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## Interior Differentiability of Weak Solutions to Parabolic Systems with Quadratic Growth Nonlinearities.

J. NAUMANN - J. WOLF

SUNTO - Sia  $u \in W_2^{1,0}(Q; \mathbb{R}^N) \cap C^{0,\gamma}(\bar{Q}; \mathbb{R}^N)$  ( $Q = \Omega \times (0, T)$ ,  $\Omega \subset \mathbb{R}^n$  aperto) una soluzione debole di un sistema nonlineare parabolico ad adamento quadratico. Utilizzando nuove stime sulle differenze di  $u$  rispetto a  $t \in (0, T)$ , dimostriamo che  $\nabla u \in L_{loc}^{A(1+\gamma/n)}(Q; \mathbb{R}^{nN})$ . Da questo risultato si deduce facilmente l'esistenza delle derivate seconde spaziale di  $u$  in  $L_{loc}^2(Q)$ .

### 1. - Introduction. Statement of the main result.

Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be a bounded open set, and let  $0 < T < +\infty$ . In the cylinder  $Q = \Omega \times (0, T)$  we consider the following system of nonlinear PDE's:

$$(1.1) \quad u_i^i - D_\alpha A_i^\alpha(\nabla u) = B_i(\nabla u) \quad (i = 1, \dots, N),$$

where

$$u = \{u^1, \dots, u^N\},$$

$$\nabla u = \{D_\alpha u^i\} (= \text{matrix of spatial derivatives}).$$

Throughout the whole paper, the functions  $A_i^\alpha$  and  $B_i$  are assumed to

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(<sup>1</sup>)  $\varphi_t = \partial\varphi/\partial t$ ,  $D_\alpha \varphi = \partial\varphi/\partial x_\alpha$  ( $\alpha = 1, \dots, n$ ). - In what follows, repeated Greek (resp. Latin) indices imply summation over  $1, \dots, n$  (resp.  $1, \dots, N$ ).

satisfy the following conditions:

$$(1.2) \quad A_i^\alpha, B_i \quad \text{continuous on } \mathbb{R}^{nN};$$

$$(1.3) \quad \begin{cases} (A_i^\alpha(\xi) - A_i^\alpha(\eta))(\xi_a^i - \eta_a^i) \geq \nu_0 |\xi - \eta|^2, & \forall \xi, \eta \in \mathbb{R}^{nN}, \\ (\nu_0 = \text{const} > 0); \end{cases}$$

$$(1.4) \quad \begin{cases} |A_i^\alpha(\xi) - A_i^\alpha(\eta)| \leq c_1 |\xi - \eta|, & \forall \xi, \eta \in \mathbb{R}^{nN}, \\ |B_i(\xi)| \leq c_2(1 + |\xi|^2), & \forall \xi \in \mathbb{R}^{nN}, \\ (c_1, c_2 = \text{const} < +\infty) \end{cases}$$

$$(\alpha = 1, \dots, n; i = 1, \dots, N).$$

REMARK. For the sake of technical simplicity and clarity of the presentation of our method, we assumed that  $A_i^\alpha, B_i$  depend on  $\xi (\in \mathbb{R}^{nN})$  only. Indeed, an inspection of the proofs below shows that our main result continues to hold for Carathéodory functions

$$A_i^\alpha(x, t, u, \xi), \quad B_i(x, t, u, \xi),$$

$$((x, t, u, \xi) \in \Omega \times (0, T) \times \mathbb{R}^N \times \mathbb{R}^{nN}; \alpha = 1, \dots, n, i = 1, \dots, N),$$

obeying (1.3) and (1.4) uniformly for all  $(x, t) \in Q$  and all  $|u| \leq M = \text{const}$  ( $c_1, c_2 = \text{const} < +\infty$  possibly depending on  $M$ ). This is readily seen by some additional elementary calculations of our arguments below. ■

Let  $W_p^1(\Omega)$  ( $1 \leq p < +\infty$ ) denote the usual Sobolev space. Given  $0 < \theta < 1$ , let

$$W_2^\theta(\Omega) := \left\{ v \in L^2(Q) : \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^2}{|x - y|^{n+2\theta}} dx dy < +\infty \right\}.$$

Define

$$W_2^{1,0}(Q) := \left\{ v \in L^2(Q) : \frac{\partial v}{\partial x_\alpha} \in L^2(Q) (\alpha = 1, \dots, n) \right\}.$$

Next, let  $1 \leq p < +\infty$  and  $-\infty < a < b < +\infty$ , and let  $X$  be a normed vector space with norm  $\|\cdot\|_X$ . Then  $L^p(a, b; X)$  denotes the vector space of all (classes of equivalent) Bochner measurable functions

$v: (a, b) \rightarrow X$  such that

$$\|v\|_{L^p(a, b; X)} := \left( \int_a^b \|v(t)\|_X^p dt \right)^{1/p} < +\infty.$$

In what follows, we identify the spaces  $L^p(0, T; L^p(\Omega))$  and  $L^p(Q)$  <sup>(2)</sup>.

Finally, set

$$L^p(Q; \mathbb{R}^N) := [L^p(Q)]^N \quad (1 \leq p \leq +\infty), \quad W_2^{1,0}(Q; \mathbb{R}^N) := [W_2^{1,0}(Q)]^N$$

etc. ■

We introduce the notion of weak solution to (1.1) regardless of whether or not the solution under consideration is subject to any boundary or initial condition.

Let (1.2) and (1.4) be satisfied. A vector function  $u \in W_2^{1,0}(Q; \mathbb{R}^N)$  is called a *weak solution* to (1.1) if

$$(1.5) \quad - \int_Q u^i \varphi_i^i dx dt + \int_Q A_i^\alpha(\nabla u) D_\alpha \varphi^i dx dt = \int_Q B_i(\nabla u) \varphi^i dx dt$$

for all  $\varphi \in W_2^1(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$  with  $\text{supp}(\varphi) \subset Q$ . ■

The local square integrability of the second order spatial derivatives of a weak solution to (1.1) can be easily proved by the method of difference quotient when

$$(1.6) \quad \nabla u \in L_{\text{loc}}^4(Q; \mathbb{R}^{nN})$$

is known.

