Omega = \mathcal{H}_{\mu_0}(x, \cdot) \llcorner \Omega$ for almost every $x \in \Omega$. We consider the solution $\tilde{g}$ to the classical Dirichlet problem

$$\begin{cases} \tilde{g} \in H_0^1(\Omega), \\ \langle L\tilde{g}, \varphi \rangle = -\int_{\Omega} \varphi\,dx, \qquad \forall \varphi \in H_0^1(\Omega). \end{cases} \tag{5.5}$$

Since $\tilde{g} \in H^1(\Omega) \cap C(\overline{\Omega})$, then the solutions u, u_0 to the problems

$$\begin{cases} u - \tilde{g} \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu, \varphi \rangle + \int\limits_{\Omega} (u - \tilde{g}) \varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

and

$$\begin{cases} u_0 - \tilde{g} \in H_0^1(\Omega) \cap L^2(\Omega, \mu_0), \\ \langle Lu_0, \varphi \rangle + \int\limits_{\Omega} (u_0 - \tilde{g}) \varphi \, d\mu_0 = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu_0), \end{cases}$$

coincide. Since $u - \tilde{g} = w_\mu$, and $u_0 - \tilde{g} = \mu_0$, then, by Proposition 2.9(d) we obtain that $\mu = \mu_0$. ■

We are now in a position to investigate the asymptotic behaviour of a sequence of μ-harmonic measures associated to a γ^L-converging sequence of measures in $\mathcal{M}_0(\Omega)$.

THEOREM 5.5. *Let $\{\mu_n\}$ and μ be measures belonging to $\mathcal{M}_0(\Omega)$. The following conditions are equivalent:*

(a) $\{\mu_n\}$ γ^L-converges to μ;

(b) for almost every $x \in \Omega$ the measures $\{\mathcal{H}_{\mu_n}(x, \cdot)\}_{n \in N}$ converge weakly in $\overline{\Omega}$ to $\mathcal{H}_\mu(x, \cdot)$;*

(c) for almost every $x \in \Omega$ the measures $\{\mathcal{H}_{\mu_n}(x, \cdot) \llcorner \Omega\}_{n \in N}$ converge weakly in Ω to $\mathcal{H}_\mu(x, \cdot) \llcorner \Omega$.*

PROOF. Let us suppose that $\{\mu_n\}$ γ^L-converges to μ. Thus, by Theorem 3.3, for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ there exists a subset $N(g)$ of Ω with Lebesgue measure zero such that

$$\lim_{n \to \infty} \int\limits_{\overline{\Omega}} g(y) \, \mathcal{H}_{\mu_n}(x, dy) = \int\limits_{\overline{\Omega}} g(y) \, \mathcal{H}_\mu(x, dy), \tag{5.6}$$

for every $x \in \Omega \backslash N(g)$. Let $\mathcal{N}(\Omega)$ be a countable subset of $H^1(\Omega) \cap C(\overline{\Omega})$ which is dense in $H^1(\Omega) \cap C(\overline{\Omega})$ with respect to the uniform norm, $\mathcal{N}(\Omega) = \{g_n\}$, and let $N = \bigcup_n N(g_n)$. Fixed $g \in H^1(\Omega) \cap C(\overline{\Omega})$ and $\varepsilon > 0$, let $n \in N$ be such that $\|g - g_n\|_{C(\overline{\Omega})} \leq \varepsilon$. Then we obtain

$$\left| \int_{\overline{\Omega}} g(y)\,\mathcal{H}_{\mu_n}(x, dy) - \int_{\overline{\Omega}} g(y)\,\mathcal{H}_{\mu}(x, dy) \right| \leq$$

$$\leq \left| \int_{\overline{\Omega}} [g(y) - g_n(y)]\,\mathcal{H}_{\mu_n}(x, dy) \right| + \left| \int_{\overline{\Omega}} [g_n(y) - g(y)]\,\mathcal{H}_{\mu}(x, dy) \right| +$$

$$+ \left| \int_{\overline{\Omega}} g_n(y)\,\mathcal{H}_{\mu_n}(x, dy) - \int_{\overline{\Omega}} g_n(y)\,\mathcal{H}_{\mu}(x, dy) \right|$$

which implies

$$\lim_{n \to \infty} \left| \int_{\overline{\Omega}} g(y)\,\mathcal{H}_{\mu_n}(x, dy) - \int_{\overline{\Omega}} g(y)\,\mathcal{H}_{\mu}(x, dy) \right| \leq 2\varepsilon,$$

for every $x \in \Omega \setminus N$. Thus (5.6) holds for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ and for every $x \in \Omega \setminus N$. Finally, since $H^1(\Omega) \cap C(\overline{\Omega})$ is dense in $C(\overline{\Omega})$ with respect to the uniform norm we extend (5.6) to every $g \in C(\overline{\Omega})$, obtaining that $\{\mathcal{H}_{\mu_n}(x, \cdot)\}_{n \in N}$ converge weakly* in $\overline{\Omega}$ to $\mathcal{H}_{\mu}(x, \cdot)$ for almost every $x \in \Omega$. Thus condition (a) implies (b). Clearly (b) implies (c), and then it remains to prove that (c) implies (a). Let us suppose that $\{\mathcal{H}_{\mu_n}(x, \cdot) \llcorner \Omega\}_{n \in N}$ converges weakly* in Ω to $\mathcal{H}_{\mu}(x, \cdot) \llcorner \Omega$ for almost every $x \in \Omega$. By Theorem 2.11 we have that, up to a subsequence, $\{\mu_n\}$ γ^L-converges to a measure $\mu_0 \in \mathcal{M}_0(\Omega)$. Thus the previous part of the proof guarantees that $\{\mathcal{H}_{\mu_n}(x, \cdot)\}_{n \in N}$ converges weakly* in $\overline{\Omega}$ to $\mathcal{H}_{\mu_0}(x, \cdot)$ for almost every $x \in \Omega$. Then by Theorem 5.4 we obtain that $\mu = \mu_0$. ∎

