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Connections on Infinite Dimensional Manifolds with Corners.

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Abstract - In this paper we study connections for surjective C^p-submersions on manifolds with corners, invariant by C^p-actions of Lie groups which are compatible with the equivalence relation defined by the submersion. In this context the principal connections and linear connections are studied as particular cases of these connections. Previously we adapt the vector bundle theory to be used in the paper, to the field of infinite dimensional manifolds with corners.

1. - Introduction.

In [5], P. Liberman defined connections for surjective submersions $\pi\colon M \to B$, as excisions of the exact sequence of vector bundles

$$0 \to VM \to TM \to \pi^*(TM) \to 0 .$$

An analogous definition of connections on smooth vector bundles

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was given by J. Vilms in [11], and a systematic study of this type of connections on fibre bundles can be found in [3].
The present paper concerns on G-connections, i.e., connections on surjective C^p-submersions π: $M \to B$ that are invariant by C^p-actions of Lie groups on M which are compatible with the equivalence relation defined by π in M.

A first original feature of this construction is that has been made in the realm of Banach differentiable manifolds with corners. A second main feature of this viewpoint is that principal connections and linear connections become particular cases of G-connections.

First we introduce a paragraph 2. about vector bundles on Banach manifolds with corners in order to have the results that will be used later. The paragraph 3. concerns with G-connections and the main result is Proposition 3.2 which establishes several characterizations of this type of connections. An existence theorem in this general context is an open problem.

In Paragraph 4. the principal connections has been studied as particular cases of G-connections and an important characterization by 1-forms is established.

Finally, in paragraph 5, linear connections are considered as $(\mathbb{R} - \{0\})$-connections, where the $(\mathbb{R} - \{0\})$-action is the scalar product in the fibers. Although a general existence theorem is not available, in 4 and 5, we prove existence theorems for principal connections and linear connections respectively.

2. – Vector bundles on manifolds with corners.

Firstly we recall the definition of a vector bundle of class p on infinite dimensional manifolds with corners.
Let M be a set, B a C^p-manifold, π: $M \to B$ a surjective map, $r = (M, B, \pi)$ and $p \in N \cup \{\infty\}$.

We say that (U, ψ, F) is a vector chart on r if F is a real Banach space, U is an open set of B and ψ: $U \times F \to \pi^{-1}(U)$ is a bijective map such that $\pi \circ \psi = p_1$, where p_1 is the first projection from $U \times F$ onto U. In this case the map ψ_b: $F \to \pi^{-1}(b) = M_b$, defined by $\psi_b(v) = \psi(b, v)$, is bijective.

Let $(U, \psi, F), (U', \psi', F')$ be vector charts of r. We say that they are C^p-compatible if there is a C^p-map μ: $U \cap U' \to L(F, F')$ such that $\psi'_b \circ \mu(b) = \psi_b$ for every $b \in U \cap U'$. Note that $\mu(b)$ is a linear homeomorphism for every $b \in U \cap U'$.

A set $\mathcal{V}$ of vector charts on r is called a vector atlas of class p if the domains of the charts of $\mathcal{V}$ cover B and any two of them are C^p-compatible.

Two vector atlases $\mathcal{V}$, V' of class p on r are called C^p-equivalent ($\tilde{C}^p$), if $\mathcal{V} \cup \mathcal{V}'$ is a vector atlas of class p on r.

One proves that $\tilde{C}^p$ is an equivalence relation over the vector atlases of class p on r.

If $\mathcal{V}$ is a vector atlas of class p on r, then the class of equivalence $[\mathcal{V}]$ is called a C^p-structure of vector bundle on r, and the pair $(r, [\mathcal{V}])$ is called a vector bundle of class p or C^p-vector bundle. The vector bundle $(r, [\mathcal{V}])$ will be denoted by (M, B, π).

Let $r = (M, B, \pi)$ be a vector bundle of class p. For every $b \in B$ there is a unique structure of topological real linear space on $\pi^{-1}(b)$ such that for every vector chart (U, ψ, E) of r with $b \in U$, the map

$$\psi_b \colon E \to \pi^{-1}(b) = M_b$$

is a linear homeomorphism. This topological real vector space M_b is a real banachable space. Moreover, there is a unique structure of differentiable manifold of class p on M such that for every chart $c = (U, \varphi, (E, \Delta))$ of B and every vector chart (U, ψ, F) of r, $c' = (\pi^{-1}(U), \alpha, (E \times F, \Delta p_1))$ is a chart of M, being

$$\alpha \colon \pi^{-1}(U) \to E_\Delta^+ \times F$$

the map given by $\alpha(x) = \big(\varphi(\pi(x)), \psi_{\pi(x)}^{-1}(x)\big)$. Thus $\pi \colon M \to B$ is a surjective C^p-submersion that preserves the boundary ($\pi(\partial M) = \partial B$), for every $x \in M$ $\mathrm{ind}_M(x) = \mathrm{ind}_B(\pi(x))$, $\partial M = \pi^{-1}(\partial B)$ and $B_k M = \pi^{-1}(B_k(B))$ for every $k \in \mathbb{N} \cup \{0\}$. Therefore for every $b \in B$, M_b is a C^p-submanifold of M without boundary whose C^p-differentiable structure coincides with the usual differentiable structure of the banachable space M_b. Let $r = (M, B, \pi)$ and $r' = (M', B', \pi')$ be vector bundles of class p and $f \colon B \to B'$ a C^p-map. A map $g \colon M \to M'$ will be called f-morphism of class p if for every $b_0 \in B$ there are vector charts (U, ψ, F) and (U', ψ', F') of r and r' respectively, with $b_0 \in U$ and $f(U) \subseteq U'$ and there is a C^p-map $h \colon U \to L(F, F')$ such that for every $b \in U$ the diagram

$$\begin{tikzcd} F \arrow[r, "\psi_b"] \arrow[d, "\lambda(b)"'] & M_b \arrow[d, "g_{|M_b} = b_b"] \\ F' \arrow[r, "\psi'_{f(b)}"] & M'_{f(b)} \end{tikzcd}$$

is commutative. If $B = B'$, $f = 1_B$ and g is an f-morphism of class p, one says that g is a B-morphism of class p. If $g \colon M \to M'$ is a bijective B-

morphism of class p, one says that g is a B-isomorphism of class p. In this case g is a C^p-diffeomorphism, g^{-1} is a B-isomorphism of class p and $(g^{-1})_b = g_b^{-1}$ for every $b \in B$.

Note that if g: $M \to M'$ is an f-morphism of class p, then

(1) g *is a* C^p-map and $f \circ \pi = \pi' \circ g$

and

(2) g_b: $M_b \to M'_{f(b)}$ is a linear continuous map for every $b \in B$.

The converse is not always true. We can only prove that g is an f-morphism of class $p-1, (p \geqslant 2)$. If (1) and (2) hold true and $p = \infty$ or $r = (M, B, \pi)$ has finite range ($\dim M_b < \infty$ for every $b \in B$), then g is an f-morphism of class p.

Every f-morphism g of class p verifies that $f(\partial B) \subseteq \partial B'$ if and only if $g(\partial M) \subseteq \partial M'$. Moreover, f preserves the index if and only if g preserves the index. Consequently, every B-morphism of class p preserves the index.

Let $r = (M, B, \pi)$ be a vector bundle of class p. One says that r is trivializable if there is a real Banach space F such that the trivial vector bundle $(B \times F, B, p_1)$ is B-isomorphic to r.

Let $r = (M, B, \pi)$ be a vector bundle of class p, B' a C^p-manifold and f: $B' \to B$ a C^p-map. Consider the set

$$B' \times_B M = \{(b', x) \in B' \times M / f(b') = \pi(x)\},$$

and the maps π': $B' \times_B M \to B'$ and g: $B' \times_B M \to M$ given by $\pi'(b', x) = b'$, $g(b', x) = x$. Then there is a unique vector bundle structure of class p on $r' = (B' \times_B M, B', \pi')$ such that g is an f-morphism of class p from r' to r. This vector bundle is also denoted by $f^*(r) = (f^*(M), B', f^*(\pi))$ and called pullback of r by f. If $t = (U, \psi, E)$ is a vector chart of r, then $t' = (f^{-1}(U), \psi', E)$ is a vector chart of $f^*(r)$, where

$$\psi': f^{-1}(U) \times E \to (f^*(\pi))^{-1}(f^{-1}(U))$$

is given by $\psi'(b', v) = (b', \psi(f(b'), v))$.

Finally it can be easily proved that for every $b' \in B'$

$$g_{b'}: f^*(M)_{b'} \to M_{f(b')}$$

is a linear homeomorphism,

$$\partial f^*(M) = (\partial B' \times M) \cap f^*(M), \qquad \mathrm{ind}_{f^*(M)}(b', x) = \mathrm{ind}_{B'}(b')$$

for every $(b', x) \in f^*(M)$ and $(f^*(M), f^*(\pi), g)$ is a fibered product of the family $\{f, \pi\}$.

Note that, in general, the set $f^*(M)$ is not a submanifold of $B' \times M$. For example: If $B = B' = [0, \infty)$, $f(b') = (b')^2$ for every $b' \in B'$ and we consider the trivial C^∞ vector bundle $(M = B \times \mathrm{R}, B, p_1)$, then $f^*(M) = \{(b', (b')^2, t)/t \in \mathrm{R}, b' \in [0, \infty)\}$ is not a submanifold of $B' \times B \times \mathrm{R}$. Nervertheless if $r = (M, B, \pi)$ is a C^p-vector bundle and f: $B' \to B$ is a C^p-map such that $\{f, \pi\}$ is transversal (these hypotheses imply that $(\mathrm{Int}(B') \times \partial M) \cap f^*(M) = \emptyset$. Indeed, if $(b', x) \in$ $\in (\mathrm{Int}(B') \times \partial M) \cap f^*(M)$, then $f(b') = \pi(x) = b \in B_{k'}(B)$, $k' > 0$ and $T_{b'}f - T_x(\pi_{|B_{k'}M})$: $T_{b'}B' \times T_x(B_{k'}M) \to T_b(B)$ is not a surjective map since $T_{b'}f(T_{b'}B') \subsetneq T_b(B_{k'}(B))$ and $T_x(\pi_{|B_{k'}M})(T_x B_{k'}M) \subsetneq T_b(B_{k'}(B))$, which contradicts the transversality of $\{f, \pi\}$) then the set $f^*(M)$ is a totally neat C^p-submanifold of $B' \times M$ ([7], 7.2.7) and $\{f^*(M), (p_1)_{|f^*(M)}, (p_2)_{|f^*(M)}\}$ is the fibered product of $\{f, \pi\}$. Thus, this structure of C^p-manifold coincides with the structure of manifold induced by the structure of vector bundle on $f^*(M)$.

Proposition 2.1. *Let (M, B, π) and (M', B', π') be vector bundles of class p and f: $B' \to B$ a C^p-map. If h: $M' \to M$ is an f-morphism of class p then there is a unique B'-morphism h_*: $M' \to f^*(M)$ of class p such that $g \circ h_* = h$ being g: $f^*(M) \to M$ the map defined by $g(b', x) = x$.* ■

Now if (M, B, π) is a C^p-vector bundle and B' is a C^p-submanifold of B, then the C^p-manifolds $\pi^{-1}(B')$ and $j^*(M)$ are C^p-diffeomorphic, where j: $B' \to B$ is the inclusion map, and there is a unique C^p-vector bundle structure $[\mathcal{V}_1]$ on $(\pi^{-1}(B'), B', \pi_{|\pi^{-1}(B')})$ such that the inclusion map j_1: $\pi^{-1}(B') \to M$ is a j-morphism of class p. Moreover the map h: $j^*(M) \to \pi^{-1}(B')$ defined by $h(b', x) = x$, is a B'-isomorphism and the C^p-differentiable structure given by $[\mathcal{V}_1]$ is the one given by the submanifold $\pi^{-1}(B')$.