By adapting the method from [2], it has been proved in [3] that for any weak solution  $u \in W_2^{1,0}(Q; \mathbb{R}^N) \cap C^\gamma(\bar{Q}; \mathbb{R}^N)$  ( $0 < \gamma < 1$ ) <sup>(3)</sup> there holds

$$(1.7) \quad \begin{cases} \nabla u \in L^{2(1+\theta)}(t_0, t_1; L^p(\Omega'; \mathbb{R}^{nN})), \\ \forall 0 < \theta < 1, \quad \forall 2 \leq p < \frac{2(1+\theta)n}{n-2\theta\gamma}, \\ (\Omega' \subset\subset \Omega \text{ } ^{(4)}, \quad 0 < t_0 < t_1 < T). \end{cases}$$

<sup>(2)</sup> This identification is possible by virtue of the linear isometry  $L^p(0, T; L^p(\Omega)) \cong L^p(Q)$ .

<sup>(3)</sup> A function  $v$  is in  $C^\gamma(\bar{Q})$  if there exists a constant  $K < +\infty$  (depending on  $v$ ) such that  $|v(x, t) - v(y, s)| \leq K(|x - y|^\gamma + |t - s|^{\gamma/2})$ ,  $\forall (x, t), (y, s) \in \bar{Q}$ .

<sup>(4)</sup>  $\Omega' \subset\subset \Omega$  means:  $\Omega'$  open,  $\bar{\Omega}' \subset \Omega$ .

These techniques have been further developed in [4] (however, without achieving (1.6)).

In [5], the integrability property (1.6) has been proved under additional differentiability properties on  $B_i$  and the (generally non verifiable and unpleasant) condition  $1/2 < \gamma < 1$  on the Hölder exponent  $\gamma$  of the weak solution under consideration. The aim of the present paper is to remove these two restrictions. Our main result is the following

**THEOREM.** *Let (1.2)-(1.4) be satisfied. Let  $u \in W_2^{1,0}(\bar{Q}; \mathbb{R}^N) \cap C^\gamma(\bar{Q}; \mathbb{R}^N)$  ( $0 < \gamma < 1$ ) be a weak solution to (1.1).*

*Then, for any  $\Omega' \subset\subset \Omega$  and  $0 < t_0 < t_1 < T$ ,*

$$(1.8) \quad \nabla u \in L^4(t_0, t_1; L^s(\Omega'; \mathbb{R}^{nN})) \quad \forall 4 \leq s < \frac{4n}{n-2\gamma},$$

$$(1.9) \quad \nabla u \in L^\sigma(\Omega' \times (t_0, t_1); \mathbb{R}^{nN}) \quad \forall 4 \leq \sigma < 4\left(1 + \frac{\gamma}{n}\right).$$

Obviously, the growth condition (1.4) on  $B_i$ , and (1.9) imply  $B_i(\nabla u) \in L_{loc}^{2(1+\gamma/n)}(Q)$  ( $i = 1, \dots, N$ ). Then the method of difference quotient gives straightforwardly:

**COROLLARY 1.** *Let (1.2)-(1.4) be satisfied. Let  $u \in W_2^{1,0}(\bar{Q}; \mathbb{R}^N) \cap C^\gamma(\bar{Q}; \mathbb{R}^N)$  ( $0 < \gamma < 1$ ) be a weak solution to (1.1). Then:*

$$(1.10) \quad D_\alpha D_\beta u^i \in L_{loc}^2(Q) \quad (\alpha, \beta = 1, \dots, n; i = 1, \dots, N). \quad \blacksquare$$

We impose the following stronger conditions upon  $A_i^\alpha$ :

$$(1.11) \quad A_i^\alpha \in C^1(\mathbb{R}^{nN}),$$

$$(1.12) \quad \frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi) \eta_\alpha^i \eta_\beta^j \geq \nu_0 |\eta|^2, \quad \forall \xi, \eta \in \mathbb{R}^{nN} (\nu_0 = \text{const} > 0),$$

$$(1.13) \quad \left| \frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi) \right| \leq c_3 = \text{const} < +\infty, \quad \forall \xi \in \mathbb{R}^{nN}$$

( $\alpha, \beta = 1, \dots, n; i, j = 1, \dots, N$ ). Clearly, (1.11)-(1.13) imply the respective conditions upon  $A_i^\alpha$  in (1.3) and (1.4).

**COROLLARY 2.** *Let  $A_i^\alpha$  and  $B_i$  satisfy (1.11)-(1.13) and (1.2), (1.4), respectively.*

*Let  $u \in W_2^{1,0}(\bar{Q}; \mathbb{R}^N) \cap C^\gamma(\bar{Q}; \mathbb{R}^N)$  ( $0 < \gamma < 1$ ) be a weak solution to (1.1). Then:*

$$(1.14) \quad u_i^i \in L_{loc}^2(Q) \quad (i = 1, \dots, N).$$

Indeed, observing (1.10) we obtain from (1.5) by integration by parts

$$\int_Q u^i \frac{\partial \varphi^i}{\partial t} dx z dt = - \int_Q \left( \frac{\partial A_i^\alpha}{\partial \xi_\beta^j} (\nabla u) D_\alpha D_\beta u^j + B_i(\nabla u) \right) \varphi^i dx dt$$

for all  $\varphi \in C_c^\infty(Q; \mathbb{R}^N)$ . Whence (1.14). ■

The technical preliminaries for the proof of our theorem are presented in Section 2. In Section 3 we derive  $L^2$ -estimates on appropriate differences of  $u$  with respect to  $t$ . These estimates are substantial refinements of those in [5] and play the key role for the proof of our theorem. This proof will be given in Section 4. It makes use of (1.7) and our results from Section 3, combined with an interpolation argument.

REMARK. An inspection of the proof of our theorem (as well as the proof in [3]) shows that (1.8), (1.9) remain true for any weak solution  $u \in W_2^{1,0}(Q; \mathbb{R}^N)$  (resp.  $u \in W_2^{1,0}(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$ ) in the more general situation of [3] such that  $u \in C^r(\overline{Q'}; \mathbb{R}^N)$  for any subcylinder  $Q' \subset \subset \overline{Q'} \subset Q$ . This local character of Hölder continuity is well-known from the theory of partial regularity of weak solutions to nonlinear elliptic and parabolic systems. ■

## 2. – Preliminaries.

In this section, we present some elementary technical results which will be used in our subsequent discussion.

2.1. *Change of variables.* Let  $t_0 \in (0, T), f \in L^2(Q)$ .