REMARK 5.6. It is possible to construct a sequence $\{\mu_n\} \in \mathcal{M}_0(\Omega)$ and a measure $\mu \in \mathcal{M}_0(\Omega)$ such that $\{\mu_n\}$ γ^L-converges to μ, but there exists $x_0 \in \Omega$ such that the sequence $\{\mathcal{H}_{\mu_n}(x_0, \cdot)\}_{n \in N}$ does not converge weakly* in $\overline{\Omega}$ to $\mathcal{H}_{\mu}(x_0, \cdot)$. It is enough to take $\mu = \infty_E$ where E is a closed subset of Ω such that $x_0 \in \partial E$ and x_0 is not a Wiener point, that is, the solution u of (3.3) corresponding to $\tilde{g}$ defined by (5.5) has pointwise value $u(x_0) \neq \tilde{g}(x_0)$ (see Theorem 7.3 below). If we consider the measures $\mu_n = \infty_{E_n}$, where E_n are closed sets with smooth boundary which contain x_0 in their interior and which uniformly approximate the set E, then for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$, the solutions u_n to the problems (3.3) corresponding to μ_n satisfy $u_n(x_0) = g(x_0)$ for every $n \in N$. Thus for every $n \in N$ the measure $\mathcal{H}_{\mu_n}(x_0, \cdot)$ coincides with the Dirac mass δ_{x_0} carried by x_0, while $\mathcal{K}_\mu(x_0, \cdot) \neq \delta_{x_0}$, because $\tilde{g} \in H^1(\Omega) \cap C(\overline{\Omega})$ and $\tilde{g}(x_0) \neq \neq \int_{\overline{\Omega}} \tilde{g}(y) \mathcal{K}_\mu(x, dy)$.

Let g be a function in $H^1(\Omega)$. The values of g on $\partial\Omega$ can be defined as follows. We fix a bounded open set Ω' such that $\Omega \subset\subset \Omega'$, and we consider a function $g_* \in H^1(\Omega')$ which extends g. The existence of such a function g_* is due to the regularity of the boundary of Ω. Since $g_* \in \in H^1(\Omega')$, it is defined in Ω' except possibly a set of capacity zero, and then it is defined almost everywhere with respect to the $(N-1)$-dimensional Hausdorff measure supported by $\partial\Omega$. Then for quasi every $x \in \partial\Omega$ we can define $g(x) = g_*(x)$, and this representative does not depend on the choice of the extension g_* (for more details, see [19], Section 4.4). With this convention about the pointwise values of g on $\partial\Omega$, g is well defined in $\overline{\Omega}$, up to a set N of capacity zero. Thus, by Lemma 5.2, the integral

$$\int_{\overline{\Omega}} g(y) \mathcal{K}_\mu(x, dy)$$

is well defined for every $x \in \Omega \backslash N$. The following result shows that this function coincides q.e. in Ω with the solution to the problem (3.3) corresponding to the datum $g \in H^1(\Omega)$.

THEOREM 5.7. *Let $g \in H^1(\Omega)$, and let u be the solution to the problem*

$$\begin{cases} u - g \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu, \varphi \rangle + \int_\Omega (u-g)\varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

Then

$$u(x) = \int_{\overline{\Omega}} g(y) \, \mathcal{K}_\mu(x, dy), \tag{5.7}$$

for quasi every x in Ω.

PROOF. As a first step, we consider $g \in H^1(\Omega)$ such that $\operatorname{ess\,sup}_{\overline{\Omega}} g \leq M$. If we define pointwise g on $\partial\Omega$ as before, then we can find a sequence $g_n \in H^1(\Omega) \cap C(\overline{\Omega})$, such that $g_n \leq M$ for every $n \in \boldsymbol{N}$, and which converges strongly to g in $H^1(\Omega)$ and q.e. in $\overline{\Omega}$. Let us now consider the solutions u_n to the problems

$$(5.8)\qquad \begin{cases} u_n - g_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu_n, \varphi\rangle + \int\limits_{\Omega} (u_n - g_n)\varphi \, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

Since $g_n \in H^1(\Omega) \cap C(\overline{\Omega})$, then

$$(5.9)\qquad u_n(x) = \int\limits_{\overline{\Omega}} g_n(y)\, \mathcal{H}_\mu(x, dy),$$

for every $x \in \Omega$. The strong convergence of $\{g_n\}$ to g in $H^1(\Omega)$ implies, by Theorem 3.3, that $\{u_n\}$ converges to u strongly in $H^1(\Omega)$. Thus, by Proposition 2.1, we can find a subset N of $\overline{\Omega}$ with capacity zero such that, up a subsequence, both $\{g_n\}$ converges to g pointwise in $\overline{\Omega}\setminus N$ and $\{u_n\}$ converges to u pointwise in $\Omega\setminus N$. By Lemma 5.2 and by dominate convergence theorem, we can pass to the limit in (5.9), obtaining (5.7).

As a second step, we consider $g \in H^1(\Omega)$ such that $g \geqslant 0$ q.e. in $\overline{\Omega}$. Then for every $n \in \mathbf{N}$ the functions $g_n = \min(g, n)$ belong to $H^1(\Omega)$, and they converge to g both pointwise and in the strong topology of $H^1(\Omega)$. Then, up to a subsequence, the solutions u_n of the inhomogeneous relaxed Dirichlet problems associated to μ and with datum g_n converge both strongly in $H^1(\Omega)$ and q.e. in Ω to the solution u of the same equation with datum g. Thus, since $u(x)$ is finite for quasi every $x \in \Omega$, and since, by the previous step, the representation formula (5.7) holds for g_n, we have

$$\sup_n \int\limits_{\overline{\Omega}} g_n(y)\mathcal{H}_\mu(x, dy) = \sup_n u_n(x) < +\infty,$$

for quasi every $x \in \Omega$. Thus we can apply the monotone convergence theorem, obtaining

$$(5.10)\quad u(x) = \lim_{n\to\infty} u_n(x) = \lim_{n\to\infty} \int\limits_{\overline{\Omega}} g_n(y)\,\mathcal{H}_\mu(x, dy) = \int\limits_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy),$$

for quasi every $x \in \Omega$. Notice that, in particular, (5.10), implies that every nonnegative function $g \in H^1(\Omega)$ is integrable with respect to $\mathcal{H}_\mu(x, \cdot)$ for quasi every $x \in \Omega$.