As a particular case, for every $k \in N \cup \{0\}$ there is a unique C^p-vector bundle structure on $(B_k(M), B_k(B), \pi_{|B_k(M)})$ such that j_1: $B_k(M) \to$ $\to M$ is a j-morphism (j: $B_k(B) \to B$) of class p.

Let $r = (M, B, \pi)$ be a vector bundle of class p and $M' \subseteq M$. One says that M' is a C^p-subbundle of r if for every $b \in B$ there is a vector chart $t = (U, \psi, E)$ of r with $b \in U$ and there is a closed linear subspace F of E which admits topological supplement in E such that

$$\psi^{-1}(\pi^{-1}(U) \cap M') = U \times F.$$

In this case there is a unique C^p-vector bundle structure on

$(M', B, \pi_{|M'})$ such that the inclusion $j: M' \to M$ is a B-morphism of class p. Moreover for every $b \in B$, $M'_b = M' \cap M_b$ is a closed linear subspace of M_b which admits a topological supplement in M_b, the set M' is a closed totally neat C^p-submanifold of M (Indeed, for every $x \in M'$, $\pi(x) = b \in B$ and there is a vector chart (U, ψ, E) of r with $b \in U$ and there is a closed linear subspace F of E which admits a topological supplement in E such that $\psi^{-1}(\pi^{-1}(U) \cap M') = U \times F$. Let $(U, \varphi, (H, \Delta))$ be a chart of B. Then $(\pi^{-1}(U), \alpha, (H \times E, \Delta \circ p_1))$ is a chart of M where $\alpha(y) = (\varphi(\pi(y)), \psi^{-1}_{\pi(y)}(y))$, $\alpha(\pi^{-1}(U) \cap M') = \alpha(\pi^{-1}(U)) \cap H^+_\Delta \times F$ and $\alpha(\pi^{-1}(U)) \cap H^+_\Delta \times F$ is an open set in $H^+_\Delta \times F$. Thus M' is a closed totally neat submanifold of M) and M' as submanifold of M coincides with M' as manifold induced by $(M', B, \pi_{|M'})$.

Let $r'' = (M'', B, \pi'')$, $r = (M, B, \pi)$ be C^p-vector bundles, M' a C^p-subbundle of r and f: $M'' \to M$ a map such that $f(M'') \subseteq M'$. Then, f is a B-morphism of class p of r'' into r if and only if f is a B-morphism of class p of r'' into $(M', B, \pi_{|M'})$.

Let $r = (M, B, \pi)$ be a C^p-vector bundle, M' a C^p-subbundle of r and R the equivalence relation on M defined by

$$xRy \text{ if and only if there is } b \in B \text{ such that } x, y \in M_b \text{ and } x - y \in M'_b .$$

Then there is a unique C^p-vector bundle structure on $(M/R, B, \bar{\pi})$ where $\bar{\pi}([x]) = \pi(x)$, such that the natural projection p: $M \to M/R$ is a B-morphism of class p. This vector bundle is called quotient vector bundle. If $b \in B$, there is a vector chart (U, ψ, E) of r with $b \in U$ and there is a closed linear subspace F of E which admits a topological supplement in E such that $\psi^{-1}(\pi^{-1}(U) \cap M') = U \times F$. Then $(U, \psi_{|U \times F}, F)$ is a vector chart of $(M', B, \pi_{|M'})$ and $(U, \psi'', E/F)$ is a vector chart of $(M/R, B, \bar{\pi})$, where $\psi''(b, [v]) = [\psi(b, v)]$. For every $b \in B$, ϕ_b: $M_b/M'_b \to (M/R)_b$ defined by $\phi_b(x + M'_b) = [x]_R$ is a linear homeomorphism. Finally p: $M \to M/R$ is a C^p-submersion (with the differentiable structures induced by the vector bundles), $p(\partial M) = \partial(M/R)$ and R is a regular relation.

Let $r = (M, B, \pi)$, $r_1 = (M_1, B, \pi_1)$ be vector bundles of class p and g: $M \to M_1$ a B-morphism of class p. One says that g is locally direct if $\ker(g) = \bigcup_{b \in B} \ker(g_b)$ is a vector subbundle of r and $\operatorname{im}(g) = \bigcup_{b \in B} \operatorname{im}(g_b)$ is a vector subbundle of r_1 (it is clear that $\pi(\ker(g)) = B$ and $\pi_1(im(g)) = B$). In this case $\bar{g}$: $M/\ker(g) \to \operatorname{im}(g)$ defined by $\bar{g}([x]) = g(x)$ is a B-isomorphism.

Let $(M, B, \pi), (M_1, B, \pi_1)$ and (M_2, B, π_2) be vector bundles of class p and $f\colon M_1 \to M$, $g\colon M \to M_2$ B-morphisms of class p. One says that

$$M_1 \xrightarrow{f} M \xrightarrow{g} M_2$$

is an exact sequence if f and g are locally direct and $\ker(g) = \operatorname{im}(f)$. (If more than two B-morphisms are considered, the generalized definition is the obvious one).

We have the following results:

PROPOSITION 2.2. *Let $r = (M, B, \pi)$, $r' = (M', B, \pi')$ be C^p-vector bundles and $f\colon M' \to M$ a B-morphism of class p. Then the following statements are equivalent:*

1) $\bar{0} \xrightarrow{\Theta} M' \xrightarrow{f} M$ *is exact, where $\bar{0}$ is the trivial vector bundle $(B \times 0, B, p_1)$ and $\Theta\colon B \times 0 \to M'$ is the locally direct B-morphism given by $\Theta(b, 0) = 0_b \in M'_b$.*

2) *f is injective and* $\operatorname{im}(f)$ *is a C^p-vector subbundle of r.*

3) $\operatorname{im}(f)$ *is a C^p-vector subbundle of r and $f\colon M' \to \operatorname{im}(f)$ is a B-isomorphism.*

4) *For every $b \in B$, there exist (U, ψ', E') vector chart of r' with $b \in U$ and (U, ψ, E) vector chart of r such that E' is a closed linear subspace of E that admits a topological supplement in E and $f \circ \psi'(x, v) = \psi(x, v), (x, v) \in U \times E'$.*

5) *For every $b \in B$, $f_b\colon M'_b \to M_b$ is injective and* $\operatorname{im}(f_b)$ *admits a topological supplement in M_b.* ■

PROPOSITION 2.3. *Let B be a paracompact C^p-manifold which admits partitions of unity of class p, $(M, B, \pi), (M', B, \pi')$ vector bundles of class p and $f\colon M' \to M$ a B-morphism of class p such that $B \times 0 \xrightarrow{\Theta} M' \xrightarrow{f} M$ is exact. Then there is a B-morphism $g\colon M \to M'$ of class p such that $g \circ f = 1_{M'}$ and* $\ker g_b$ *admits topological supplement in M_b for every $b \in B$.* ■

PROPOSITION 2.4. *Let $r = (M, B, \pi)$, $r' = (M', B, \pi')$ be vector bundles of class p and $f\colon M \to M'$ a B-morphism of class p. Then the following statements are equivalent:*

1) $M \xrightarrow{f} M' \to B \times 0$ *is exact.*

2) *f is surjective and* $\ker(f)$ *is a C^p-vector subbundle of r.*

3) $\ker(f)$ *is a C^p-vector subbundle of r and $\bar{f}\colon M/\ker(f) \to M'$, given by $\bar{f}([x]) = f(x)$, is a B-isomorphism of class p.*

4) *For every* $b \in B$, *there exist* (U, ψ, E) *vector chart of* r *with* $b \in$ $\in U$ *and* $(U, \psi', E/F)$ *vector chart of* r', *where* F *is a closed linear subspace of* E *that admits a topological supplement in* E, *such that* $f \circ \psi(x, v) = \psi'(x, p(v))$, *where* $p\colon E \to E/F$ *is the natural projection,* $(x, v) \in U \times E$.

5) *For every* $b \in B, f_b\colon M_b \to M_b'$ *is surjective and* $\ker(f_b)$ *admits a topological supplement in* M_b. ■

Notice that the sequence of vector bundles and B-morphisms of class p

$$\begin{tikzcd} M' \arrow[r, "f"] \arrow[dr, "\pi'"'] & M \arrow[r, "g"] \arrow[d, "\pi"] & M'' \arrow[dl, "\pi''"] \\ & B & \end{tikzcd}$$

where M has finite rank, is exact if and only if $M_b' \xrightarrow{f_b} M_b \xrightarrow{g_b} M_b''$ is exact for every $b \in B$.

PROPOSITION 2.5. *Let us consider the sequence of* C^p*-vector bundles and* B*-morphisms of class* p

$$\begin{tikzcd} B \times 0 \arrow[r] \arrow[drr, "p_1"'] & M' \arrow[r, "f"] \arrow[dr, "\pi'"'] & M \arrow[r, "g"] \arrow[d, "\pi"] & M'' \arrow[r] \arrow[dl, "\pi''"] & B \times 0 \arrow[dll, "p_1"] \\ & & B & & \end{tikzcd}$$

Then this sequence is exact if and only if $\operatorname{im}(f)$ *is a* C^p*-vector subbundle of* (M, B, π), $f\colon M' \to \operatorname{im}(f)$ *is a* B*-isomorphism of class* p, $\operatorname{im}(f) = \operatorname{Ker}(g)$ *and* $\bar{g}\colon M/\operatorname{im}(f) \to M''$ *is a* B*-isomorphism of class* p. ■

PROPOSITION 2.6. *Let* B *be a paracompact* C^p*-manifold which admits partitions of unity of class* p, $(M, B, \pi), (M', B, \pi'), (M'', B, \pi'')$ *vector bundles of class* p *and* $f\colon M' \to M$, $h\colon M \to M''$ B*-morphisms of class* p *such that*

$$B \times 0 \to M' \xrightarrow{f} M \xrightarrow{h} M'' \to B \times 0$$

is exact. Then M'' *is* B*-isomorphic to a vector subbundle of* M *and there* *is a B-morphism* $s\colon M'' \to M$ *of class* p *such that* $h \circ s = 1_{M''}$. ∎

Let $f\colon X \to X'$ be a C^p-map ($p \geqslant 2$). Then $Tf\colon TX \to TX'$ is a f-morphism of class $p-1$ and there is a unique X-morphism $T_* f\colon TX \to \to f^*(TX')$ such that $g \circ T_* f = Tf$ where g is the second projection. Moreover $X \times 0 \to TX \xrightarrow{T_* f} f^*(TX')$ is exact if and only if $T_x f$ is injective and $\operatorname{im}(T_x f)$ admits a topological supplement in $T_{f(x)}X'$ for every $x \in X$ (if $f(\partial X) \subseteq \partial X'$ and $\operatorname{ind}(v) = \operatorname{ind} T_x f(v)$ for every $v \in (T_x X)^i$, then $X \times \times 0 \to TX \xrightarrow{T_* f} f^*(TX')$ is exact if and only if f is an immersion (see 3.2.11 of [7])). Now if $T_x f$ is injective and $\operatorname{im}(T_x f)$ admits a topological suppplement in $T_{f(x)}(X')$, we can consider the quotient vector bundle $(f^*(TX'))/T_* f(TX), X, \overline{f^*(\pi')})$ where (TX', X', π') is the tangent vector bundle of X'. This quotient vector bundle will be called normal vector bundle of f.