1) Let  $\lambda \in (-t_0, 0)$ . Then:

$$(2.1) \quad \int_Q f(x, t) \left( \int_t^{t-\lambda} g(x, s) ds \right) dx dt = \int_{t_0}^T \int_\Omega \left( \int_{t+\lambda}^t f(x, s) ds \right) g(x, t) dx dt,$$

$$\forall g \in L^2(\Omega \times \mathbb{R}), \quad g = 0 \text{ a.e. in } \Omega \times ((-\infty, t_0) \cup (T, +\infty)).$$

2) Let  $\lambda \in (0, T - t_0)$ . Then:

$$(2.2) \quad \int_Q f(x, t) \left( \int_{t-\lambda}^t g(x, s) ds \right) dx dt = \int_0^{t_0} \int_\Omega \left( \int_t^{t+\lambda} f(x, s) ds \right) g(x, t) dx dt,$$

$$\forall g \in L^2(\Omega \times \mathbb{R}), \quad g = 0 \text{ a.e. in } \Omega \times ((-\infty, 0) \cup (t_0, +\infty)).$$

Statements 1) and 2) can be easily proved by combining the change of variables theorem and Fubini's theorem. ■

**2.2. Steklov mean.** Let  $f \in L^1(\Omega \times \mathbb{R})$ ,  $f = 0$  a.e. in  $\Omega \times ((-\infty, 0) \cup (T, +\infty))$ . The Steklov mean of  $f$  (with respect to  $t$ ) is defined by

$$f_\lambda(x, t) = -\frac{1}{\lambda} \int_{t+\lambda}^t f(x, s) ds, \quad \text{for a.a. } (x, t) \in Q (\lambda < 0),$$

$$f_\lambda(x, t) = \frac{1}{\lambda} \int_t^{t+\lambda} f(x, s) ds \quad \text{for a.a. } (x, t) \in Q (\lambda > 0).$$

Let  $f \in L^p(\Omega \times \mathbb{R})$  ( $1 \leq p < +\infty$ ),  $f = 0$  a.e. in  $\Omega \times ((-\infty, 0) \cup (T, +\infty))$ . By Hölder's inequality and Fubini's theorem,

$$(2.3) \quad \int_Q |f_\lambda|^p dx dt \leq \int_Q |f|^p dx dt, \quad \forall \lambda \in \mathbb{R} (\lambda \neq 0).$$

Observing the density of  $C(\overline{Q})$  in  $L^p(Q)$  we obtain  $f_\lambda \rightarrow f$  in  $L^p(Q)$  as  $\lambda \rightarrow 0$ .

The function  $f_\lambda$  possesses a weak derivative  $\partial f_\lambda / \partial t \in L^p(Q)$ ; more precisely, there holds

$$(2.4) \quad \begin{cases} \frac{\partial f_\lambda}{\partial t}(x, t) = -\frac{1}{\lambda} (f(x, t) - f(x, t + \lambda)) & \text{for a.a. } (x, t) \in Q, \quad \forall \lambda < 0, \\ \frac{\partial f_\lambda}{\partial t}(x, t) = \frac{1}{\lambda} (f(x, t + \lambda) - f(x, t)) & \text{for a.a. } (x, t) \in Q, \quad \forall \lambda > 0. \end{cases}$$

Indeed, let  $\varphi \in C_c^\infty(Q)$ , and assume  $\lambda > 0$ . We have

$$\begin{aligned} \int_Q f_\lambda(x, t)(\varphi(x, t - h) - \varphi(x, t)) dx dt &= \\ &= \frac{1}{\lambda} \int_Q \left( \int_{t+\lambda}^{t+\lambda+h} f(x, s) ds - \int_t^{t+h} f(x, s) ds \right) \varphi(x, t) dx dt \\ &= -\frac{h}{\lambda} \int_Q (f_h(x, t + \lambda) - f_h(x, t)) \varphi(x, t) dx dt \end{aligned}$$

for all  $0 < h < \lambda$ . An analogous reasoning is true when  $\lambda < 0$ . Whence (2.4).

In addition, assume  $D_\alpha f \in L^p(\Omega \times \mathbb{R})$  ( $\alpha = 1, \dots, n$ ). Then, for any  $\lambda \in \mathbb{R}$  ( $\lambda \neq 0$ ),

$$(2.5) \quad (D_\alpha f_\lambda)(x, t) = (D_\alpha f)_\lambda(x, t) \quad \text{for a.a. } (x, t) \in Q. \quad \blacksquare$$

**2.3. Differences.** Let  $f \in L^1(\Omega \times \mathbb{R})$ ,  $f = 0$  a.e. in  $\Omega \times ((-\infty, 0) \cup (T, +\infty))$ . For any  $h \in \mathbb{R}$ , define

$$(\Delta_h f)(x, t) = f(x, t + h) - f(x, t) \quad \text{for a.a. } (x, t) \in Q.$$

The Steklov mean (with  $\lambda = h$ ) of the function  $\Delta_h f$  is

$$(2.6) \quad (\Delta_h f)_h = \Delta_h f_h, \quad (\Delta_{-h} f)_h = \Delta_{-h} f_h \quad \text{a.e. in } Q.$$

Next, let  $f \in L^2(\Omega \times \mathbb{R})$ ,  $f = 0$  a.e. in  $\Omega \times ((-\infty, 0) \cup (T, +\infty))$ . Let  $\tau \in C_c^1(\mathbb{R})$ . An elementary calculation gives

$$(2.7) \quad |\Delta_h(\Delta_{-h}(\tau f_h - (\tau f)_h))| \leq 2\sqrt{3|h|} \max_{\mathbb{R}} |\tau'| \left( \int_{t-2|h|}^{t+2|h|} f^2 ds \right)^{1/2}$$

for a.a.  $(x, t) \in Q$  and all  $h \in \mathbb{R}$  ( $h \neq 0$ ).

Finally, let  $g \in L^2(\Omega \times \mathbb{R})$ ,  $g = 0$  a.e. in  $\Omega \times ((-\infty, 0) \cup (T, +\infty))$ .