Let now g be a function in $H^1(\Omega)$. We can split $g = g^+ - g^-$, and we already know that for the solutions u_1 and u_2, corresponding to the positive and negative part of g, formula (5.7) holds up to a set of capacity zero. Moreover we have $u = u_1 - u_2$ q.e. in Ω, due to the linearity of problem (3.3). Thus (5.7) holds for u. ■

In the last part of this section we show that, in some cases, it is possible to exhibit an elliptic equation solved by the μ-harmonic measures.

THEOREM 5.8. *Let $\mu \in \mathcal{M}_0(\Omega)$, and let B be a quasi open subset of Ω such that $\mu(B) < \infty$. Then $u(x) = \mathcal{H}_\mu(x, B) \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and it solves*

$$\langle Lu, \varphi \rangle + \int_\Omega u\varphi \, d\mu = \int_B \varphi \, d\mu ,$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$.

PROOF. Since B is a quasi open subset of Ω, then, by Lemma 2.1 in [4], there exists a sequence $\{g_n\}$ of functions belonging to $H_0^1(\Omega)$, such that $0 \leq g_n \leq 1_B$, and increasing to 1_B q.e. in Ω. Let u_n be the solution to the problem

$$\text{(5.11)} \qquad \begin{cases} u_n - g_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu_n, \varphi \rangle + \displaystyle\int_\Omega (u_n - g_n) \varphi \, d\mu = 0 , \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

By Theorem 5.7 we have that

$$\text{(5.12)} \qquad u_n(x) = \int_{\overline{\Omega}} g_n(y) \mathcal{H}_\mu(x, dy),$$

for quasi every $x \in \Omega$. Moreover, by Corollary 3.8, the sequence $\{u_n\}$ increases to a function u. Thus, taking the limit in (5.12) as n goes to ∞, we obtain that $u(x) = \mathcal{H}_\mu(x, B)$ q.e. in Ω. Since for every $n \in N$, $g_n \leq 1_B$ and $\mu(B)$ is finite, both g_n and u_n belong to $H_0^1(\Omega) \cap L^2(\Omega, \mu)$, so that (5.11) can be written as

$$\langle Lu_n, \varphi \rangle + \int_\Omega u_n \varphi \, d\mu = \int_\Omega g_n \varphi \, d\mu ,$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$. Choosing $\varphi = u_n$ as test function in (5.11), and using the Young inequality in the left-hand side, we obtain

$$\lambda \int_\Omega |Du_n|^2 dx + \int_\Omega u_n^2 d\mu \leq 1/2 \left(\mu(B) + \int_\Omega u_n^2 d\mu \right),$$

which implies that $\{u_n\}$ is a bounded sequence in $H_0^1(\Omega) \cap L^2(\Omega, \mu)$. Thus, up to a subsequence, $\{u_n\}$ converges to the function u in the weak topology of $H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and we can pass to the limit in the equations solved by u_n, obtaining that

$$\langle Lu, \varphi \rangle + \int_\Omega u\varphi \, d\mu = \int_B \varphi \, d\mu,$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$. ∎

In order to state a more precise result in the case when μ is a finite measure, we need the following result.

THEOREM 5.9. *Let $\mathcal{E}$ be a class of Borel subsets of Ω such that the following holds:*

(i) *if $E_n \in \mathcal{E}$, and E_n increases to a Borel subset E of Ω, then $E \in \mathcal{E}$;*

(ii) *if $E_n \in \mathcal{E}$, and E_n decreases to a Borel subset E of Ω, then $E \in \mathcal{E}$;*

(iii) *$\mathcal{E}$ contains all the open subsets of Ω.*

Then $\mathcal{E}$ coincides with the class of all Borel subsets of Ω.

PROOF. See [8], Theorem II.6. ∎

COROLLARY 5.10. *Let $\mu \in \mathcal{M}_0(\Omega)$ be a finite measure. Then for every $B \in \mathcal{B}(\Omega)$ the function $u(x) = \mathcal{H}_\mu(x, B)$ belongs to $H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and it solves*

$$\langle Lu, \varphi \rangle + \int_\Omega u\varphi \, d\mu = \int_B \varphi \, d\mu, \tag{5.13}$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$.

PROOF. In order to obtain the result it is enough to prove that the class $\mathcal{E}$ of all Borel subsets of Ω such that the function $\mathcal{H}_\mu(x, B)$ belongs to $H_0^1(\Omega) \cap L^2(\Omega, \mu)$ and solves (5.13) has the properties required in Theorem 5.9. In Theorem 5.8 we have proved that (iii) holds.

Let us consider a sequence $\{E_n\}$ of elements of $\mathcal{E}$ which increases to a Borel subset E of Ω. Then $\mathcal{H}_\mu(x, E_n)$ increases to $\mathcal{H}_\mu(x, E)$ for every $x \in \Omega$. Moreover the function $u_n = \mathcal{H}_\mu(x, E_n)$ solves

$$
\begin{cases}
u_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\
\langle Lu_n, \varphi\rangle + \int\limits_\Omega u_n \varphi \, d\mu = \int\limits_{E_n} \varphi \, d\mu, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu),
\end{cases}
\tag{5.14}
$$

so that, choosing u_n as test function in (5.14), we have

$$
\|u_n\|_{H_0^1(\Omega)} + \|u_n\|_{L^2(\Omega, \mu)} \leq c\mu(E).
$$

Thus u_n converge to $\mathcal{H}_\mu(x, E)$ weakly both in $H_0^1(\Omega)$ and in $L^2(\Omega, \mu)$, and a passage to the limit in (5.14) implies that E belongs to $\mathcal{E}$. Similarly one can check that the property (ii) holds. Thus we can apply Theorem 5.9, obtaining that $\mathcal{E} = \mathcal{B}(\Omega)$. ■

As a corollary of Theorem 5.10 we have that for every finite measure μ and for every Borel subset B of Ω, the function $\mathcal{H}_\mu(x, B)$ is the solution of the relaxed Dirichlet problem associated to the measure μ and with datum $\nu = \mu \llcorner B$. The notion of solution of a relaxed Dirichlet problem with a right hand side measure was introduced in [16], and it was proved a representation formula for this solution in terms of μ-Green functions (see [16] Theorem 7.6). Thus, for every $B \subseteq \Omega$, we have that

$$
\mathcal{H}_\mu(x, B) = \int\limits_B G_\mu(x, y) \, d\mu
$$

for quasi every $x \in \Omega$, which implies that

$$
H_\mu(g)(x) = \int\limits_\Omega g(y) G_\mu(x, y) \, d\mu + \int\limits_{\partial\Omega} g(y) \, \mathcal{H}_\mu(x, dy)
$$

for quasi every $x \in \Omega$. In the special case when $\mu = 0$, then $\mathcal{H}_\mu(x, \cdot) \llcorner \Omega = 0$, and $\mathcal{H}_\mu(x, \cdot)$ coincides with the classical harmonic measure in Ω. If μ is the Lebesgue measure, then G_μ is the Green function corresponding to the operator $Lu + u$ and with homogeneous boundary conditions in Ω, and $\mathcal{H}_\mu(x, \cdot)$ is the harmonic measure in Ω relative to the same operator, so that we recover the well known representation formula of the potential theory.