If $f\colon X \to X'$ is a C^p-map ($p \geqslant 2$), then $TX \xrightarrow{T_* f} f^*(TX') \to X \times 0$ is exact if and only if $T_x f$ is surjective and $\ker(T_x f)$ admits a topological supplement in $T_x X$ for every $x \in X$ (If $f(\partial X) \subseteq \partial X'$, then $TX \xrightarrow{T_* f} f^*(T(X')) \to X \times 0$ is exact if and only if f is a C^p-submersion). If, for every $x \in X$, $T_x f$ is surjective and $\ker(T_x f)$ admits a topological supplement in $T_x X$ then $\ker(T_* f)$ is a C^{p-1}-subbundle of (TX, X, π) (which will be called the relative tangent vector bundle and will be denoted $T(X/X')$ or $V(X)$) and if moreover $f(\partial X) \subseteq \partial X'$ we have that $T_x(f^{-1}(f(x))) \equiv T(X/X')_x$ for every $x \in X$ and $0 \to \to T(X/X')_x \xrightarrow{i} T_x X \xrightarrow{T_x f} T_{f(x)}X' \to 0$ is exact.

Let $\pi\colon M \to B$ be a C^p-submersion ($p \geqslant 2$). Then there exists a unique C^{p-1}-M-morphism $T_* \pi\colon TM \to \pi^*(TB)$ such that $g \circ T_* \pi = T\pi$, where g is the second projection and

$$M \times 0 \to \ker(T_* \pi) = \sum_{x \in M} \operatorname{Ker}(T_x M) \xrightarrow{i} TM \xrightarrow{T_* \pi} \pi^*(TB) \to M \times 0$$

is exact $(T_* \pi(x, v) = (x, \pi(x), T_x \pi(v)))$.

Let I be a finite set such that $I = I_- \cup I_+$ and $I_- \cap I_+ = \emptyset$, C_I the class of objects $\{\varepsilon = \{E_i\}_{i \in I} / E_i$ is a real Banach space$\}$ and $\operatorname{Hom}(\varepsilon, \varepsilon')$ the set $\prod_{i \in I_+} L(E_i, E_i') \times \prod_{i \in I_-} L(E_i', E_i)$, for every $\varepsilon, \varepsilon' \in C_I$. For every $(\bar{g}, \bar{f}) = ((g_i)_{i \in I}, (f_i)_{i \in I}) \in \operatorname{Hom}(\varepsilon, \varepsilon') \times \operatorname{Hom}(\varepsilon', \varepsilon'')$ one defines $\bar{f} \circ \bar{g} = ((f_i \circ g_i)_{i \in I_+}, (g_i \circ f_i)_{i \in I_-}) \in \operatorname{Hom}(\varepsilon, \varepsilon'')$. It is clear that $\mathfrak{C}_I = = (C_I, \bigcup_{(\varepsilon, \varepsilon') \in C_I \times C_I} \operatorname{Hom}(\varepsilon, \varepsilon'))$ with the preceding composition is a category, where $1_\varepsilon = (1_{E_i})_{i \in I}$ for every $\varepsilon = \{E_i\} \in C_I$.

If $\tau\colon \mathfrak{C}_I \to \mathfrak{B}$ is a covariant functor, where $\mathfrak{B}$ is the category of real

Banach spaces and linear continuous maps, such that for every $(\varepsilon, \varepsilon') \in \in C_I \times C_I$, the map

$$\sigma\colon \operatorname{Hom}(\varepsilon, \varepsilon') \to L(\tau(\varepsilon), \tau(\varepsilon'))\sigma(\bar{f}) = \tau(\bar{f})$$

is of class p, one says that τ is a vector functor of type I and class p.

Let $\tau\colon \mathfrak{C}_I \to \mathfrak{B}$ be a vector functor of type I and class p and $\mathfrak{M} = = \{(M^i, B_i, \pi_i)\}_{i \in I}$ a family of vector bundles of class p. Then there is a unique vector bundle structure of class p on

$$\left(\tau(\mathfrak{M}) = \sum_{b \in \prod_{i \in I} B_i} \tau(\mathfrak{M}_b), \prod_{i \in I} B_i, \pi\right)$$

where $\mathfrak{M}_b = \{M^i_{b_i}\}_{i \in I}$, and $\pi(\tau(\mathfrak{M}_b)) = \{b\}$, such that if (U_i, ψ_i, E_i) is a vector chart of (M^i, B_i, π_i), $i \in I$ then $(\prod_{i \in I} U_i, \psi, \tau(\{E_i\}_{i \in I}))$ is a vector chart of this structure, where

$$\psi\colon \prod_{i \in I} U_i \times \tau(\{E_i\}_{i \in I}) \to \sum_{b \in \prod_{i \in I} U_i} \tau(\mathfrak{M}_b)$$

is defined by $\psi(b, v) = \big(b, \tau(\{((\psi_i)_{b_i}^{-1})_{i \in I_+}, ((\psi_i)_{b_i})_{i \in I_-}\})^{-1}(v)\big)$.

If $B_i = B$ for all $i \in I$, then there is a unique vector bundle structure of class p on $(\tau(\mathfrak{M}) = \sum_{b \in B} \tau(\mathfrak{M}_b), B, \pi)$, where $\mathfrak{M}_b = \{M^i_b\}_{i \in I}$ and $\pi(\tau(\mathfrak{M}_b)) = \{b\}$, such that if (U, ψ_i, E_i) is a vector chart of (M^i, B, π_i), $i \in I$ then $(U, \psi, \tau(\{E_i\}_{i \in I}))$ is a vector chart of this structure, where

$$\psi\colon U \times \tau(\{E_i\}_{i \in I}) \to \sum_{b \in U} \tau(\mathfrak{M}_b)$$

is defined by $\psi(b, v) = \big(b, \tau(\{((\psi_i)_b^{-1})_{i \in I_+}, ((\psi_i)_b)_{i \in I_-}\})^{-1}(v)\big)$.

As a particular case we obtain the vector bundle of linear continuous maps. Indeed, let us consider $I = \{1, 2\}$, $I_+ = \{2\}$, $I_- = \{1\}$,

$$\tau\colon \mathfrak{C}_I \to \mathfrak{B}$$

the covariant functor defined by $\tau(\{E_1, E_2\}) = L(E_1, E_2)$ and $\tau(f_2, f_1)(h) = f_2 \circ h \circ f_1$ for every $(f_2, f_1) \in \operatorname{Hom}(\{E_1, E_2\}, \{E_1', E_2'\})$ and the family of vector bundles of class p $\mathfrak{M} = = \{(M^1, B_1, \pi_1), (M^2, B_2, \pi_2)\}$. Then τ is a vector functor of type I and class ∞ and there is a unique vector bundle structure of class p on $(\tau(\mathfrak{M}) = \sum_{b \in B_1 \times B_2} L(M^1_{b_1}, M^2_{b_2}), B_1 \times B_2, \pi)$ such that if (U_1, ψ_1, E_1) and (U_2, ψ_2, E_2) are vector charts of (M^1, B_1, π_1) and (M^2, B_2, π_2) respectively, then $(U_1 \times U_2, \psi, L(E_1, E_2))$ is a vector chart of this structure, where

$$\psi\colon U_1 \times U_2 \times L(E_1, E_2) \to \sum_{b \in U_1 \times U_2} L(M^1_{b_1}, M^2_{b_2})$$

is defined by $\psi(b, v) = (b, (\psi_2)_{b_2} \circ v \circ (\psi_1)^{-1}_{b_1})$. We adopt the notation $\sum_{b \in B_1 \times B_2} L(M^1_{b_1}, M^2_{b_2}) = L(M^1, M^2)$. If $B_1 = B_2 = B$ there is a unique vector bundle structure of class p on $(\tau(\mathfrak{M}) = \sum_{b \in B} L(M^1_b, M^2_b) = L(M^1, M^2), B, \pi)$ such that if (U, ψ_1, E_1) and (U, ψ_2, E_2) are vector charts of (M^1, B, π_1) and (M^2, B, π_2) respectively, then $(U, \psi, L(E_1, E_2))$ is a vector chart of this structure, where

$$\psi\colon U \times L(E_1, E_2) \to \sum_{b \in U} L(M^1_b, M^2_b)$$

is defined by $\psi(b, v) = (b, (\psi_2)_b \circ v \circ (\psi_1)^{-1}_b)$.

Let I be a finite set such that $I = I_+$ and $I = \emptyset$, $\tau\colon \mathfrak{C}_I \to \mathfrak{B}$ the covariant functor defined by $\tau(\{E_i\}_{i \in I}) = \prod_{i \in I} E_i$ and $\tau((f_i)_{i \in I}) = \prod_{i \in I} f_i$ and $\mathfrak{M} = \{(M^i, B, \pi_i)\}_{i \in I}$ a family of vector bundles of class p. Then τ is a vector functor of type I and class p and there is a unique vector bundle structure of class p on $(\tau(\mathfrak{M}) = \oplus_{i \in I} M^i, B, \pi)$, where $\pi(\oplus_{i \in I} M^i_b) = \{b\}$, such that if (U, ψ_i, E_i) is a vector chart of (M^i, B, π_i), $i \in I$, then $(U, \psi, \prod_{i \in I} E_i)$ is a vector chart of this structure, where $\psi(b, v) = (b, \prod_{i \in I} (\psi_i)_b)(v)$. This vector bundle will be called Whitney sum of $\mathfrak{M}$ and τ will be called Whitney vector functor. Notice that $p_i\colon \oplus_{i \in I} M^i \to M^i$ defined by $p_i(b, (x_i)_{i \in I}) = x_i \in M^i_b$ is a B-morphism for every $i \in I$ and $j_i\colon M^i \to \oplus_{i \in I} M^i$ defined by $j_i(x_i) = (\pi_i(x_i), (0\ldots, x_i, \ldots, 0))$ is a B-morphism for every $i \in I$.

Let τ be the Whitney vector functor, $f\colon B \to B'$ a C^p-map, $\mathfrak{M}' = \{((M')^i, B', \pi'_i)\}_{i \in I}$ a family of vector bundles of class p, (M, B, π) a vector bundle of class p and $u_i\colon M \to (M')^i$ an f-morphism of class p for every $i \in I$. Then $u = (u_i)_{i \in I}\colon M \to \oplus_{i \in I} (M')^i$, defined by $u(x) = (f(\pi(x)), (u_i(x))_{i \in I})$, is an f-morphism of class p.

Let τ be the Whitney vector functor, $f\colon B \to B'$ a C^p-map, $\mathfrak{M} = \{(M^i, B, \pi_i)\}_{i \in I}$ a family of vector bundles of class p, (M', B', π') a vector bundle of class p and $v_i\colon M^i \to M'$ an f-morphism of class p for every $i \in I$. Then $v = \sum_{i \in I} v_i\colon \oplus_{i \in I} M^i \to M'$ defined by $v(b, (x_i)_{i \in I}) = \sum_{i \in I} v_i(x_i) \in M'_{f(b)}$, is an f-morphism of class p.

Let τ be the Whitney vector functor, where $I = I_+ = \{1, 2\}$ ($I_- = \emptyset$)

and

$$\mathfrak{M} = \{(M^1, B, \pi_1), (M^2, B, \pi_2)\}$$

a family of vector bundles of class p. Then

$$B \times 0 \to M^1 \xrightarrow{j_1} M^1 \oplus M^2 \xrightarrow{p_2} M^2 \to B \times 0$$

is exact.