Let  $0 < t_0 < t_1 < T$  and let  $\tau \in C(\mathbb{R})$ ,  $\text{supp } (\tau) \subset (t_0, t_1)$ . Then it is easily seen that

$$(2.8) \quad \int_Q \tau f(\Delta_h g) \, dx \, dt = \int_Q (\Delta_{-h}(\tau f)) g \, dx \, dt,$$

$$(2.9) \quad \int_Q f \Delta_{-h}(\tau \Delta_h(\Delta_{-h}(\tau \Delta_h f))) \, dx \, dt = \int_Q (\Delta_{-h}(\tau \Delta_h f))^2 \, dx \, dt$$

for all  $|h| < \min\{t_0, T - t_1\}$ . ■

### 3. - Difference estimates.

Let  $u \in W_2^{1,0}(Q; \mathbb{R}^N)$  be a weak solution to (1.1). Let  $\Omega' \subset \subset \Omega$  and  $0 < t_0 < t_1 < T$  be arbitrary. We localize (1.5) with respect to  $t$  as follows:

$$(3.1) \quad \int_{\Omega'} \frac{\partial u_\lambda^i}{\partial t}(x, t) \psi^i(x) \, dx + \int_{\Omega'} (A_i^\alpha(\nabla u))_\lambda(x, t) D_\alpha \psi^i(x) \, dx = \\ = \int_{\Omega'} (B_i(\nabla u))_\lambda(x, t) \psi^i(x) \, dx,$$

for a.a.  $t \in (t_0, t_1)$ ,  $\forall 0 < |\lambda| < \min\{t_0, T - t_1\}$ ,

$\forall \psi \in W_2^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ ,  $\psi = 0$  a.e. in  $\Omega \setminus \Omega'$

(notice that the exceptional set in  $(t_0, t_1)$  depends neither on  $\psi$  nor on  $\lambda$ ).

To prove (3.1), we first assume  $0 < \lambda < T - t_1$ . Let  $\varphi \in W_2^{1,0}(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$  have its support in  $\Omega \times (0, t_1)$ . We extend  $\varphi$  by zero onto  $\Omega \times ((-\infty, 0) \cup (t_1, +\infty))$  and denote this extension again by  $\varphi$ . Then the function

$$\varphi_\lambda(x, t) = \frac{1}{\lambda} \int_{t-\lambda}^t \varphi(x, s) \, ds \quad \text{for a.a. } (x, t) \in Q \ (0 < \lambda < T - t_1)$$

is admissible in (1.5). Observing (2.2) (with  $t_1$  in place of  $t_0$ ) and (2.4),

(2.5) we obtain

$$(3.2) \quad \int_0^{t_1} \int_{\Omega} \frac{\partial u_\lambda^i}{\partial t} \varphi^i dx dt + \int_0^{t_1} \int_{\Omega} (A_i^\alpha)_\lambda D_\alpha \varphi^i dx dt = \int_0^{t_1} \int_{\Omega} (B_i)_\lambda \varphi^i dx dt .$$

Next, fix  $\Omega^*$  such that  $\Omega' \subset \subset \Omega^* \subset \subset \Omega$  and  $\partial\Omega^*$  is of class  $C^1$ . Let  $\psi \in \overset{\circ}{W}_p^1(\Omega^*; \mathbb{R}^N)$  <sup>(5)</sup> ( $n < p < +\infty$ ). We extend  $\psi$  by zero onto  $\Omega \setminus \Omega^*$  and keep the notation  $\psi$  for this extension. Thus  $\psi \in W_p^1(\Omega; \mathbb{R}^N)$ . Finally, let  $\eta \in C_c^1((0, t_1))$ . Inserting  $\varphi(x, t) = \psi(x)\eta(t)$  ( $(x, t) \in Q$ ) in (3.2) we get by a standard argument

$$(3.1') \quad \int_{\Omega^*} \frac{\partial u_\lambda^i}{\partial t} \psi^i dx + \int_{\Omega^*} (A_i^\alpha)_\lambda D_\alpha \psi^i dx = \int_{\Omega^*} (B_i)_\lambda \psi^i dx$$

for a.a.  $t \in (t_0, t_1)$ , where the exceptional set possibly depends on  $\psi$  and  $\lambda$ . Observing the separability of  $\overset{\circ}{W}_p^1(\Omega^*)$  and  $\mathbb{R}$  we obtain (3.1') for all  $\psi \in \overset{\circ}{W}_p^1(\Omega^*; \mathbb{R}^N)$  and a.a.  $t \in (t_0, t_1)$  with an exceptional set independent of  $\psi$  and  $\lambda$ .

Let  $\psi \in W_2^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ ,  $\psi = 0$  a.e. in  $\Omega \setminus \Omega'$ . Let  $\psi_m$  ( $m = 1, 2, \dots$ ) denote the standard mollification of  $\psi$ . We have  $\psi_m \in C_c^\infty(\Omega^*; \mathbb{R}^N)$  for sufficiently large  $m$ , and  $\max_{\Omega} |\psi_m| \leq \text{ess sup}_{\Omega} |\psi|$  for all  $m$ . Inserting  $\psi = \psi_m$  into (3.1') and letting  $m \rightarrow \infty$  gives (3.1).

Assume  $-t_0 < \lambda < 0$ . Let  $\varphi \in W_2^{1,0}(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$  have its support in  $\Omega \times (t_0, T)$ . Then we insert the test function

$$\varphi_\lambda(x, t) = -\frac{1}{\lambda} \int_t^{t-\lambda} \varphi(x, s) ds \quad \text{for a.a. } (x, t) \in Q \quad (-t_0 < \lambda < 0)$$

into (1.5) and make use of (2.1) and (2.4), (2.5). By an analogous argument as above we obtain (3.1). ■

Let  $\Omega' \subset \subset \Omega$  and  $0 < t_0 < t_1 < T$ . The localized form (3.1) of the notion of weak solution to (1.1) is the point of departure for proving the following

LEMMA 1. *Let  $u \in W_2^{1,0}(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$  be a weak solution*

<sup>(5)</sup>  $\overset{\circ}{W}_p^1(\Omega^*) = \{v \in W_p^1(\Omega^*); v = 0 \text{ a.e. on } \partial\Omega^*\}$ .

to (1.1). Then:

$$(3.3) \quad \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h u|^2 dx dt \leq C_1 (1 + \|u\|_{L^\infty(Q; \mathbb{R}^N)}) \int_Q (1 + |\nabla u|^2) dx dt |h|$$

$$|h| < \min\{t_0, T - t_1\},$$

with  $C_1 = \text{const}$  not depending on  $h$ .

PROOF. Let  $\Omega' \subset \subset \Omega'' \subset \subset \Omega$ , and let  $\zeta \in C_c^\infty(\Omega'')$  be a cut-off function such that  $0 \leq \zeta \leq 1$  in  $\Omega$ ,  $\zeta \equiv 1$  on  $\Omega'$ .