6. - Generalized solutions.

Fixed a measure $\mu \in \mathcal{M}_0(\Omega)$ and a function $g \in C(\overline{\Omega})$, we have that the function $H_\mu(g)(x) = \int_{\overline{\Omega}} g(y)\,\mathcal{H}_\mu(x, dy)$ is the pointwise limit of any sequence $\{u_n\}$ of solutions of inhomogeneous relaxed Dirichlet problems corresponding to the measure μ and with datum $g_n \in H^1(\Omega) \cap \cap C(\overline{\Omega})$, converging uniformly to g. Thus it seems to be natural to define $H_\mu(g)$ as a generalized solution of the inhomogeneous relaxed Dirichlet problem associated to a measure μ and with datum $g \in C(\overline{\Omega})$. This definition is also motivated by the following result, which states that every generalized solution of an inhomogeneous relaxed Dirichlet problem associated to a finite measure μ, is a local solution of (3.4).

PROPOSITION 6.1. *Let $\mu \in \mathcal{M}_0(\Omega)$ be a finite measure. Then for every $g \in C(\overline{\Omega})$ the function $u = H_\mu(g)$ belongs to $H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu)$ and it solves*

$$\langle Lu, \varphi\rangle + \int_{\Omega} (u - g)\,\varphi\, d\mu = 0,$$

for all $\varphi \in H^1_0(\Omega) \cap L^2(\Omega, \mu)$ with compact support in Ω.

PROOF. Let g be a function belonging to $C(\overline{\Omega})$. If we consider a sequence $\{g_n\}$ of functions in $H^1(\Omega) \cap C(\overline{\Omega})$ converging to g in the uniform norm, then by Corollary 3.9, the corresponding solutions u_n to the problems

$$\begin{cases} u_n - g_n \in H^1_0(\Omega) \cap L^2(\Omega, \mu), \\ \langle Lu_n, \varphi\rangle + \int_{\Omega} (u_n - g_n)\,\varphi\, d\mu = 0, \qquad \forall \varphi \in H^1_0(\Omega) \cap L^2(\Omega, \mu), \end{cases}$$

converge uniformly q.e. in Ω to a function u. Moreover the limit u does not depend on the choice of the approximating sequence, and a passage to the limit in

$$u_n(x) = \int_{\overline{\Omega}} g_n(y)\mathcal{H}_\mu(x, dy), \qquad \forall x \in \Omega,$$

gives $u = H_\mu(g)$ q.e. in Ω.

Since μ is finite, every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ belongs to $L^2(\Omega, \mu)$, and, by (4.4), problem (3.3) can be written as

$$(6.1)\quad \begin{cases} u - g \in H_0^1(\Omega), \\ \langle Lu, \varphi\rangle + \int_\Omega u\varphi\, d\mu = \int_\Omega g\varphi\, d\mu, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

Fixed $x_0 \in \Omega$, and $0 < r < R$ such that $B_r(x_0) \subset B_R(x_0) \subset \Omega$, let us consider $\psi \in C_0^\infty(B_R(x_0))$ such that $0 \leq \psi \leq 1$ in Ω, $\psi = 1$ in $B_r(x_0)$, and $|D\psi| \leq 2/(R-r)$ in $B_R(x_0)$. If we put $u\psi^2$ as test function in the equation (6.1), we obtain

$$\lambda \int_{B_R(x_0)} |Du|^2 \psi^2 \leq \int_{B_R(x_0)} gu\psi^2 \, d\mu + 2\Lambda \int_{B_R(x_0)} |D\psi||Du||u|\,\psi\, dx.$$

By Young inequality we have that

$$\int_{B_R(x_0)} |D\psi||Du||u|\,\psi\, dx \leq \varepsilon \int_{B_R(x_0)} |Du|^2 \psi^2\, dx + \frac{1}{\varepsilon(R-r)^2} \int_{B_R(x_0)} |u|^2\, dx.$$

Choosing $0 < \varepsilon \leq \lambda/2\Lambda$, we obtain

$$(6.2)\quad \int_{B_r(x_0)} |Du|^2 dx \leq c\left(\|g\|_{C(\overline{\Omega})} \int_{B_R(x_0)} |u|\, d\mu + \int_{B_R(x_0)} u^2\, dx \right).$$

Now we can apply formula (6.2) to $u_n - u_k$, which is the solution of problem (6.1) corresponding to the datum $g_n - g_k \in H^1(\Omega) \cap C(\overline{\Omega})$. Since g_n converge uniformly to g and u_n converge uniformly q.e. in Ω to u, then (6.2) implies that $\{u_n\}$ is a Cauchy sequence in $H^1(B_r(x_0))$, so that it converges weakly to u in this space. Thus we can pass to the limit in

$$\langle Lu_n, \varphi\rangle + \int_\Omega u_n \varphi\, d\mu = \int_\Omega g_n \varphi\, d\mu,$$

provided that we choose test functions φ with compact support in Ω, obtaining the result. ■

REMARK 6.2. The previous result does not hold if μ is not a finite measure. Namely, if we consider $\mu = \infty_{\Omega \setminus \Omega'}$, where Ω' is an open subset of Ω, then for every $g \in C(\overline{\Omega})$ the function $H_\mu(g)$ coincides with g a.e. in $\Omega \setminus \Omega'$. Thus, if g does not belong to $H^1_{\mathrm{loc}}(\Omega)$, then there is no hope to obtain that $H_\mu(g)$ is a local solution of the inhomogeneous relaxed Dirichlet problem.

In the following propositions we single out the main properties of the generalized solutions.