PROPOSITION 2.7. *Let*

$$\begin{tikzcd}
B\times 0 \arrow[r] \arrow[drr, "p_1"'] & M' \arrow[r, "f"] \arrow[dr, "\pi'"] & M \arrow[r, "g"] \arrow[d, "\pi"] & M'' \arrow[r] \arrow[dl, "\pi''"'] & B\times 0 \arrow[dll, "p_1"] \\
& & B & &
\end{tikzcd}$$

be an exact sequence of C^p-vector bundles. Then the following statements are equivalent:

a) There exists a B-morphism of class p, s: $M'' \to M$ such that $g \circ s = 1_{M''}$.

b) There exists a B-morphism of class p, r: $M \to M'$ such that $r \circ f = 1_{M'}$.

c) There exists a C^p-vector subbundle M''' of (M, B, π) such that (M, B, π) is B-isomorphic of class p to $(f(M') \oplus M''', B, \pi_*)$ by means of the map $\psi(b, (x_1, x_2)) = x_1 + x_2 \in M_b$ (consequently $\mathrm{im}(f_b) \oplus_T M'''_b = M_b$ for every $b \in B$).

Moreover if a), then $f + s$: $M' \oplus M'' \to M$ is a B-isomorphism of class p.

PROOF. $a) \to c)$. We have that $\alpha = s \circ g$: $M \to M$ is a B-morphism of class p, $\alpha \circ \alpha = \alpha$ and $\ker(\alpha) = \mathrm{im}(f)$ is a C^p-vector subbundle of (M, B, π). On the other hand $\mathrm{im}(\alpha) = s(M'')$ is a C^p-vector subbundle of (M, B, π). Indeed, for every $b \in B$ we have $g_b \circ s_b = 1_{M''_b}$, s_b is injective, by (3.2.18 of [7]) $\mathrm{im}(s_b)$ is closed in M_b and $\mathrm{im}(s_b)$ admits a topological supplement, $\ker(g_b)$, in M_b. Thus s: $M'' \to M$ is injective and $s(M'')$ is a C^p-vector subbundle of (M, B, π).

Then $\mathrm{im}(f_b) \oplus_T \mathrm{im}(\alpha_b) = \ker(\alpha_b) \oplus_T \mathrm{im}(\alpha_b) = M_b$ for every $b \in B$ and $(f(M') \oplus \mathrm{im}\,\alpha = \ker\alpha \oplus \mathrm{im}\,\alpha, B, \pi_*)$ is B-isomorphic to (M, B, π).

$b) \to a)$ We have that $\beta = f \circ r$: $M \to M$ is a B-morphism of class p, $\beta \circ \beta = \beta$ and $\mathrm{im}(\beta) = f(M')$ is a C^p-vector subbundle of (M, B, π). On the other hand, since $r_b \circ f_b = 1_{M'_b}$, $\ker(r)$ is a C^p-vector subbundle of (M, B, π) and $\ker(\beta) = \ker(r)$. Moreover the map φ: $\mathrm{im}(f) \oplus \oplus \ker(r) \to M$, defined by $\varphi(b, (x_1, x_2)) = x_1 + x_2 \in M_b$, is a B-isomorphism, $\ker(r) \xrightarrow{j} M \xrightarrow{g} M''$ is a B-isomorphism, $s = j \circ (g \circ j)^{-1}$: $M'' \to M$ is a B-morphism and $g \circ s = 1_{M''}$.

$c) \to b)$ Let σ: $M \to f(M') \oplus M'''$ be the B-isomorphism given by $c)$. Then

$$r\colon M \xrightarrow{\sigma} f(M') \oplus M''' \xrightarrow{p_1} f(M') \xrightarrow{f^{-1}} M'$$

is a B-morphism which verifies $b)$.

We know that $f + s$: $M' \oplus M'' \to M$ is a B-morphism of class p. But, if $a)$ holds (see proof $a) \to c)$) $f + s$ is bijective. Thus $f + s$ is a B-isomorphisms of class p. ■

Let $\mathfrak{M} = \{(M^i, B, \pi_i)\}_{i \in I}$ be a finite family of vector bundles of class p, U an open set of B and s_i: $U \to M^i$ a C^p-section of (M^i, B, π_i) for every $i \in I$. Then $\sum_{i \in I} s_i$: $U \to \oplus_{i \in I} M^i$ defined by $(\sum_{i \in I} s_i)(b) = (b, (s_i(b))_{i \in I})$ is a C^p-section of $(\oplus_{i \in I} M^i, B, \pi)$ and ϕ: $\prod_{i \in I} S^p_{M^i}(U) \to S^p_{\oplus_{i \in I} M^i}(U)$ defined by $\phi((s_i)_{i \in I}) = \sum_{i \in I} s_i$ is a $C^p(U, \mathbb{R})$-module isomorphism.

Let $\mathfrak{M} = \{(M^i, B, \pi_i)\}_{i \in I = \{1, \ldots, n\}}$ be a finite family of vector bundles of class p such that $\{\pi_i\colon M^i \to B\}_{i \in \{1, \ldots, n\}}$ is a transversal family of maps. Then $P = M^1 \times_B \ldots \times_B M^n$ is a totally neat submanifold of class p of $M^1 \times \ldots \times M^n$, $\{P, \{p_i\colon P \to M^i\}_{i \in I}\}$ and $\{\oplus_{i = 1, \ldots, n} M^i, \{p_i\colon \oplus_{i = 1, \ldots, n} M^i \to M^i\}_{i = 1, \ldots, n}\}$ are fibered products of $\{\pi_i\colon M^i \to B\}_{i = 1, \ldots, n}$ and l: $\oplus_{i = 1, \ldots, n} M^i \to P$, defined by $l(b, (x_i)_{i = 1, \ldots, n}) = (x_i)_{i = 1, \ldots, n}$ is a C^p-diffeomorphism.

Let us consider X_1, X_2 manifolds of class p, $(TX_1, X_1, \pi_1), (TX_2, X_2, \pi_2)$ the tangent vector bundles of X_1 and X_2 respectively, p_1: $X_1 \times X_2 \to X_1$, p_2: $X_1 \times X_2 \to X_2$ the projections. Then $(T(X_1 \times X_2), X_1 \times X_2, \pi)$ and $(p_1^*(TX_1) \oplus p_2^*(TX_2), X_1 \times X_2, \bar{\pi})$ are $X_1 \times X_2$-isomorphic by means of $(\bar{p}_1, \bar{p}_2)$, where

$$\bar{p}_1((x_1, x_2), v) = ((x_1, x_2), T_{(x_1, x_2)} p_1(v))$$

and

$$\bar{p}_2((x_1, x_2), v) = ((x_1, x_2), T_{(x_1, x_2)} p_2(v)).$$

Let $I = \{0, 1, \ldots, d\}$ such that $I_+ = \{0\}, I_- = \{1, \ldots, d\}$ and η_d: $\mathfrak{C}_I \to \mathfrak{B}$ the covariant functor defined by

$$\eta_d(\{E_i\}_{i \in I}) = L(E_1, \ldots E_d; E_0)$$

(continuous multilinear maps) and $\eta_d(f_0, (f_i)_{i \in I_-})(u) = f_0 \circ u \circ \prod_{i \in I_-} f_i$. Then η_d is a vector functor of type I and class ∞ and if we have the family $\mathfrak{M} = \{(M^i, B, \pi_i)\}_{i \in I}$ of C^p-vector bundles, we can consider the C^p-vector bundle of class p

$$\left((\eta_d(\mathfrak{M}) = \sum_{b \in B} L(M_b^1, \ldots, M_b^d; M_b^0), B, \pi\right).$$

This vector bundle will be called multilinear vector bundle and will be denoted by

$$(L(M^1, \ldots, M^d; M^0), B, \pi).$$

If (U, ψ_i, E_i) is a vector chart of (M^i, B, π_i), $i \in I$, then

$$(U, \psi, L(E_1, \ldots E_d; E_0))$$

is a vector chart of the multilinear vector bundle where

$$\psi\colon\ U \times L(E_1, \ldots, E_d; E_0) \to \sum_{b \in U} L(M_b^1, \ldots, M_b^d; M_d^0)$$

is given by $\psi(b, v) = \left(b, (\psi_0)_b \circ v \circ \prod_{i \in I_-} (\psi_i)_b^{-1}\right)$. The particular case $I = \{0, 1\}$, $I_+ = \{0\}$ and $I_- = \{1\}$ has been already considered. In this case if $\mathfrak{M} = \{(M, B, \pi), (B \times \mathbb{R}, B, p_1)\}$ then $(L(M, B \times \mathbb{R}), B, \pi_*)$ will be called dual vector bundle of (M, B, π) and will be denoted by (M^*, B, π_*), $(M_b^* = L(M_b, \mathbb{R}))$.

Let X be a C^p-manifold, (TX, X, π) the tangent C^{p-1}-vector bundle of X and

$$(T(X)^*, X, \pi_*)$$

the dual vector bundle of (TX, X, π) (Also called cotangent vector bundle of X). Then $T(X)_x^* = L(T_xX, \mathbb{R}) = (T_xX)'$. Moreover if $c = (U, \psi, (E, \Delta))$is a chart of X then $t_c = (U, \phi, E)$ is a vector chart of (TX, X, π), where $\phi\colon\ U \times E \to \sum_{b \in U} T_bX$ is given by $\phi(b, v) = (b, \theta_c^b(v))$, and from the vector charts t_c and $t^0 = (U, 1_{|U \times \mathbb{R}}, \mathbb{R})$ of (TX, X, π) and $(X \times \mathbb{R}, X, p_1)$ respectively we have that $(U, \psi', L(E, \mathbb{R}))$ is a vector chart of $((TX)^*, X, \pi_*)$, where $\psi'(b, v) = (b, v(\theta_c^b)^{-1})$.

Proposition 2.8. *Let* $\mathfrak{M} = \{(M^1, B, \pi_1), (M^0, B, \pi_0)\}$ *be a family of* C^p*-vector bundles,*

$$(L(M^1; M^0), B, \pi)$$

the associated linear vector bundle, $S^p_{L(M^1, M^0)}(B)$ *the* $C^p(B, \mathbb{R})$*-module of* C^p*-sections,*

$$\mathrm{Morph}^p(M^1, M^0)$$

the $C^p(B, \mathbb{R})$*-module of* B*-morphisms of class* p *and* $\eta\colon S^p_{L(M^1, M^0)}(B) \to \to \mathrm{Morph}^p(M^1, M^0)$ *the map defined by* $\eta(s)(x) = (s(\pi_1(x))(x)$. *Then* η *is an isomorphism of* $C^p(B, \mathbb{R})$*-modules.* ($\eta^{-1}(f) = s_f$, $s_f(x) = = (x, f_x)$). ■

Lemma 2.1. *Let* $\{(M^i, B, \pi_i)\}_{i=1,\dots,d}$ *be a family of vector bundles of class* p. *Then there is a differentiable structure of class* p *on* $P = = M^1 \times_B \dots \times_B M^d$ *such that* $(P, (p_1)_{|P}, \dots, (p_d)_{|P})$ *is a fibered product of* $\{\pi_i\colon M^i \to B\}_{i=1,\dots,d}$. ■

Let $\{(M^i, B, \pi_i)\}_{i=1,\dots,d} \cup \{(M^0, B, \pi_0)\}$ be a family of C^p-vector bundles and $u\colon M^1 \times_B \dots \times_B M^d \to M^0$ a map. One says that u is a multilinear morphism if for every $b_0 \in B$ there are $t^i = (U, \psi_i, E_i)$ vector chart of (M^i, B, π_i) for every $i = 0, 1, \dots, d$ with $b_0 \in U$ and a C^p-map $\lambda\colon U \to L(E_1, \dots, E_d; E_0)$ such that for every $b \in U$, the diagram

$$\begin{tikzcd}
E_1 \times \dots \times E_d \arrow[r, "\prod_{i=1}^{d} (\psi_i)_b"] \arrow[d] & M^1_b \times \dots \times M^d_b \arrow[d] \\
E_0 \arrow[r, "(\psi_0)_b"] & M^0_b
\end{tikzcd}$$

is commutative. In this case $u_b\colon M^1_b \times \dots \times M^d_b \to M^0_b$ is a d-linear continuous map, u is a C^p-map, $\varphi_i\colon M^i \to M^0$, defined by $\varphi_i(x_i) = = u(0, \dots, x_i, \dots, 0)$, is a B-morphism of class p, $i = 1, \dots, d$, and, if $\{\pi_i\colon M^i \to B\}_{i=1,\dots,d}$ is transversal (which is equivalent to $\partial B = \emptyset$, 4.1.19 and 7.2.4 of [7]), $M^1 \times_B \dots \times_B M^d$ is a totally neat submanifold of $M^1 \times \dots \times M^d$ and both differentiable structures coincide.