Let  $0 < h < T - t_1$ . Setting  $\lambda = h$ , (2.4) gives

$$\frac{\partial u_h}{\partial t}(x, t) = \frac{1}{h} (\Delta_h u)(x, t) \quad \text{for a.a. } (x, t) \in \Omega \times (0, t_1),$$

and (3.1) (with  $\Omega''$  in place of  $\Omega'$ ) implies

$$(3.4) \quad \frac{1}{h} \int_{\Omega''} (\Delta_h u^i)(x, t) \psi^i(x) dx + \int_{\Omega''} (A_i^\alpha)_h D_\alpha \psi^i dx = \int_{\Omega''} (B_i)_h \psi^i dx$$

for a.a.  $t \in (t_0, t_1)$  and all  $\psi \in W_2^1(\Omega; \mathbb{R}^N) \cap L^\infty(\Omega; \mathbb{R}^N)$ ,  $\psi = 0$  a.e. in  $\Omega \setminus \Omega''$ . The function  $\psi(x) = (\Delta_h u)(x, t) \zeta(x)$  ( $t \in (t_0, t_1)$ ) being admissible in (3.4) it follows that

$$(3.5) \quad \frac{1}{h} \int_{t_0}^{t_1} \int_{\Omega''} |\Delta_h u|^2 \zeta dx dt =$$

$$= - \int_{t_0}^{t_1} \int_{\Omega''} (A_i^\alpha)_h [(\Delta_h D_\alpha u^i) \zeta + (\Delta_h u^i) D_\alpha \zeta] dx dt + \int_{t_0}^{t_1} \int_{\Omega''} (B_i)_h (\Delta_h u^i) \zeta dx dt.$$

By (1.4) and (2.3),

$$- \int_{t_0}^{t_1} \int_{\Omega} (A_i^\alpha)_h (\Delta_h D_\alpha u^i) \zeta dx dt \leq$$

$$\leq \left( \int_Q (A_i^\alpha)^2 dx dt \right)^{1/2} \left( \int_{t_0}^{t_1} \int_{\Omega''} (D_\alpha u^i(x, t+h) - D_\alpha u^i(x, t))^2 dx dt \right)^{1/2} \leq$$

$$\leq c \int_Q (1 + |\nabla u|^2) dx dt \quad (6).$$

(6) By  $c$  we denote positive constants which may change their numerical value from line to line, but do not depend on  $h$ .

The estimation of the other terms on the right of (3.5) is readily seen by the same arguments. Whence (3.3).

Let  $-t_0 < h < 0$ . From (2.4) it follows that

$$\frac{\partial u_h}{\partial t}(x, t) = \frac{1}{h} (\Delta_h u)(x, t) \quad \text{for a.a. } (x, t) \in \Omega \times (t_0, T).$$

Then (3.1) gives

$$\begin{aligned} -\frac{1}{h} \int_{t_0}^{t_1} \int_{\Omega''} |\Delta_h u|^2 \zeta \, dx \, dt &= \\ &= \int_{t_0}^{t_1} \int_{\Omega''} (A_i^\alpha)_h [(\Delta_h D_\alpha u^i) \zeta + (\Delta_h u^i) D_\alpha \zeta] \, dx \, dt - \int_{t_0}^{t_1} \int_{\Omega''} (B_i)_h (\Delta_h u^i) \zeta \, dx \, dt. \end{aligned}$$

Inequality (3.3) is now easily seen. ■

As above, let  $\Omega' \subset \subset \Omega$  and  $0 < t_0 < t_1 < T$ . We fix cut-off functions  $\zeta \in C^\infty(\mathbb{R}^n)$ ,  $\text{supp}(\zeta) \subset \Omega'$ ,  $0 \leq \zeta \leq 1$  in  $\mathbb{R}^n$ , and  $\tau \in C^\infty(\mathbb{R})$ ,  $\text{supp}(\tau) \subset (t_0, t_1)$ ,  $0 \leq \tau \leq 1$  in  $\mathbb{R}$ . The following result plays the key role for the proof of our theorem.

LEMMA 2. *Let  $u \in W_2^{1,0}(Q; \mathbb{R}^N) \cap L^\infty(Q; \mathbb{R}^N)$  be a weak solution to (1.1). Assume*

$$(3.6) \quad \nabla u \in L^p(\Omega' \times (t_0, t_1); \mathbb{R}^{nN}) \quad (2 < p < 4).$$

Then:

$$\begin{aligned} (3.7) \quad &\int_{t_0}^{t_1} \int_{\Omega'} |\Delta_{-h}(\tau^2 \Delta_h u)|^2 \zeta^2 \, dx \, dt \leq \\ &\leq C_2 \left( |h|^{p/2} + \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 \, dx \, dt |h| \right) \forall |h| < \min \left\{ 1, \frac{t_0}{4}, \frac{T-t_1}{4} \right\}, \end{aligned}$$

with  $C_2 = \text{const}$  not depending on  $h$ .

PROOF. We extend  $u$  by zero onto  $\Omega \times ((-\infty, 0) \cup (T, +\infty))$ . Let  $0 < |h| < \min \{1, (t_0/4), (T-t_1)/4\}$ . The function

$$\varphi = \Delta_{-h}(\tau^2 \Delta_h(\Delta_{-h}(\tau^2 u_h))) \zeta^2$$

is admissible in (1.5). By (2.4),

$$\begin{aligned} \varphi_t &= 2\Delta_{-h}(\tau\tau' \Delta_h(\Delta_{-h}(\tau^2 u_h))) \xi^2 + \\ &\quad + 2\Delta_{-h}(\tau^2 \Delta_h(\Delta_{-h}(\tau\tau' u_h))) \xi^2 + \frac{1}{h} \Delta_{-h}(\tau^2 \Delta_h(\Delta_{-h}(\tau^2 \Delta_h u))) \xi^2 \end{aligned}$$

a.e. in  $Q$ . We insert  $\varphi$  into (1.5) (and multiply by  $-1$  in the case  $h > 0$ ). Using (2.8) and (2.9) it follows that