PROPOSITION 6.3. *If $\{\mu_n\}$ is a sequence of measures belonging to $\mathcal{M}_0(\Omega)$ which γ^L-converges to μ, then for every $g \in C(\overline{\Omega})$ the generalized solutions $H_{\mu_n}(g)$ converge pointwise a.e. in Ω to the generalized solution $H_\mu(g)$.*

PROOF. The result follows directly from Theorem 5.5. ■

PROPOSITION 6.4. *For every $\mu \in \mathcal{M}_0(\Omega)$ there exists a sequence $\{E_n\}$ of compact subsets of Ω such that, if we denote by $\{\mathcal{H}_n(x, \cdot)\}_{x \in \Omega_n}$ the harmonic measures of $\Omega_n = \Omega \backslash E_n$, then for every $g \in C(\overline{\Omega})$, the generalized solutions*

$$u_n(x) = \int_{\partial \Omega_n} g(y)\, \mathcal{H}_n(x, dy) \tag{6.3}$$

extended to Ω by $u_n = g$ in E_n, converge pointwise a.e. in Ω to the generalized solution $H_\mu(g)$.

PROOF. By Proposition 2.12 there exists a sequence $\{E_n\}$ of compact subsets of Ω such that $\mu_n = \infty_{E_n}$ γ^L-converge to μ. Thus, by Proposition 6.3, for every $g \in C(\overline{\Omega})$ the sequence $\{H_{\mu_n}(g)\}$ converges a.e. in Ω to $H_\mu(g)$. It remains to prove that $H_{\mu_n}(g)$ coincides a.e. in Ω with the generalized solution (6.3) extended to Ω by setting $u_n = g$ in $\Omega \backslash E_n$. But this is a direct consequence of Remark 3.4, which implies that

$$\int_{\overline{\Omega}} g(y)\, \mathcal{H}_{\mu_n}(x, dy) = \int_{\partial \Omega_n} g(y)\, \mathcal{H}_n(x, dy) + (\mathbf{1}_{E_n} g)(x),$$

for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$, and for quasi every $x \in \Omega$. ■

7. – A new formulation of the Wiener criterion.

Let μ be a measure in $\mathcal{M}_0(\Omega)$. Fixed $x_0 \in \Omega$, let $R > 0$ be such that $B_R(x_0) \subseteq \Omega$. Here and henceforth we set $B_\varrho = B_\varrho(x_0)$ for every $\varrho > 0$. For every $0 < \varrho \leq R$, the μ-capacitary potential z_ϱ of B_ϱ in Ω relative to the operator L is the unique solution to the problem

$$(7.1)\qquad \begin{cases} z_\varrho \in H_0^1(\Omega), \qquad z_\varrho - 1 \in L^2(B_\varrho, \mu), \\ \langle L z_\varrho, \varphi\rangle + \int\limits_{B_\varrho} (z_\varrho - 1)\varphi\, d\mu = 0, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

By the maximum principle, $0 \leq z_\varrho \leq 1$ q.e. in Ω. Moreover, by (7.1) and Proposition 2.6, $L(1 - z_\varrho) \leq 0$ in the sense of distributions in Ω. Thus z_ϱ is a L-superharmonic function belonging to $H_0^1(\Omega)$, so that z_ϱ coincides with the limit of its averages at each point $x \in \Omega$, and it is a lower semicontinuous function.

The μ-capacity of B_ϱ in Ω relative to the operator L is defined by

$$\operatorname{cap}_\mu^L(B_\varrho, \Omega) = \langle L z_\varrho, z_\varrho\rangle + \int\limits_{B_\varrho} z_\varrho^2\, d\mu .$$

The notions of μ-capacitary potential and of μ-capacity were introduced in [6], in order to obtain the following Wiener criterion for relaxed Dirichlet problems.

THEOREM 7.1. *Fixed $\mu \in \mathcal{M}_0(\Omega)$ and a point $x_0 \in \Omega$, the following properties are equivalent:*

(*a*) *if $v \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu)$ satisfies*

$$\langle Lv, \varphi\rangle + \int\limits_{\Omega} v\varphi\, d\mu = 0,$$

for every test function $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ with compact support in a neighborhood of x_0, then $\lim\limits_{\varrho \to 0^+} \operatorname*{ess\,sup}\limits_{B_\varrho(x_0)} |v| = 0$;

(*b*) *x_0 is a Wiener point of μ, that is*

$$(7.2)\qquad \int\limits_0^R \frac{\operatorname{cap}_\mu^L(B_\varrho(x_0), B_{2\varrho}(x_0))}{\varrho^{N-1}}\, d\varrho = +\infty .$$

PROOF. See Theorem 5.5 in [6], and Theorem 2.1 in [11]. ■

Theorem 7.1 is the generalization to the case of relaxed Dirichlet problems of the classical Wiener criterion established in [18] and in [14], which characterizes those points $x_0 \in \partial\Omega$ such that every local solution of an elliptic equation defined in a neighborhood of x_0 is continuous and vanishes at x_0 (regular points of $\partial\Omega$), as being those points

(Wiener points) $x_0 \in \partial\Omega$ such that

$$\int_0^R \frac{\operatorname{cap}(B_\varrho(x_0)\backslash\Omega, B_{2\varrho}(x_0))}{\varrho^{N-1}}\, d\varrho = +\infty .$$

We want to prove that x_0 is a Wiener point for μ if and only if for every $g \in C(\overline{\Omega})$ the generalized solution $u = H_\mu(g)$ of the inhomogeneous relaxed Dirichlet problem corresponding to the measure μ and with datum g, satisfies

$$\lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} |H_\mu(g) - g| = 0 .$$

A first step in this direction is to characterize the Wiener points by using the μ-capacitary potentials z_ϱ.

PROPOSITION 7.2. *Let us fix $\mu \in \mathcal{M}_0(\Omega)$, $x_0 \in \Omega$, and $R > 0$ such that $B_R \subset \Omega$. For every $0 < \varrho \leq R$, let z_ϱ be the μ-capacitary potential of B_ϱ in Ω relative to the operator L. If x_0 is a Wiener point for the measure μ, then $z_\varrho(x_0) = 1$ for every $0 < \varrho \leq R$. Conversely, if x_0 is not a Wiener point for the measure μ, then $\lim_{\varrho \to 0^+} z_\varrho(x_0) = 0$.*

PROOF. If x_0 is a Wiener point for the measure μ, then it is a Wiener point for the measures $\mu \llcorner B_\varrho$ for every $0 < \varrho \leq R$ (see Lemma 2.6 in [3]). Since $1 - z_\varrho$ is a local solution of the relaxed Dirichlet problem (2.4) corresponding to the measure $\mu \llcorner B_\varrho$ and with datum $f = 0$, then, by Theorem 7.1, we obtain that $z_\varrho(x_0) = 1$ for every $0 < \varrho \leq R$.