Let $\{(M^i, B, \pi_i)\}_{i=1,\dots,d} \cup \{(M^0, B, \pi_0)\}$ be a family of C^p-vector bundles,

$$(L(M^1, \dots M^d; M^0), B, \pi)$$

the multilinear vector bundle and $\mathrm{MultMorph}(M^1 \times_B \dots \times_B M^d; M^0)$ the set of multilinear morphisms. Then there is a bijective map

$$\phi\colon S^p_{L(M^1, \dots M^d; M^0)}(B) \to \mathrm{MultMorph}(M^1 \times_B \dots \times_B M^d; M^0)$$

such that $\phi(s)(x_1, \dots, x_d) = s(\pi_1(x_1))(x_1, \dots, x_d)$.

Let (M, B, π) be a C^p-vector bundle, $(L(M, B \times \mathbb{R}), B, \pi_*)$ the dual vector bundle of (M, B, π) and $s\colon B \to M$, $s^*\colon B \to L(M, B \times \mathbb{R})$ sections of class p. Then $\sigma\colon B \to B \times \mathbb{R}$, $\sigma(b) = ((b, s^*(b)(s(b)))$ is a C^p-section of $(B \times \mathbb{R}, B, p_1)$.

Let us consider $I = I_+ \cup I_-$, $I_+ = \{0\}$, $I_- = \{1\}$, $d \geqslant 1$, $m_d\colon \mathfrak{C}_I \to \mathfrak{B}$ the covariant functor defined by $m_d(\{E_0, E_1\}) = \mathrm{Mult}^d(E_1, E_0)$, $m_d((f_0, f_1))(u) = f_0 \circ u \circ f_1^d$ and $\mathfrak{M} = \{(M^0, B, \pi_0), (M^1, B, \pi_1)\}$ a family of C^p-vector bundles. Then m_d is a vector functor of type I and class p, which will be called d-linear functor and the C^p-vector bundle $((m_d(\mathfrak{M}) = \sum_{b \in B} \mathrm{Mult}^d(M_b^1, M_b^0), B, \pi)$ will be called d-linear vector bundle and will be denoted by

$$(\mathrm{Mult}^d(M^1, M^0), B, \pi).$$

Let us consider $I = I_+ \cup I_-$, $I_+ = \{0\}$, $I_- = \{1\}$, $d \geqslant 1$, $a_d\colon \mathfrak{C}_I \to \mathfrak{B}$ the covariant functor defined by

$$a_d(\{E_0, E_1\}) = \mathrm{Alt}^d(E_1, E_0) = \{f \in \mathrm{Mult}^d(E_1, E_0) / f(x_1, \dots, x_d) = 0$$
$$\text{whenever there exist } i, j \in \{1, \dots, d\} \text{ with } i \neq j \text{ and } x_i = x_j\},$$

$a_d(f_0, f_1)(u) = f_0 \circ u \circ f_1^d$ and $\mathfrak{M} = \{(M^0, B, \pi_0), (M^1, B, \pi_1)\}$ a family of C^p-vector bundles. Then a_d is a vector functor of type I and class p, which will be called d-linear alternating functor, and the C^p-vector bundle $(a_d(\mathfrak{M}) = \sum_{b \in B} \mathrm{Alt}^d(M_b^1, M_b^0), B, \pi)$, will be called d-linear alternating vector bundle and will be denoted by $(\mathrm{Alt}^d(M^1, M^0), B, \pi)$.

Moreover $\mathrm{Alt}^d(M^1, M^0)$ is a C^p-vector subbundle of $(\mathrm{Mult}^d(M^1, M^0), B, \pi)$ and the associated vector bundle structure coincides with one given by the d-linear alternating vector bundle. Finally if $w\colon B \to \mathrm{Alt}^d(M^1, M^0)$ is a C^p-section and $s_1, \dots, s_d$ are C^p-sections of (M^1, B, π_1), there exists a unique C^p-section $w(s_1, \dots, s_d)\colon B \to M^0$ of (M^0, B, π_0) such that $w(s_1, \dots, s_d)(b) = w(b)(s_1(b), \dots, s_d(b))$.

Let $\{(M^i, B, \pi_i)\}_{i=1,\dots,n} \cup \{(M, B, \pi)\}$ be a family of C^p-vector bundles, $u\colon M^1 \times_B \dots \times_B M^n \to M$ a multilinear morphism and $s_1, \dots, s_n$ C^p-sections of $(M^1, B, \pi_1), \dots, (M^n, B, \pi_n)$ respectively. Then there is a unique C^p-section $u(s_1, \dots, s_n)\colon B \to M$ of (M, B, π) such that $u(s_1, \dots, s_n)(b) = u(s_1(b), \dots, s_n(b))$.

Let $(M, B, \pi), (M', B, \pi'), (M'', B, \pi'')$ and (M^0, B, π_0) be vector bundles of class p, ψ: $M \times_B M' \to M''$ a bilinear morphism and $d \geqslant \geqslant 1$, $l \geqslant 1$. Then u_ψ: $\mathrm{Alt}^d(M^0, M) \times_B \mathrm{Alt}^l(M^0, M') \to \mathrm{Alt}^{d+l}(M^0, M'')$, defined by

$$u_\psi(h_1, h_2)(x_1, \ldots, x_d, x_{d+1}, \ldots, x_{d+l}) =$$

$$= \sum_{\sigma \in P_{d+l}} i(\sigma)\psi_b(h_1(x_{\sigma(1)} \ldots x_{\sigma(d)}), h_2(x_{\sigma(d+1)}, \ldots, x_{\sigma(d+l)}))$$

where $\sigma(1) < \ldots < \sigma(d)$, $\sigma(d+1) < \ldots < \sigma(d+l)$, $(x_1 \ldots x_{d+l}) \in (M_b^0)^{d+l}$, $b = \bar{\pi}(h_1) = \bar{\bar{\pi}}(h_2)$, is a bilinear morphism and if $\{\pi, \pi'\}$ is transversal, then $\{\bar{\pi}, \bar{\bar{\pi}}\}$ is transversal. Here $\bar{\pi}$: $\mathrm{Alt}^d(M^0, M) \to B$ and $\bar{\bar{\pi}}$: $\mathrm{Alt}^l(M^0, M') \to B$ denote the projections of the respective vector bundles.

Let $(M, B, \pi), (M', B, \pi'), (M'', B, \pi'')$ and (M^0, B, π_0) be vector bundles of class p, ψ: $M \times_B M' \to M''$ a bilinear morphism, $d \geqslant 1$, $l \geqslant 1$, u_ψ: $\mathrm{Alt}^d(M^0, M) \times_B \mathrm{Alt}^l(M^0, M') \to \mathrm{Alt}^{d+l}(M^0, M'')$, the bilinear morphism previously defined , w: $B \to \mathrm{Alt}^d(M^0, M)$ and w_1: $B \to \to \mathrm{Alt}^l(M^0, M')$ C^p-sections and s_i: $B \to M^0$ a C^p-section $i = 1, \ldots, d+l$. Now we consider the C^p-section $u_\psi(w, w_1) = w \wedge_\psi w_1$: $B \to \to \mathrm{Alt}^{d+l}(M^0, M'')$ defined by $(w \wedge_\psi w_1)(b) = u_\psi(w(b), w_1(b))$ and the C^p-section $(w \wedge_\psi w_1)(s_1, \ldots s_{d+l})$: $B \to M''$ defined by

$$(w \wedge_\psi w_1)(s_1, \ldots s_{d+l})(b) = (w \wedge_\psi w_1)(b)(s_1(b), \ldots s_{d+l}(b)).$$

Then

$$(w \wedge_\psi w_1)(s_1, \ldots s_{d+l}) = \sum_\sigma i(\sigma)\psi(w(s_{\sigma(1)}, \ldots, s_{\sigma(d)}), w_1(s_{\sigma(d+1)}, \ldots, s_{\sigma(d+l)}))$$

where $\sigma(1) < \ldots < \sigma(d)$, $\sigma(d+1) < \ldots < \sigma(d+l)$.

3. – Connections associated to regular equivalence relations.

Let M be a C^p-manifold and S a regular equivalence relation on M. Then there exists a unique C^p-differentiable structure on M/S such that q: $M \to M/S$ is a C^p-submersion (q is the natural projection).

Thus the sequence $TM \xrightarrow{T_*q} q^*(T(M/S)) \to M \times \{0\}$ of C^{p-1}-vector bundles is exact, $V^S(M) = \mathrm{Ker}\, T_*q = \sum_{x \in M} \mathrm{Ker}(T_x q) = \sum_{x \in M} V_x^S(M)$ is a C^{p-1}-vector subbundle of TM and

$$M \times 0 \to V^S(M) \xrightarrow{j} TM \xrightarrow{T_*q} q^*(T(M/S)) \to M \times 0$$

is an exact sequence of C^{p-1}-vector bundles.

$$(T_* q(x, v) = (x, q(x), T_x q(v))).$$

Note that if $S(\partial M) = \partial M$, for every $x \in M$, $q^{-1}(q(x))$ is a C^p-submanifold of M without boundary and $\operatorname{Ker} T_y q = T_y j_x (T_y) q^{-1}(q(x)))$ for every $y \in q^{-1}((q(x))$, where j_x: $q^{-1}(q(x)) \to M$ is the inclusion map. Hence $V^S(M) = \sum_{x \in M} T_x j_x (T_x q^{-1}(q(x)))$.

PROPOSITION 3.1. *Let M be a C^p-manifold, S a regular equivalence relation on M and η: $M \times G \to M$ a C^p-action on the right over M of a Lie group G of class p, which is compatible with S, i.e., $(x, \eta(x, g)) \in S$ for every $(x, g) \in M \times G$. For every $g \in G$, we consider the C^p-diffeomorphism η_g: $M \to M$ defined by $\eta_g(x) = \eta(x, g)$. Then, for every $(x, g) \in \in M \times G$, $T_x \eta_g (V_x^S(M)) = V_{x.g}^S(M)$.*

PROOF. Let q: $M \to M/S$ be the natural projection. Then $T_x q = = T_{x.g} q \circ T_x \eta_g$, $V_x^S(M) = \operatorname{Ker} T_x q = (T_x \eta_g)^{-1} (\operatorname{Ker} T_{x.g} q) = (T_x \eta_g)^{-1} \cdot \cdot (V_{x.g}^S(M))$ and $V_{x.g}^S(M) = T_x \eta_g (V_x^S(M))$. ■

DEFINITION 3.1. *Let M be a C^p-manifold, S a regular equivalence relation on M and η: $M \times G \to M$ a C^p-action on the right over M of a Lie group of class p, compatible with S. We say that $H^S(M) \subsetneq TM$ is a G-connection on M associated to S if*

(i) *$H^S(M)$ is a C^{p-1}-vector subbundle of TM and the map θ_S^G: $V^S(M) \oplus H^S(M) \to TM$ defined by $\theta_S^G(x, (v_1, v_2)) = (x, v_1 + v_2)$ is a M-isomorphism of class $p - 1$.*

(ii) *For every $(x, g) \in M \times G$, $T_x \eta_g (H_x^S(M)) = H_{x.g}^S(M)$ where $(H^S(M))_x = \{x\} \times H_x^S(M)$.*

The Proposition 2.7 suggests the following characterizations of G-connections:

PROPOSITION 3.2. *Let M be a C^p-manifold, S a regular equivalence relation on M and η: $M \times G \to M$ a C^p-action on the right over M of a Lie group G of class p, compatible with S. Then the following statements are equivalent:*

(i) *There is a G-connection $H^S(M)$ on M associated to S.*

(ii) *There is a M-morphism of class $p - 1$*

$$\psi\colon\ q^*(T(M/S)) \to TM$$

such that:

(*a*) $T_* q \circ \psi = 1_{q^*(T(M/S))}$. *(Hence,* $\operatorname{im}\psi$ *is a vector subbundle of class* $p-1$ *of TM).*

(*b*) *For every* $(x, g) \in M \times G$, $\psi_{\eta(x,g)} = T_x \eta_g \circ \psi_x$ *where* ψ_x: $T_{q(x)}(M/S) \to T_x M$ *is the continuous linear map induced by* ψ *at* x $(\psi_x(v) = p_2 \circ \psi(x, q(x), v))$.