$$\begin{aligned} (3.8) \quad & \frac{1}{|h|} \int_Q |\Delta_{-h}(\tau^2 \Delta_h u)|^2 \xi^2 dx dt = \\ & = \pm 2 \int_Q (\Delta_h u^i) \tau\tau' \Delta_h(\Delta_{-h}(\tau^2 u_h^i)) \xi^2 dx dt \pm \\ & \pm 2 \int_Q (\Delta_h u^i) \tau^2 \Delta_h(\Delta_{-h}(\tau\tau' u_h^i)) \xi^2 dx dt \mp \\ & \mp \int_Q (\Delta_h A_i^\alpha(\nabla u)) \tau^2 \Delta_h(\Delta_{-h}(\tau^2 D_\alpha u_h^i)) \xi^2 dx dt \mp \\ & \mp 2 \int_Q ((\Delta_h A_i^\alpha(\nabla u)) \tau^2 \Delta_h(\Delta_{-h}(\tau^2 u_h^i))) \xi D_\alpha \xi dx dt \pm \\ & \pm \int_Q (\Delta_h B_i(\nabla u)) \tau^2 \Delta_h(\Delta_{-h}(\tau^2 u_h^i)) \xi^2 dx dt = I_1 + \dots + I_5. \end{aligned}$$

We estimate the integrals  $I_1, \dots, I_5$ . First, we have

$$\begin{aligned} I_1 &= \pm 2 \int_{t_0}^{t_1} \int_{\Omega'} (\Delta_h u^i) \tau\tau' \Delta_h(\Delta_{-h}(\tau^2 u_h^i)) \xi^2 dx dt \leq \\ & \leq \max_{\mathbb{R}} |\tau'| \left( \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h u|^2 dx dt \right)^{1/2} \left( \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h(\Delta_{-h}(\tau^2 u_h))|^2 dx dt \right)^{1/2}. \end{aligned}$$

Here the first integral on the right can be bounded by using (3.41). Clearly,  $\Delta_h(\Delta_{-h}(\cdot)) = \Delta_{-h}(\Delta_h(\cdot))$ . Thus, by (2.6),

$$(3.9) \quad \Delta_h(\Delta_{-h}(\tau^2 u_h)) = (\Delta_{-h}(\Delta_h(\tau^2 u)))_h + \Delta_h(\Delta_{-h}(\tau^2 u_h - (\tau^2 u)_h)).$$

Observing (2.3) and (2.7) we obtain

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h(\Delta_{-h}(\tau^2 u_h))|^2 dx dt &\leq \\ &\leq 2 \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_{-h}(\Delta_h(\tau^2 u))|^2 dx dt + c \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 h^2 \leq \\ &\leq 16 \int_{3t_0/4}^{(T+3t_1)/4} \int_{\Omega'} |\Delta_h u|^2 dx dt + c \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 h^2, \end{aligned}$$

where the last integral can be bounded again by using (3.3) (with  $3t_0/4$  and  $(T + 3t_1)/4$  in place of  $t_0$  and  $t_1$ , respectively). Hence

$$I_1 \leq c \left\{ \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 + (1 + \|u\|_{L^\infty(Q; \mathbb{R}^N)}) \int_Q (1 + |\nabla u|^2) dx dt \right\} |h|.$$

From the structure of  $I_2$  it is easily seen that this integral can be estimated by the same bound as  $I_1$ .

To estimate  $I_3$  we make use of (1.4):

$$\begin{aligned} I_3 &\leq \left( \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 dx dt \right)^{1/2} \times \\ &\quad \times \left( \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h(\Delta_{-h}(\tau^2 \nabla u_h))|^2 \zeta^2 dx dt \right)^{1/2}. \end{aligned}$$

Again combining (2.6) and (2.3), (2.7) (cf. (3.9)) it follows that

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h(\Delta_{-h}(\tau^2 \nabla u_h))|^2 \zeta^2 dx dt &\leq \\ &\leq 2 \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_{-h}(\Delta_h(\tau^2 \nabla u))|^2 \zeta^2 dx dt + c \int_Q |\nabla u|^2 dx dt |h| \quad (7) \leq \\ &\leq 16 \int_{3t_0/4}^{(T+3t_1)/4} \int_{\Omega'} |\Delta_h(\tau^2 \nabla u)|^2 \zeta^2 dx dt + c \int_Q |\nabla u|^2 dx dt |h| \leq \\ &\leq 32 \int_{t_0/2}^{(T+t_1)/2} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 dx dt + c \int_Q |\nabla u|^2 dx dt |h|, \end{aligned}$$

(7) Note that  $f_\lambda \zeta = (f\zeta)_\lambda$ .

and thus

$$I_3 \leq c \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 dx dt + c \int_Q |\nabla u|^2 dx dt |h|.$$

The estimation of  $I_4$  parallels the one of  $I_1$  and  $I_3$ :

$$I_4 \leq c \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 dx dt + c \left\{ \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 + (1 + \|u\|_{L^\infty(Q; \mathbb{R}^N)}) \int_Q (1 + |\nabla u|^2) dx dt \right\} |h|.$$

Finally, the estimation of  $I_5$  makes essential use of (3.6). First, by (1.4) and Hölder's inequality,

$$I_5 \leq c \|u\|_{L^\infty(Q; \mathbb{R}^N)}^{(4-p)/(p-2)} \left( \int_{t_0}^{t_1} \int_{\Omega'} (1 + |\nabla u|^p) dx dt \right)^{2/p} \times \left( \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h (\Delta_{-h}(\tau^2 u_h))|^2 \zeta^2 dx dt \right)^{(p-2)/p}.$$

Next, using once more (3.9) and observing that

$$\begin{aligned} (\Delta_{-h}(\Delta_h(\tau^2 u)))(x, t) &= \\ &= (\Delta_{-h}(\tau^2 \Delta_h u))(x, t) - (\Delta_{-h} \tau^2)(t)u(x, t) - (\Delta_h \tau^2)(t)u(x, t+h) \end{aligned}$$

(for a.a.  $(x, t) \in Q$ ), we find

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\Omega'} |\Delta_h (\Delta_{-h}(\tau^2 u_h))|^2 \zeta^2 dx dt &\leq \\ &\leq 2 \int_Q |\Delta_{-h}(\Delta_h(\tau^2 u))|^2 \zeta^2 dx dt + c \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 h^2 \leq \\ &\leq 4 \int_Q |\Delta_{-h}(\tau^2 \Delta_h u)|^2 \zeta^2 dx dt + c \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 h^2. \end{aligned}$$

Thus,

$$I_5 \leq \frac{1}{2|h|} \int_Q |\Delta_{-h}(\tau^2 \Delta_h u)|^2 \xi^2 dx dt + \\ + c \left\{ \|u\|_{L^\infty(Q; \mathbb{R}^N)}^2 |h| + \left( \int_{t_0}^{t_1} \int_{\Omega'} (1 + |\nabla u|^p) dx dt \right)^{(p-2)/p} |h|^{(p-2)/2} \right\}.$$

Inserting the estimates on  $I_1, \dots, I_5$  into (3.8) and multiplying by  $|h|$  gives (3.7). ■

#### 4. - Proof of the Theorem.

We divide the proof of the theorem into four steps.