Conversely, let us suppose that x_0 is not a Wiener point for the measure μ. This implies, as it was shown in the proof of Theorem 5.5 in [6], that there exists a sequence $\varrho_i > 0$, decreasing to zero, such that

$$\sum_{i=1}^\infty \operatorname{cap}_\mu^L(B_{\varrho_i}\backslash B_{\varrho_{i+1}}, \Omega)\varrho_{i+1}^{2-N} < +\infty$$

if $N \geq 3$, or

$$\sum_{i=1}^\infty \operatorname{cap}_\mu^L(B_{\varrho_i}\backslash B_{\varrho_{i+1}}, \Omega)\log\left(\frac{2R}{\varrho_{i+1}}\right) < +\infty$$

if $N = 2$. If we denote by z_n the μ-capacitary potential of B_{ϱ_n} in Ω, and by $z_{i,i+1}$ the μ-capacitary potential of $B_{\varrho_i}\backslash B_{\varrho_{i+1}}$ in Ω, then by Proposition 3.8 of [6], we have

$$z_n \leq \sum_{i=n}^\infty z_{i,i+1} ,$$

a.e. in Ω. Since $z_{i,i+1}$ is an L-superharmonic function, then there exists a constant $c > 0$ depending only on λ, Λ, such that

$$\frac{1}{|B_r|}\int_{B_r} z_{i,i+1}(y)\,dy \leq c z_{i,i+1}(x_0),$$

for every $0 < r \leq R$. Thus we can pass to the limit in the inequality

$$\frac{1}{|B_r|}\int_{B_r} z_n(y)\,dy \leq \sum_{i=n}^{\infty} \frac{1}{|B_r|}\int_{B_r} z_{i,i+1}(y)\,dy,$$

as r tends to zero, obtaining

$$z_n(x_0) \leq \sum_{i=n}^{\infty} z_{i,i+1}(x_0).$$

On the other hand, by Proposition 3.9 of [6],

$$z_{i,i+1}(x_0) \leq c\,\mathrm{cap}_\mu^L(B_{\varrho_i} \backslash B_{\varrho_{i+1}}, \Omega)\varrho_{i+1}^{2-N}$$

if $N \geq 3$, and

$$z_{i,i+1}(x_0) \leq c\,\mathrm{cap}_\mu^L(B_{\varrho_i} \backslash B_{\varrho_{i+1}}, \Omega)\log\left(\frac{2R}{\varrho_{i+1}}\right),$$

if $N = 2$. Thus, for every $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$0 \leq z_n(x_0) \leq c\sum_{i=n}^{\infty} \mathrm{cap}_\mu^L(B_{\varrho_i} \backslash B_{\varrho_{i+1}}, \Omega) \leq \varepsilon,$$

for every $n \geq n_0$, which concludes the proof. ■

Now we give a new version of the Wiener criterion which involves the behaviour of the solution w_μ of problem (2.7).

THEOREM 7.3. *Let μ be a measure in $\mathcal{M}_0(\Omega)$, and let w_μ the solution of the relaxed Dirichlet problem* (2.7) *corresponding to the measure μ and with right-hand side $f = 1$. The following properties are equivalent:*

(*a*) *x_0 is a Wiener point;*

(*b*) $\lim\limits_{\varrho \to 0^+} \operatorname*{ess\,sup}\limits_{B_\varrho(x_0)} w_\mu = 0$;

(*c*) $w_\mu(x_0) = 0$.

PROOF. If x_0 is a Wiener point, then for every $f \in L^\infty(\Omega)$, every local solution v of the equation (2.4) in a neighborhood of x_0 satisfies

$\lim_{\varrho \to 0^+} \operatorname{ess\,sup}_{B_\varrho(x_0)} |v| = 0$ (see [6], Theorem 6.4). Thus (a) implies (b). Moreover, by Proposition 2.9(a), w_μ is a nonnegative upper semicontinuous function, so that (b) is equivalent to (c). It remains to show that (b) implies (a).

Let us now consider a function $f \in L^2(\Omega)$ such that there exists an open set $\Omega' \subset\subset \Omega$, with $x_0 \in \Omega'$ and $f = 0$ a.e. in Ω'. The function f can be approximate in the strong topology of $L^2(\Omega)$ by a sequence $\{f_n\}$ of functions in $L^\infty(\Omega)$ such that $f_n = 0$ a.e. in Ω'. Let v_n be the solution to the problem

$$\begin{cases} v_n \in H_0^1(\Omega) \cap L^2(\Omega, \mu), \\ \langle L v_n, \varphi \rangle + \int_\Omega v_n \varphi \, d\mu = \int_\Omega f_n \varphi \, dx, \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu). \end{cases}$$

It is easy to see that $\{v_n\}$ converges strongly in $H_0^1(\Omega)$ to the solution v of the relaxed Dirichlet problem corresponding to μ and with right-hand side f. Since both v_n and v are local solutions of (2.4) with right-hand side zero in Ω', then, by Theorem 4.2 of [15], they belong to $L^\infty_{\mathrm{loc}}(\Omega')$, and, if we fix an open set $\Omega'' \subset\subset \Omega'$, there exists a constant $c > 0$ such that

$$\|v_n - v\|_{L^\infty(\Omega'')} \leq c \int_{\Omega'} |v_n - v|^2 \, dx .$$

Thus, fixed $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\|v_{n_0} - v\|_{L^\infty(\Omega'')} \leq \varepsilon .$$

On the other hand, by (b) and Proposition 2.9(c), there exists $\varrho_0 > 0$ such that

$$\sup_{B_\varrho} |v_{n_0}| \leq \|f_{n_0}\|_{L^\infty(\Omega)} \sup_{B_\varrho} w_\mu \leq \varepsilon ,$$

for every $0 < \varrho \leq \varrho_0$. Then

$$\sup_{B_\varrho} |v| \leq \sup_{B_\varrho} |v_{n_0}| + \sup_{\Omega''} |v_{n_0} - v| \leq 2\varepsilon ,$$

for every $0 < \varrho \leq \varrho_0$, which implies that $\lim_{\varrho \to 0^+} \operatorname{ess\,sup}_{B_\varrho(x_0)} |v| = 0$.