(iii) *There is a M-morphism of class* $p-1$ ϕ: $TM \to V^S(M)$ *such that:*

(*a*) $\phi \circ j = 1_{V^S(M)}$. *(Hence,* $\ker\phi$ *is a vector subbundle of class* $p-1$ *of TM).*

(*b*) *For every* $(x, g) \in M \times G$, $\phi_{\eta(x,g)} \circ T_x \eta_g = T_x \eta_g \circ \phi_x$.

PROOF. (i) $\to$ (iii) We have that $\phi = p_1 \circ (\theta^G_S)^{-1}$: $TM \to V^S(M)$ is a M-morphism of class $p-1$, $\phi \circ j = 1_{V^S(M)}$ and $\ker\phi = H^S(M)$. Let (x, g) be an element of $M \times G$ and $u \in T_x M$. Then $u = u_1 + u_2$ with $u_1 \in V^S_x(M)$, $u_2 \in H^S_x(M)$, $T_x\eta_g(u) = T_x\eta_g(u_1) + T_x\eta_g(u_2)$, $T_x\eta_g(u_1) \in V^S_{x.g}(M)$ and $T_x\eta_g(u_2) \in H^S_{x.g}(M)$. Therefore, $T_x\eta_g\phi_x(u) = T_x\eta_g(u_1) = \phi_{x.g} T_x \eta_g(u)$.

(iii) $\to$ (ii) We have that $\operatorname{Ker}\phi$ is a subbundle of TM, φ: $V^S(M) \oplus \operatorname{Ker}\phi \to TM$, defined by $\varphi(x, (v_1, v_2)) = (x, v_1 + v_2)$, is a M-isomorphism of class $p-1$ and $(T_* q)_{|\operatorname{Ker}\phi}$: $\operatorname{Ker}\phi \to q^*(T(M/S))$ is a M-isomorphism of class $p-1$. We define ψ by $\psi = i \circ ((T_* q)_{|\operatorname{Ker} f})^{-1}$: $q^*(T(M/S)) \to TM$. Then ψ is a M-morphism of class $p-1$, $\psi_x = ((T_x q)_{|\operatorname{Ker}\phi_x})^{-1}$ and $T_* q \circ \psi = 1_{q^*(T(M/S))}$.
Let (x, g) be an element of $M \times G$ and $u \in T_{q(x)}(M/S)$. Then $\psi_x(u) = v \in T_x M$ where $T_x q(v) = u$ and $v \in \operatorname{Ker}\phi_x$. On the other hand the equality $\phi_{x.g} \circ T_x\eta_g(v) = T_x\eta_g(\phi_x(v)) = 0$ implies that $T_x\eta_g(v) \in \operatorname{Ker}(\phi_{x.g})$ and $T_x q(v) = u = T_{\eta(x,g)} q \circ T_x \eta_g(v)$. Therefore $\psi_{x.g}(u) = T_x\eta_g(v) = T_x\eta_g\psi_x(u)$.

(ii) $\to$ (i) The condition $T_* q \circ \psi = 1_{q^*(T(M/S))}$ implies that $\operatorname{im}\psi$ is a C^{p-1}-vector subbundle of TM and the map θ^G_S: $V^S(M) \oplus \operatorname{im}(\psi) \to TM$, defined by $\theta^G_S(x, (v_1, v_2)) = (x, v_1 + v_2)$, is a M-isomorphism of class $p-1$.
Let (x, g) be an element of $M \times G$, then by (*b*) of (ii), $\operatorname{im}(\psi_{x.g}) = T_x\eta_g(\operatorname{im}(\psi_x))$. ∎

REMARK 3.1. *With the hypotheses of Proposition* 3.2, *we have:*

(1) *If* ψ: $q^*(T(M/S)) \to TM$ *verifies the conditions of* (ii), *then* $\operatorname{im}(\psi)$ *is a G-connection on M associated to S.*

(2) *If* ϕ: $TM \to V^S(M)$ *verifies the conditions of* (iii), *then* $\ker(\phi)$ *is a G-connection on M associated to S.*

4. - Connections on principal bundles with corners.

Let $\lambda = (P, G, \eta)$ be a principal bundle of class p and $\pi_\lambda \colon P \to P/G$ the canonical projection. In [9] we proved that π_λ is a surjective C^p-submersion that preserves the boundary ($\pi_\lambda(\partial P) = \partial(P/G)$) and R_G is a regular equivalence relation on P that verifies $R_G(\partial P) = \partial P$ and $\eta \colon P \times G \to P$ is a C^p-action compatible with R_G. Hence R_G is a neat C^p-submanifold of $P \times P$ and $(p_1)_{|R_G} \colon R_G \to P$ is a surjective C^p-submersion.

Then we have the exact sequence of C^{p-1}-vector bundles

$$P \times 0 \to V^{R_G}(P) \xrightarrow{j} TP \xrightarrow{T_*\pi_\lambda} \pi_\lambda^*(T(P/G)) \to P \times 0 .$$

In this case the C^{p-1}-vector bundle $(V^{R_G}(P), P, (\tau_P)_{|V^{R_G}(P)})$ is trivializable.

PROPOSITION 4.1. *Let $\lambda = (P, G, \eta)$ be a principal bundle of class p. For every $x \in P$, consider the C^p-diffeomorphism $\eta_x \colon G \to \pi_\lambda^{-1}(\pi_\lambda(x))$ defined by $\eta_x(g) = x.g$. Then $\Psi \colon P \times T_e G \to V^{R_G}(P)$ defined by $\Psi(x, u) = (x, T_x j_x T_e \eta_x(u))$ is a P-isomorphism of class $p-1$, where $j_x \colon \pi_\lambda^{-1}(\pi_\lambda(x)) \to P$ is the inclusion map.*

PROOF. Firstly we note that $T_x \pi_\lambda T_x j_x T_e \eta_x(u) = 0$. Let x be an element of P, $c_1 = (U_1, \varphi_1, E_1)$ a chart of G centred at e, $c = (U, \varphi, (E, \Delta))$ a chart of P centred at x and V an open neighbourhood of x in U such that $\eta(V \times U_1) \subseteq U$. We consider $t = (V, 1_{V \times T_e G}, T_e G)$ vector chart of $(P \times T_e G, P, p_1)$ and $t_c = (V, \varphi_c, E)$ vector chart of (TP, P, τ_P) where $\varphi_c \colon V \times E \to \tau_P^{-1}(V)$ is defined by $\varphi_c(y, v) = (y, \theta_c^y(v))$.

Let $\lambda \colon V \to L(T_e G, E)$ be the C^{p-1}-map defined by

$$\lambda(y) = D_2(\varphi \circ \eta \circ (\varphi^{-1} \times \varphi_1^{-1}))(\varphi(y), 0) \circ (\theta_{c_1}^e)^{-1}$$

for every $y \in V$. Then, for every $y \in V$, the diagram

$$\begin{tikzcd} T_e G \arrow[r, "1_{T_e G}"] \arrow[d, "\lambda(y)"'] & T_e G \arrow[d, "\Psi_y"] \\ E \arrow[r, "(\varphi_c)_y"] & T_y P \end{tikzcd}$$

is commutative. This proves that $\Psi \colon P \times T_e G \to TP$ is a P-morphism of class $p-1$. Since $\Psi(P \times T_e G) \subseteq V^{R_G}(P)$ and $\Psi \colon P \times T_e G \to V^{R_G}(P)$

is a bijective map, we have that Ψ is a P-isomorphism of class $p-1$. ■

DEFINITION 4.1. *Let $\lambda = (P, G, \eta)$ be a principal bundle of class p. A principal connection on λ is a G-connection $H(P)$ on P associated to R_G.*

If $H(P)$ is a principal connection on λ, then $\phi = p_1 \circ (\theta^G_{R_G})^{-1}$: $TP \to \to V^{R_G}(P)$ is a P-morphism of class $p-1$, $\operatorname{Ker}\phi = H(P)$, $(T_* \pi_\lambda)_{|\operatorname{Ker}\phi}$: $\operatorname{Ker}\phi \to \pi^*_\lambda(T(P/G))$ is a P-isomorphism of class $p-1$ and $\psi = i \circ [(T_* \pi_\lambda)_{|\operatorname{Ker}\phi}]^{-1}$: $\pi^*_\lambda(T(P/G)) \to TP$ is a P-morphism of class $p-1$, which will be called P-morphism associated to $H(P)$, and $\operatorname{im}(\psi) = H(P)$, $T_*(\pi_\lambda) \circ \psi = 1_{\pi^*_\lambda(T(P/G))}$ and $\psi_{x.g} = T_x \eta_g \circ \psi_x$. In the sequel ψ will be also called principal connection on λ.

From Remark 3.1, if

$$\psi\colon \pi^*_\lambda(T(P/G)) \to TP$$

is a P-morphism of class $p-1$ such that $T_* \pi_\lambda \circ \psi = 1_{\pi^*_\lambda(T(P/G))}$ and for every $(x, g) \in P \times G$

$$\psi_{\eta(x,g)} = (T_x \eta_g) \circ \psi_x ,$$

then $\operatorname{im}\psi$ is a principal connection on λ, whose associated P-morphism is ψ.

Let $\lambda = (P, G, \eta)$ be a principal bundle of class p, $H(P)$ a principal connection of λ and X: $P \to TP$ a vector field on P of class $p-1$. Then the maps X^V: $P \to V^{R_G}(P)$ and X^H: $P \to H(P)$, defined by $X^V = p_1 \circ (\theta^G_{R_G})^{-1} \circ X$, and $X^H = p_2 \circ (\theta^G_{R_G})^{-1} \circ X$ are vector fields of class $p-1$ which verify $X(x) = X^V(x) + X^H(x)$ for every $x \in P$.

Let $\lambda = (P, G, \eta)$ be a principal bundle of class p, $H(P)$ a principal connection of λ and η_x: $G \to \pi_\lambda^{-1}(\pi_\lambda(x))$ the C^p-diffeomorphism defined by $\eta_x(g)) = \eta(x, g)$. Then the map w: $P \to L(TP, P \times T_e G)$ defined by $w(x) = (x, w_x)$ with w_x: $T_x P \to T_e G$, $w_x(v) = (T_e \eta_x)^{-1} \circ (p_1)_x \circ \circ ((\theta^G_{R_G})_x)^{-1}(v)$ will be called connection form of $H(P)$.