1) Let  $0 < \theta < 1$  and  $2 < q < 2(1 + \theta)n/(n - 2\theta\gamma)$ . From [3] it follows that  $\nabla u \in L^{2(1+\theta)}(t_0, T; L^q(\Omega'; \mathbb{R}^{nN}))$  ( $0 < t_0 < T$ ) and

$$(4.1) \quad \int_{t_0}^T \|\nabla u\|_{L^{2(1+\theta)}(\Omega'; \mathbb{R}^{nN})}^2 dt \leq c \left( 1 + \int_Q |\nabla u|^2 dx dt \right)$$

(cf. also [5; p. 60]).

In particular,  $q = 2(1 + \theta)$  gives

$$(4.2) \quad \nabla u \in L^{2(1+\theta)}(\Omega' \times (t_0, T); \mathbb{R}^{nN}) \quad \forall 0 < \theta < 1.$$

Whence (3.52). ■

2) Let  $\Omega' \subset \subset \Omega$  and  $0 < t_0 < t'_0 < t'_1 < t_1 < T$ , and let  $\zeta \in C^\infty(\mathbb{R}^n)$ ,  $\tau \in C^\infty(\mathbb{R})$  be cut-off functions such that  $\text{supp}(\zeta) \subset \Omega'$ ,  $0 \leq \zeta \leq 1$  in  $\mathbb{R}^n$  and  $\text{supp}(\tau) \subset (t'_0, t'_1)$ ,  $0 \leq \tau \leq 1$  in  $\mathbb{R}$ . Set

$$h_0 = \frac{1}{2} \min \left\{ 1, \frac{t_0}{4}, \frac{T - t_1}{4}, t'_0 - t_0, t_1 - t'_1 \right\}.$$

Let  $0 < |h| \leq h_0$ . We consider (3.1) for a.a.  $t \in (t'_0, t'_1)$  and  $0 < |\lambda| \leq |h|$ , form the difference  $\Delta_h$  and insert the test function  $\psi =$



$= (\Delta_h u_\lambda) \zeta^2 \tau^2$ . Integration over the interval  $(t'_0, t'_1)$  then gives

$$\begin{aligned} & \int_{t'_0}^{t'_1} \int_{\Omega'} (\Delta_h (A_i^\alpha)_\lambda) (\Delta_h D_\alpha u_\lambda^i) \zeta^2 \tau^2 \, dx \, dt \leq \\ & \leq -2 \int_{t'_0}^{t'_1} \int_{\Omega'} (\Delta_h (A_i^\alpha)_\lambda) (\Delta_h u_\lambda^i) \zeta D_\alpha \zeta \tau^2 \, dx \, dt + \\ & + \int_{t'_0}^{t'_1} \int_{\Omega'} (\Delta_h (B_i)_\lambda) (\Delta_h u_\lambda^i) \zeta^2 \tau^2 \, dx \, dt + \int_{t'_0}^{t'_1} \int_{\Omega'} |\Delta_h u_\lambda|^2 \zeta^2 \tau \tau' \, dx \, dt. \end{aligned}$$

We let  $\lambda \rightarrow 0$  and make use of (1.3). It follows that

$$\begin{aligned} (4.3) \quad & \nu_0 \int_{t'_0}^{t'_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 \, dx \, dt \leq \\ & \leq -2 \int_{t'_0}^{t'_1} \int_{\Omega'} (\Delta_h A_i^\alpha) (\Delta_h u^i) \zeta D_\alpha \zeta \tau^2 \, dx \, dt + \int_{t'_0}^{t'_1} \int_{\Omega'} (\Delta_h B_i) (\Delta_h u^i) \zeta^2 \tau^2 \, dx \, dt \times \\ & \quad \times \int_{t'_0}^{t'_1} \int_{\Omega'} |\Delta_h u|^2 \zeta^2 \tau \tau' \, dx \, dt = I_1 + I_2 + I_3. \end{aligned}$$

By (1.4)

$$I_1 \leq \frac{\nu_0}{4} \int_{t'_0}^{t'_1} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 \, dx \, dt + c \int_{t'_0}^{t'_1} \int_{\Omega'} |\Delta_h u|^2 \, dx \, dt,$$

where the second integral on the right can be estimated by the aid of Lemma 1.

The estimation of  $I_2$  by an appropriate power of  $|h|$  is the crucial point. To this end, we combine (4.2) and Lemma 2 above. Observing

(1.4) and (2.8) we obtain

$$\begin{aligned}
 I_2 &= \int_{t_0'}^{t_1'} \int_{\Omega'} B_i \Delta_{-h} (\tau^2 \Delta_h u^i) \zeta^2 dx dt \leq \\
 &\leq c \|u\|_{C(\bar{Q}; \mathbb{R}^N)}^{(4-p)/p} \left( \int_{t_0'}^{t_1'} \int_{\Omega'} (1 + |\nabla u|^p) dx dt \right)^{2/p} \times \\
 &\times \left( \int_{t_0'}^{t_1'} \int_{\Omega'} |\Delta_{-h} (\tau^2 \Delta_h u)|^2 \zeta^2 dx dt \right)^{(p-2)/p} \leq \frac{\nu_0}{4} \int_{t_0'}^{t_1'} \int_{\Omega'} |\Delta_h \nabla u|^2 \zeta^2 \tau^2 dx dt + \\
 &\quad + c \left( 1 + \|u\|_{C(\bar{Q}; \mathbb{R}^N)}^{p(4-p)/8} \right) \left( 1 + \int_{t_0'}^{t_1'} \int_{\Omega'} |\nabla u|^p dx dt \right) |h|^{(p-2)/2}
 \end{aligned}$$

( $2 < p < 4$ ; cf. (3.6), (4.2)).

Finally, we estimate  $I_3$  by using once more Lemma 1.