Finally, let $v \in H^1_{\mathrm{loc}}(\Omega) \cap L^2_{\mathrm{loc}}(\Omega, \mu)$ be such that

$$\langle L v, \varphi \rangle + \int_\Omega v \varphi \, d\mu = 0$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ with compact support in Ω. Fixed $\varrho > 0$ such that $v \in H^1(B_{2\varrho}) \cap L^2(B_{2\varrho}, \mu)$, let us consider a function $\psi \in C_0^\infty(B_{2\varrho})$ such that $\psi = 1$ in B_ϱ, and $0 \leq \psi \leq 1$ in Ω. For every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$ we have

$$\langle L(v\psi), \varphi\rangle = \langle Lv, (\varphi\psi)\rangle - \int_\Omega \varphi \sum_{i,j=1}^N a_{ij} D_j v \cdot D_i \psi \, dx + \int_\Omega v \sum_{i,j=1}^N a_{ij} D_j \psi \cdot D_i \varphi \, dx =$$

$$= -\int_\Omega v\varphi\psi \, d\mu - \int_\Omega \varphi \sum_{i,j=1}^N a_{ij} D_j v \cdot D_i \psi \, dx - \langle L\psi, v\varphi\rangle - \int_\Omega \varphi \sum_{i,j=1}^N a_{ji} D_j v \cdot D_i \psi \, dx \, .$$

Then we obtain

$$\langle L(v\psi,) \varphi\rangle + \int_\Omega (v\psi) \varphi \, d\mu = -\int_\Omega \varphi \sum_{i,j=1}^N (a_{ij} + a_{ji}) D_j v \cdot D_i \psi \, dx - \int_\Omega \varphi v L\psi \, dx \, ,$$

for every $\varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu)$. Moreover the function $\sum_{i,j=1}^N (a_{ij} + a_{ji}) D_j v \cdot D_i \psi + vL\psi$ belongs to $L^2(\Omega)$, and it is zero a.e. in B_ϱ. Thus, by the previous step,

$$\lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} |v| = \lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} |v\psi| = 0 \, ,$$

which concludes the proof.

REMARK 7.4. Let A_μ be the set of σ-finiteness of μ, that is the union of all quasi open subset A of Ω such that $\mu(A) < +\infty$, and let S_μ be the complement of A_μ in Ω. In [3] it was proved that S_μ is contained in the set of all Wiener points and coincides with that set up to a set of capacity zero. On the other hand, in [4] it was proved that S_μ coincides with the set $\{w_\mu = 0\}$ up to a set of capacity zero. By Theorem 7.3, these two results are the same.

The characterization of the Wiener points in terms of μ-harmonic measures is the following.

THEOREM 7.5. *Let $\mu \in \mathcal{M}_0(\Omega)$, and $x_0 \in \Omega$. The following properties are equivalent:*

(*a*) *x_0 is a Wiener point for the measure μ;*

(*b*) *$\lim_{x \to x_0} \mathcal{H}_\mu(x, \cdot) = \delta_{x_0}$, where the limit is taken in the weak* topology of measures in $\overline{\Omega}$;*

(*c*) $\lim_{x \to x_0} \mathcal{H}_\mu(x, \cdot) \llcorner \Omega = \delta_{x_0}$, *where the limit is taken in the weak* topology of measures in* Ω;

(*d*) $\mathcal{H}_\mu(x_0, \cdot) = \delta_{x_0}$;

(*e*) $\mathcal{H}_\mu(x_0, \{x_0\}) > 0$.

PROOF. Let us suppose that x_0 is a Wiener point for the measure μ. For every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ with $Lg \in L^\infty(\Omega)$, we consider the function $v = u - g$, where u is the solution of problem (3.3). Then v is the solution to the relaxed Dirichlet problem (3.6) with right-hand side $-Lg \in L^\infty(\Omega)$. Thus, by Proposition 2.9(*c*), we have

$$|v(x)| \leq \|Lg\|_{L^\infty(\Omega)} w_\mu(x),$$

for every $x \in \Omega$. Then, by Theorem 7.3

$$\lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} |v| = \lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} w_\mu = 0,$$

so that for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ with $Lg \in L^\infty(\Omega)$, and for every $\varepsilon > 0$ there exists $\varrho_\varepsilon > 0$ such that

$$\left| \int_{\overline{\Omega}} g(y)\, \mathcal{H}_\mu(x, dy) - g(x) \right| \leq |w_\mu(x)| \leq \varepsilon,$$

for every $x \in B_{\varrho_\varepsilon}(x_0)$. By Lemma 4.1, we obtain that (*b*) holds. Clearly (*b*) implies (*c*). Let us now suppose that (*c*) holds and let us consider the solution $\tilde{g}$ of (5.5). We already know that the solution $\tilde{u}$ to the problem (3.3) with datum $\tilde{g}$ is such that $\tilde{u} - \tilde{g} = w_\mu$. Since $\lim_{x \to x_0} \mathcal{H}_\mu(x, \cdot) \llcorner \Omega = \delta_{x_0}$, and $\tilde{g} \in H_0^1(\Omega) \cap C(\overline{\Omega})$, then

$$\lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} w_\mu = \lim_{\varrho \to 0^+} \operatorname*{ess\,sup}_{B_\varrho(x_0)} (\tilde{u} - \tilde{g}) = 0,$$

and, by Theorem 7.3, this means that x_0 is a Wiener point for the measure μ. Thus (*c*) implies (*a*).

As a direct consequence of Theorem 7.3 and Proposition 2.9(*c*), we obtain that, if x_0 is a Wiener point for μ, then for every $g \in H^1(\Omega) \cap C(\overline{\Omega})$ with $Lg \in L^\infty(\Omega)$ we have

$$|H_\mu(g)(x_0) - g(x_0)| \leq \|Lg\|_{L^\infty(\Omega)} w_\mu(x_0) = 0.$$

Thus, (*a*) implies (*d*). Conversely, if $\mathcal{H}_\mu(x_0, \cdot) = \delta_{x_0}$, then $w_\mu(x_0) = 0$, that is x_0 is a Wiener point for μ. As usual, it follows from the fact that $w_\mu = H_\mu(\tilde{g}) - \tilde{g}$, where $\tilde{g}$ is the solution of (5.5).