PROPOSITION 4.2. *Let $\lambda = (P, G, \eta)$ be a principal bundle of class p and $H(P)$ a principal connection of λ. Then the connection form w: $P \to L(TP, P \times T_e G)$ of $H(P)$ is a C^{p-1}-section.*

PROOF. By Proposition 4.1 the map f_w: $TP \to P \times T_e G$ defined by

$$f_w(x, v) = (x, (T_e \eta_x)^{-1} \circ (p_1)_x \circ (\theta^G_{R_G})^{-1}_x(v)),$$

is a P-morphism of class $p-1$ since $f_w = \Psi^{-1} \circ p_1 \circ (\theta^G_{R_G})^{-1}$. Now using Proposition 2.8 one obtains that w is a C^{p-1}-section. ■

Clearly for every $v \in T_x P$, $v \in H_x(P)$ if and only if $w_x(v) = 0$.

PROPOSITION 4.3. *Let $\lambda = (P, G, \eta)$ be a principal bundle of class p. Then for every $u \in T_e G$ the map $Z_u: P \to TP$ defined by $Z_u(x) = (x, T_e \eta_x(u))$, is a vector field of class $p-1$. (Z_u is called the Killing vector field associated to u). Hence $Z_u: P \to V^{R_G}(P)$ is a C^{p-1}-section.*

PROOF. The map $X_u: P \to P \times T_e G$ defined by $X_u(x) = (x, u)$ is a C^p-section of the C^p-vector bundle $(P \times T_e G, P, p_1)$. Then the result follows from Proposition 4.1, since $Z_u = j \circ \Psi \circ X_u$. ($j: V^{R_G}(P) \to TP$). ■

The following result establishes a bijection between the principal connections and the set of one-forms that verifies some properties.

PROPOSITION 4.4. Let *$\lambda = (P, G, \eta)$ be a principal bundle of class p and $w: P \to L(TP, P \times T_e G)$ a C^{p-1}-section. Then the following statements are equivalent:*

1) *There is a principal connection $H(P)$ on λ such that w is the connection form of $H(P)$.*

2) *w verifies the following conditions*

(*a*) *For every $(x, g) \in P \times G$*

$$w_{x.g} \circ T_x \eta_g = T_e \varphi_g \circ w_x$$

where $\varphi_g: G \to G$ is given by $\varphi_g(s) = g^{-1} s g$ ($T_e \varphi_g$ is usually denoted by $Ad(g^{-1})$).

(*b*) *For every $u \in T_e G$,*

$$w_x((Z_u)_x) = u, \qquad \textit{for every } x \in P.$$

PROOF. Suppose that w is the connection form of a principal connection $H(P)$. Then

$$\phi = p_1 \circ (\theta^G_{R_G})^{-1}: TP \to V^{R_G}(P)$$

is a P-morphism of class $p-1$ and $\phi_{x.g} \circ T_x\eta_g = T_x\eta_g \circ \phi_x$.

$$(a) \quad w_{x.g}(T_x\eta_g) = (T_e\eta_{x.g})^{-1} \circ \phi_{x.g} \circ T_x\eta_g = (T_e\eta_{x.g})^{-1} \circ T_x\eta_g \circ \phi_x = \\ = T_e\varphi_g \circ (T_e\eta_x)^{-1} \circ \phi_x = Ad(g^{-1})w_x .$$

$$(b) \quad w_x \circ T_e\eta_x(u) = (T_e\eta_x)^{-1} \circ \phi_x \circ T_e\eta_x(u) = (T_e\eta_x)^{-1}(T_e\eta_x(u)) = u .$$

Suppose that w verifies (a) and (b). Let f_w: $TP \to P \times T_eG$ be the P-morphism of class $p-1$ associated to w and Ψ: $P \times T_eG \to V^{R_G}(P)$ the P-isomorphism of class $p-1$ given by Proposition 4.1. Then $\phi = \Psi \circ f_w$: $TP \to V^{R_G}(P)$ is a P-morphism of class $p-1$. We have that:

(i) For every $(x, v) \in V^{R_G}(P)$, $\phi(x, v) = (x, T_e\eta_x(w_x(v)))$ and there is $u \in T_eG$ such that $T_e\eta_x(u) = v$. Then by (b) $\phi(x, v) = (x, T_e\eta_x w_x T_e\eta_x(u)) = (x, T_e\eta_x(u)) = (x, v)$. Therefore $\phi \circ j = 1_{V^{R_G}(P)}$.

(ii) Let $(x, g) \in P \times G$ and $v \in T_xP$. Then $\phi_{x.g} \circ T_x\eta_g = T_x\eta_g \circ \phi_x$. By Remark 3.1 (2) $H(P) = \operatorname{Ker}\phi$ is a principal connection on λ. It is easy to check that w is the connection form of $H(P)$. (If $v \in T_xP$, $v = \phi_x(v) + (v - \phi_x(v))$). ■

Now we will study the existence of principal connections on principal bundles.

Let $\lambda = (P, G, \eta)$ be a principal bundle of class p and $y \in P/G$. We consider the set

$$E_y = \{X\colon \pi_\lambda^{-1}(y) \to \tau_P^{-1}(\pi_\lambda^{-1}(y))/X \text{ is the class p} - 1,\ \tau_P \circ X = 1_{\pi_\lambda^{-1}(y)}$$
$$\text{and for every } (x, g) \in \pi_\lambda^{-1}(y) \times G,\ X_{x.g} = T_x\eta_g(X_x)\},\quad (X(x) = (x, X_x)).$$

Let us consider the set $E = \{(y, X)/y \in P/G \text{ and } X \in E_y\}$ and the map p: $E \to P/G$ defined by $p(y, X) = y$.

LEMMA 4.1. For every $a \in P$ and $w \in T_aP$, the map $X^{(a,w)}$: $\pi_\lambda^{-1}(\pi_\lambda(a)) \to \tau_P^{-1}(\pi_\lambda^{-1}(\pi_\lambda(a)))$, defined by $X^{(a,w)}(z) = (z, T_a\eta_g(w))$ with $g = \tau(a, z)$, is an element of $E_{\pi_\lambda(a)}$.

PROOF. Let z_0 be an element of $\pi_\lambda^{-1}(\pi_\lambda(a))$, $c = (U, \varphi, (E, \Delta))$ a chart of P with $\varphi(a) = 0$, $c_1 = (U_1, \varphi_1, (E_1, \Delta_1))$ a chart of P with $\varphi_1(z_0) = 0$, (V, ψ, F) a chart of G centred at $\tau(a, z_0)$ and V^* an open set of P such that $z_0 \in V^* \subseteq U_1$, $\eta(U \times V) \subseteq U_1$ and $\tau((\{a\} \times V^*) \cap R_G) \subseteq V$. Then for every

$$z \in V^* \cap \pi_\lambda^{-1}(\pi_\lambda(a)) = B,\ T_a\eta_{\tau(a,z)} = \theta_{c_1}^z D(\varphi_1 \circ \eta_{\tau(a,z)} \circ \varphi^{-1})(0)(\theta_c^a)^{-1} = \\ = \theta_{c_1}^z \circ D_1(\varphi_1 \circ \eta \circ (\varphi^{-1} \times \psi^{-1}))(0, \psi(\tau(a, z)) \circ (\theta_c^a)^{-1} = \theta_{c_1}^z v_z .$$

Consequently the map $(z, v) \in B \times T_a P \to (z, v_z(v)) \in B \times E_1$ is a B-morphism of class $p-1$.

Moreover the map $(z, v) \in B \times E_1 \to (z, \theta^z_{c_1}(v)) \in \sum_{x \in B} T_x P$ is a B-isomorphism of class $p-1$. Consequently $z \in B \to (z, \theta^z_{c_1} \circ v_z(w)) \in \sum_{x \in B} T_x P$ is a C^{p-1}-section. But this last map coincides with $X^{(a, w)}_{|B}$. The other conditions are trivial to be checked. ■

PROPOSITION 4.5. *There is a vector bundle structure of class $p-1$ on $(E, P/G, p)$ such that for every s: $U \to P$ local section of class p of π_λ: $P \to P/G$ and every chart $c = (V, \varphi, (F, \Delta))$ of P such that $s^{-1}(V) \neq \emptyset$ then $(s^{-1}(V), \psi, F)$ is a vector chart of such structure, where*

$$\psi\colon s^{-1}(V) \times F \to p^{-1}(s^{-1}(V))$$

is given by $\psi(y, v) = (y, X^{(s(y), \theta^{s(y)}_c(v))})$.

PROOF It is easy to prove that ψ is a bijective map and $p \circ \psi = p_1$.

If $t_1 = (s_1^{-1}(V_1), \psi_1, F_1)$ and $t_2 = (s_2^{-1}(V_2), \psi_2, F_2)$ are vector charts associated to the C^p-sections s_1: $U_1 \to P$, s_2: $U_2 \to P$ and the charts of P $c_1 = (V_1, \varphi_1, (F_1, \Delta_1))$, $c_2 = (V_2, \varphi_2, (F_2, \Delta_2))$, the map

$$\mu\colon s_1^{-1}(V_1) \cap s_2^{-1}(V_2) = A \to L(F_1, F_2)$$

given by $\mu(y) = (\theta^{s_2(y)}_{c_2})^{-1} \circ T_{s_1(y)} \eta_g \circ \theta^{s_1(y)}_{c_1} = D(\varphi_2 \circ \eta_g \circ \varphi_1^{-1})(\varphi_1 \circ s_1(y))$ is a C^{p-1}-map, being $g = \tau(s_1(y), s_2(y))$. Indeed, let z_0 be an element of A, (W, α, F) a chart of G with $\tau(s_1(z_0), s_2(z_0)) \in W$ and V^*, V^{**} open sets such that $s_1(z_0) \in V^* \subseteq V_1$, $s_2(z_0) \in V^{**} \subseteq V_2$, $\eta(V^* \times W) \subseteq V_2$ and $\tau((V^* \times V^{**}) \cap R_G) \subseteq W$. Let Z be an open set such that $Z \subseteq A$, $z_0 \in Z$, $s_1(Z) \subseteq V^*$, $s_2(Z) \subseteq V^{**}$. Then for every

$$y \in Z,\ \mu(y) = D_1(\varphi_2 \circ \eta \circ (\varphi_1^{-1} \times \alpha^{-1})(\varphi_1 s_1(y), \alpha(\tau(s_1(y), s_2(y)))).$$

Finally $(\psi_2)_y \circ \mu(y) = (\psi_1)_y$ and the charts t_1 and t_2 are compatible. ■

PROPOSITION 4.6. Let ϕ: $E \to T(P/G)$ be the map defined by $\phi(y, X) = (y, T_x \pi_\lambda(X_x))$ where $\pi_\lambda(x) = y$ (Note that if $\pi_\lambda(x') = y$ then $T_x \pi_\lambda(X_x) = T_{x'} \pi_\lambda(X_{x'}) \in T_y(P/G)$ because of $X_{x.g} = T_x \eta_g(X_x)$ where $g = \tau(x, x')$). Then ϕ is a surjective P/G-morphism of class $p-1$.

PROOF. If $y = \pi_\lambda(x) \in P/G$, there is an open neighbourhood V of y and there is a C^p-map s: $V \to P$ such that $s(y) = x$ and $\pi_\lambda \circ s = 1_V$.