Let  $\Omega'' \subset \subset \Omega'$  and  $t_0'' < t_0' < t_1'' < t_1'$ . Specializing the cut-off functions such that  $\zeta \equiv 1$  on  $\Omega''$ ,  $\tau \equiv 1$  on  $(t_0'', t_1'')$  from (4.3) it follows that

$$\int_{t_0''}^{t_1''} \int_{\Omega''} |\Delta_h \nabla u|^2 dx dt \leq C_3 |h|^{(p-2)/2} \quad \forall |h| \leq h_0;$$

here the constant  $C_3$  depends on  $\max_{\mathbb{R}^n} |\nabla \zeta|$ ,  $\max_{\mathbb{R}} |\tau'|$ , but is independent of  $h$ . Thus,

$$(4.4) \quad \int_{-\delta}^{\delta} \frac{1}{|h|^{1+2\varrho}} \left( \int_{t_0''}^{t_1''} \int_{\Omega''} |\Delta_h \nabla u|^2 dx dt \right) dh \leq C_4 < +\infty$$

$$\forall 0 < \delta \leq h_0, \quad \forall 0 < \varrho < \frac{p-2}{4},$$

where  $C_4 \rightarrow +\infty$  as  $\varrho \rightarrow (p-2)/4$ . ■

3) Estimate (4.4) implies the fractional differentiability of  $\nabla u$  with respect to  $t$ : there exists  $0 < \delta_0 \leq h_0$  such that

$$\begin{aligned}
 (4.5) \quad &\int_{t_0''}^{t_1''} \int_{t_0''}^{t_1''} \left( \int_{\Omega''} \frac{|\nabla u(x, s) - \nabla u(x, t)|^2}{|s-t|^{1+2\varrho}} dx \right) ds dt \leq \\
 &\leq c \int_{-\delta_0}^{\delta_0} \frac{1}{|h|^{1+2\varrho}} \left( \int_{t_0''}^{t_1''} \int_{\Omega''} |\Delta_h \nabla u|^2 dx dt \right) dh
 \end{aligned}$$

for all  $0 < \varrho < (p - 2)/4$ , the constant  $c$  being independent of  $\varrho$  (cf. [5]).

Next, set

$$\eta(t) = \left( \int_{\Omega''} |\nabla u(x, t)|^2 dx \right)^{1/2}, \quad t \in (t_0'', t_1'').$$

Then (4.47) and (4.5) imply

$$\int_{t_0''}^{t_1''} \int_{t_0''}^{t_1''} \frac{(\eta(s) - \eta(t))^2}{|s - t|^{1+2\varrho}} ds dt < +\infty \quad \left( 0 < \varrho < \frac{p-2}{4} \right).$$

Hence, by virtue of Sobolev's imbedding theorem  $\eta \in L^{2/(1-2\varrho)}(t_0'', t_1'')$  (cf. e.g. [1]), i.e.

$$(4.6) \quad \nabla u \in L^{2/(1-2\varrho)}(t_0'', t_1''; L^2(\Omega''; \mathbb{R}^{nN})) \quad \left( 0 < \varrho < \frac{p-2}{4} \right). \quad \blacksquare$$

4) *Proof of (1.8).* Let  $4 \leq s < 4n/(n - 2\gamma)$ . Then we fix  $q$  such that

$$s < q < \frac{4n}{n - 2\gamma}.$$

Letting denote

$$\mu = \frac{2(q - s)}{s(q - 2)},$$

we have

$$(4.7) \quad \mu < \frac{1}{2}, \quad \frac{1}{s} = \frac{1 - \mu}{q} + \frac{\mu}{2}.$$

Next, let  $1/4 < \varrho < (p - 2)/4$  be fixed ( $3 < p < 4$ ; cf. (4.6)). Define

$$(4.8) \quad F(\theta) = \frac{1 - \mu}{2(1 + \theta)} + \frac{\mu}{2/(1 - 2\varrho)}, \quad \theta \in [0, 1].$$

Obviously,  $F(0) > 1/4 > F(1)$ . The continuity of  $F$  implies the existence of a  $\theta_0 \in (0, 1)$  such that

$$F(\theta_0) = \frac{1}{4}.$$

Now we fix  $\theta_1 \in [\theta_0, 1)$  satisfying

$$q < \frac{2(1 + \theta_1)n}{n - 2\theta_1\gamma}.$$

The function  $F$  being decreasing on  $[0, 1]$  it follows that  $F(\theta_1) \leq 1/4$ .

From (4.1), (4.6) we get by virtue of (4.7), (4.8) ( $\theta = \theta_1$ ) via interpolation

$$\nabla u \in L^{1/F(\theta_1)}(t_0, t_1; L^s(\Omega' \mathbb{R}^{nN}))$$

(cf. e.g. [6; Th. 1.18.4, p. 128]). Whence (1.13).

*Proof of (1.9).* Let  $4 \leq \sigma < 4(1 + \gamma/n)$ , i.e.

$$(\sigma - 4) \left( \frac{4n}{n - 2\gamma} - 2 \right) < 2 \left( \frac{4n}{n - 2\gamma} - \sigma \right).$$

Hence, there exists  $q$  such that  $\sigma < q < 4n/n - 2\gamma$  and

$$(\sigma - 4)(q - 2) < 2(q - \sigma).$$

As above, let denote

$$m = \frac{2(q - \sigma)}{\sigma(q - 2)}.$$

Then

$$\frac{1}{4\mu} \left( 1 - \frac{4}{\sigma} \right) < \frac{1}{4}, \quad \frac{1}{\sigma} = \frac{1 - \mu}{q} + \frac{\mu}{2}.$$

Now we first choose  $3 < p < 4$  and then  $1/4 < \varrho < (p - 2)/4$  such that

$$\frac{1}{4} + \frac{1}{4\mu} \left( 1 - \frac{4}{\sigma} \right) < \varrho.$$

It follows that  $F(0) > 1/4 \geq 1/\sigma > F(1)$ . By an analogous argument as above we find a  $\theta^* \in (0, 1)$  such that

$$q < \frac{2(1 + \theta^*)n}{n - 2\theta^*\gamma}, \quad F(\theta^*) \leq \frac{1}{\sigma}.$$

As above, interpolation gives

$$\nabla u \in L^{1/F(\theta^*)}(t_0, t_1; L^\sigma(\Omega'; \mathbb{R}^{nN})). \quad \blacksquare$$

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