By (*d*), we have that $\mathcal{H}_\mu(x_0, \{x_0\}) = 1$, that is (*e*) holds.

Let us now suppose that $\mathcal{K}_\mu(x_0, \{x_0\}) = \alpha > 0$; we want to show that x_0 is a Wiener point. Let $R > 0$ be such that $B_R(x_0) = B_R \subset \Omega$, and, for every $0 < \varrho \leq R$, let z_ϱ be the μ-capacitary potential of B_ϱ in Ω relative to the operator L. By Proposition 7.2, in order to show that x_0 is a Wiener point it is enough to prove that $z_\varrho(x_0) > \alpha$ for every $0 < \varrho \leq R$. We consider a function $g_\varrho \in H_0^1(\Omega)$, $0 \leq g_\varrho \leq 1$ q.e. in Ω, such that $g_\varrho = 1$ q.e. in $B_{\varrho/2}$, and $g_\varrho = z_\varrho$ q.e. in $\Omega \backslash B_\varrho$. Let u_ϱ be the solution of the inhomogeneous relaxed Dirichlet problem corresponding to g_ϱ, that is

$$\begin{cases} u_\varrho - g_\varrho \in H_0^1(\Omega) \cap L^2(\Omega, \mu) \\ \langle L u_\varrho, \varphi \rangle + \int\limits_\Omega (u_\varrho - g_\varrho)\varphi \, d\mu = 0 , \qquad \forall \varphi \in H_0^1(\Omega) \cap L^2(\Omega, \mu) . \end{cases}$$

By the maximum principle, $0 \leq u_\varrho \leq 1$ q.e. in Ω, and then

$$\int\limits_{B_\varrho} [(u_\varrho - z_\varrho)^+]^2 dx \leq \int\limits_{B_\varrho \cap \{z_\varrho \leq u_\varrho\}} (1 - z_\varrho)^2, d\mu$$

which implies that the function $(u_\varrho - z_\varrho)^+$ belongs to $L^2(B_\varrho, \mu)$. Moreover $u_\varrho - z_\varrho = u_\varrho - g_\varrho$ q.e. in $\Omega \backslash B_\varrho$, so that $(u_\varrho - z_\varrho)^+$ belongs to $H_0^1(\Omega) \cap L^2(\Omega, \mu)$. If we put it as test function both in the equation solved by u_ϱ, and in the equation solved by z_ϱ, we obtain

$$\int\limits_{\{z_\varrho \leq u_\varrho\}} \sum_{i,j=1}^N a_{ij} D_j u_\varrho \cdot D_i(u_\varrho - z_\varrho) \, dx + \int\limits_{\{z_\varrho \leq u_\varrho\}} (u_\varrho - g_\varrho)(u_\varrho - z_\varrho) \, d\mu = 0 ,$$

and

$$\int\limits_{\{z_\varrho \leq u_\varrho\}} \sum_{i,j=1}^N a_{ij} D_j z_\varrho \cdot D_i(u_\varrho - z_\varrho) \, dx + \int\limits_{\{z_\varrho \leq u_\varrho\} \cap B_\varrho} (z_\varrho - 1)(u_\varrho - z_\varrho) \, d\mu = 0 .$$

Taking the difference between these two equations we obtain

$$\int\limits_{\{z_\varrho \leq u_\varrho\}} \sum_{i,j=1}^N a_{ij} D_j(u_\varrho - z_\varrho) \cdot D_i(u_\varrho - z_\varrho) \, dx +$$

$$+ \int\limits_{\{z_\varrho \leq u_\varrho\}} (u_\varrho - z_\varrho)^2 \, d\mu + \int\limits_{\{z_\varrho \leq u_\varrho\} \cap B_\varrho} (1 - g_\varrho)(u_\varrho - z_\varrho) \, d\mu = 0 .$$

Since each term of the sum is nonnegative, then we have that $u_\varrho \leq z_\varrho$

a.e. in Ω. By Theorem 5.7 we have that

$$z_\varrho(x) \geqslant u_\varrho(x) = \int_{\overline{\Omega}} g_\varrho(y)\,\mathcal{H}_\mu(x, dy) \geqslant \mathcal{H}_\mu(x, B_{\varrho/2}),$$

for almost every $x \in \Omega$. Finally, by Lemma 5.3, we get

$$z_\varrho(x_0) \geqslant \lim_{\varepsilon \to 0^+} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} z_\varrho(y)\,dy \geqslant$$

$$\geqslant \liminf_{\varepsilon \to 0^+} \frac{1}{|B_\varepsilon|} \int_{B_\varepsilon} u_\varrho(y)\,dy \geqslant \liminf_{\varepsilon \to 0^+} H_\mu^\varepsilon(x_0, B_{\varrho/2}) \geqslant \alpha\,,$$

for every $0 < \varrho \leqslant R$, which concludes the proof. ∎

Finally, we use the previous result in order to describe the mass of the μ-harmonic measures inside the sets of capacity zero.

THEOREM 7.6. *Fixed a bounded open set Ω' such that $\Omega \subset\subset \Omega'$, let B be a Borel subset of $\overline{\Omega}$ with* $\operatorname{cap}(B, \Omega') = 0$. *Then for every Wiener point $x \in \Omega$ we have $\mathcal{H}_\mu(x, B) = 1_B$, while $\mathcal{H}_\mu(x, B) = 0$ for every $x \in \Omega$ which is not a Wiener point.*

PROOF. By Theorem 7.5, $\mathcal{H}_\mu(x, \cdot) = \delta_x$ for every Wiener point $x \in \Omega$, so that the first assertion is obvious. Conversely, if $x \in \Omega$ is not a Wiener point, then, by Theorem 7.5(*c*), for every Borel subset B of Ω we have $\mathcal{H}_\mu(x, B) = \mathcal{H}_\mu(x, B \setminus \{x\})$. Thus, by Lemma 5.2, we obtain $\mathcal{H}_\mu(x, B) = 0$ for every B with capacity zero. ∎

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