Let $c = (U, \varphi, (F, \Delta))$ be a chart of P and $c' = (U', \varphi', (F', \Delta'))$ a chart of P/G such that $x \in U$ and $\pi_\lambda(U) \subseteq U'$. Then $(s^{-1}(U) \cap \cap U', \psi', F')$ and $(s^{-1}(U) \cap U', \psi, F)$ are vector charts of $T(P/G)$ and E respectively, where $\psi'(y, v) = (y, \theta^y_{c'}(v))$ and $\psi(y, v) = = (y, X^{(s(y), \theta_c^{s(y)}(v))})$, and λ: $s^{-1}(U) \cap U' \to L(F, F')$ given by $\lambda(y) = = (\theta^y_{c'})^{-1} \circ T_{s(y)}\pi_\lambda \circ \theta_c^{s(y)} = D(\varphi' \circ \pi_\lambda \circ \varphi^{-1})(\varphi(s(y)))$ is a C^{p-1}-map and $\phi_y \circ \psi_y = \psi'_y \circ \lambda(y)$ which prove that ϕ is a P/G-morphism of class $p - 1$. ■

Proposition 4.7. *The set* $E' = \{(y, X) \in E / X(\pi_\lambda^{-1}(y)) \subseteq V^{R_G}(P)\}$ *is a* C^{p-1}*-vector subbundle of* $(E, P/G, p)$.

Proof. We have that ϕ_y is surjective for every $y \in P/G$ (Prop. 4.6). From the formula $\phi_y \circ \psi_y = \psi'_y \circ \lambda(y)$ of the proof of 4.6, we have that $\operatorname{Ker} \phi_y$ admits topological supplement. Then by Proposition 2.4 $\operatorname{Ker} \phi = = E'$ is a C^{p-1}-vector subbundle of E. ■

Now the sequence

$$P/G \times 0 \to E' \xrightarrow{i} E \xrightarrow{\phi} T(P/G) \to P/G \times 0$$

is exact. By Proposition 2.6 if P/G is a paracompact manifold that admits partitions of unity of class $p - 1$, then there is a P/G-morphism g: $T(P/G) \to E$ of class $p - 1$ such that $\phi \circ g = 1_{T(P/G)}$.

This P/G-morphism g of class p-1 induces a unique P-morphism of class $p - 1$ g_*: $\pi_\lambda^*(T(P/G)) \to \pi_\lambda^*(E)$ such that $p_2 \circ g_*(x, \pi_\lambda(x), v) = = g(\pi_\lambda(x), v))$, $v \in T_{\pi_\lambda(x)}(P/G)$, where

$$g_*(x, \pi_\lambda(x), v) = (x, g(\pi_\lambda(x), v)) = (x, \pi_\lambda(x), X), X \in E_{\pi_\lambda(x)}.$$

Proposition 4.8. *The map* ϕ_2: $\pi_\lambda^*(E) \to TP$ *defined by* $\phi_2(x, \pi_\lambda(x), X) = (x, X_x)$ *is a P-isomorphism of class* $p - 1$.

Proof. Let x be an element of P, s: $V \to P$ a local section of class p of π_λ with $\pi_\lambda(x) \in V$, $s(\pi_\lambda(x)) = x$ and $c = (U, \varphi(F, \Delta))$ a chart of P with $s(V) \subseteq U$. Then $(\pi_\lambda^{-1}(s^{-1}(U)) \cap U, \psi, F)$ and $(\pi_\lambda^{-1}(s^{-1}(U)) \cap \cap U, \psi', F)$ are vector charts of $\pi_\lambda^*(E)$ and TP respectively, where $\psi(z, v) = (z, \pi_\lambda(z), X^{(s(\pi_\lambda(z)), \theta_c^{s(\pi_\lambda(z))}(v))})$ and $\psi'(z, v) = (z, \theta_c^z(v))$. Finally β: $\pi_\lambda^{-1}(s^{-1}(U)) \cap U \to L(F, F)$, defined by

$$\begin{aligned}\beta(z) &= (\theta_c^z)^{-1} \circ T_{s(\pi_\lambda(z))}\eta_g \circ \theta_c^{s(\pi_\lambda(z))} = \\ &= D(\varphi \circ \eta_g \circ \varphi^{-1})(\varphi s \pi_\lambda(z))) = \\ &= D_1(\varphi \circ \eta \circ (\varphi^{-1} \times \alpha^{-1}))(\varphi s \pi_\lambda(z), \alpha(\tau(s\pi_\lambda(z), z))\end{aligned}$$

where (W, α, L) is a convenient chart of G, being $g = \tau(s(\pi_\lambda(z)), z)$, is a C^{p-1}-map and $(\phi_2)_b \circ \psi_b = \psi'_b \circ \beta(b)$ for every $b \in \pi_\lambda^{-1}(s^{-1}(U)) \cap U$, which proves that ϕ_2 is a P-morphism. ∎

Now the map $\Psi = \phi_2 \circ g_*: \pi_\lambda^*(T(P/G)) \to TP$ is a principal connection on λ. Indeed, Ψ is a P- morphism of class $p-1$ which verifies that $T_* \pi_\lambda \circ \Psi = 1_{\pi_\lambda^*(T(P/G))}$. Moreover $\Psi_{x.g} = T_x \eta_g \circ \Psi_x$ for every $(x, g) \in P \times G$, because of $\Psi_x(v) = X_x$ where $g(\pi_\lambda(x), v) = (\pi_\lambda(x), X)$, $\Psi_{x.g}(v) = X'_{x.g}$ where $g(\pi_\lambda(x.g), v) = (\pi_\lambda(x.g), X')$ and $X = X' \in E_{\pi_\lambda(x)}$.

Then we have the following theorem.

THEOREM 4.1. *Let $\lambda = (P, G, \eta)$ be a principal bundle of class p. Suppose that P/G is a paracompact C^p-manifold which admits partitions of unity of class $p-1$. Then λ admits principal connections.*

5. - Linear connections.

In this paragraph we explain the linear connections as $(\mathbb{R} - \{0\})$-connections. To obtain this objetive we previously establish some specific properties of vector bundles.

PROPOSITION 5.1. *Let (M, B, π) be a vector bundle of class $p \geq 1$. Then $(TM, TB, T\pi)$ is a vector bundle of class $p-1$.*

PROOF. Let $c = (U, \varphi, (E, \Delta))$ be a chart of B and $t = (U, \psi, F)$ a vector chart of (M, B, π). Then $c_t = (\pi^{-1}(U), \varphi_{ct} = (\varphi \times 1_F) \circ \psi^{-1}, (E \times F, \Delta \circ p_1))$ is a chart of M, $t_c = (U, \psi_c, E)$ is a vector chart of (TB, B, τ_B) where $\psi_c(x, v) = (x, \theta_c^x(v))$ and $(\tau_B^{-1}(U), (\varphi \times 1_E) \circ \psi_c^{-1}, (E \times E, \Delta \circ p_1)$ is a chart of TB. Notice that if $c' = (U', \varphi', (E', \Delta'))$ is a chart of B and $t' = (U', \psi', F')$ is a vector chart of (M, B, π), then there is a C^p-map $\mu: U \cap U' \to L(F, F')$ such that $\psi_b = \psi'_b \circ \mu(b)$ for every $b \in U \cap U'$ and it is easy to see that

$$(3) \quad (\varphi'_{c't'} \circ \varphi_{ct}^{-1})(x, y) = (\varphi' \circ \varphi^{-1}(x), \mu(\varphi^{-1}(x)(y)),$$

$$(4) \quad D(\varphi'_{c't'} \circ \varphi_{ct}^{-1})(x, y)(x', y') =$$
$$= (D(\varphi' \circ \varphi^{-1})(x)(x'), (D(\mu \circ \varphi^{-1})(x)x')y + \mu(\varphi^{-1}(x))y'),$$

for every $(x, y) \in \varphi(U \cap U') \times F, (x', y') \in E \times F$.

Let us consider the map $\psi_2: \tau_B^{-1}(U) \times F \times F \to (T\pi)^{-1}\tau_B^{-1}(U) =$

$= \sum_{y \in \pi^{-1}(U)} T_y M$ defined by

$$\psi_2(x, v, v_1, v_2) = \big(\psi(x, v_1),\, T_{(x, v_1)}\psi \circ \theta^{(x, v_1)}_{c \times c_F}((\theta^x_c)^{-1}(v), v_2)\big)$$

where $C_F = (F, 1_F, F)$. Then $\sigma = (\tau_B^{-1}(U), \psi_2, F \times F)$ is a vector chart of $(TM, TB, T\pi)$. If $c' = (U', \varphi', (E', \Delta'))$, $t' = (U', \psi', F')$ are other charts of B and (M, B, π) respectively, then the map

$$\bar{\mu}\colon \sum_{x \in U \cap U'} T_x B \to L(F \times F, F' \times F')$$

defined by

$$\bar{\mu}(x, v) = \big(\mu(x) \circ p_1,\, \mu(x) \circ p_2 + (D(\mu\varphi^{-1})(\varphi(x))(\theta^x_c)^{-1}(v)) \circ p_1\big)$$

is a C^{p-1}-map and $\bar{\mu}(x, v) = ((\psi'_2)_{(x, v)})^{-1} \circ (\psi_2)_{(x, v)}$ (see (4)). Thus we have proved that σ and σ' are C^{p-1}-compatible vector charts and therefore we have the vector bundle $(TM, TB, T\pi)$. ■

Now using the charts $(\tau_B^{-1}(U), (\varphi \times 1_E) \circ \psi_c^{-1}, (E \times E, \Delta \circ p_1))$ and σ, one obtains the chart of TM, (as total space of $(TM, TB, T\pi)$)

$$\left(\sum_{y \in \pi^{-1}(U)} T_y M, [(\varphi \times 1_E) \circ \psi_c^{-1} \times 1_{F \times F}] \circ \psi_2^{-1}, (E \times E \times F \times F, \Delta \circ p_1)\right).$$

Finally using the chart c_t, one obtains the vector chart $(\pi^{-1}(U), \psi_{c_t}, E \times F)$ of (TM, M, τ_M) where $\psi_{c_t}(y, w) = (y, \theta^y_{c_t}(w))$, and the chart $(\tau_M^{-1}\pi^{-1}(U), [(\varphi \times 1_F) \circ \psi^{-1} \times 1_{E \times F}] \circ \psi_{c_t}^{-1}, (E \times F \times \times E \times F, \Delta \circ p_1))$ of TM (as total space of (TM, M, τ_M)). It is easy to prove that the two differentiable structures on TM coincide.

Using the vector chart σ, we have that

$$(\psi_2)_{(x, v)}\colon F \times F \to (T\pi)^{-1}(x, v) = \sum_{y \in \pi^{-1}(x)} (T_y\pi)^{-1}(v), \quad (x, v) \in \tau_B^{-1}(U),$$

is a linear homeomorphism and if $(y_1, w_1), (y_2, w_2) \in (T\pi)^{-1}(x, v)$, then

$$(y_1, w_1) + (y_2, w_2) =$$

$$= \big(y_1 + y_2,\, \theta^{y_1 + y_2}_{c_t}((\theta^x_c)^{-1}(v), p_2(\theta^{y_1}_{c_t})^{-1}(w_1)) + p_2(\theta^{y_2}_{c_t})^{-1}(w_2))\big).$$

Let (M, B, π) be a vector bundle of class p. Then we know that

$$M \times 0 \to \ker(T_*\pi) = VM \xrightarrow{j} TM \xrightarrow{T_*\pi} \pi^*(TB) \to M \times 0$$

is exact and $T_*\pi(x, v) = (x, \pi(x), T_x\pi(v))$. If $c = (U, \varphi, (E, \Delta))$ is a chart of B and $t = (U, \psi, F)$ is a vector chart of (M, B, π), then $t_{c_t} = (\pi^{-1}(U), \psi_{c_t}, E \times F)$ is a vector chart of (TM, M, τ_M) where $\psi_{c_t}(y, w) = (y, \theta^y_{c_t}(w))$ and $c_t = (\pi^{-1}(U), (\varphi \times 1_F) \circ \psi^{-1}, (E \times F, \Delta \circ p_1))$ is the associated chart of M.

Moreover $\psi_{c_t}(\pi^{-1}(U) \times 0 \times F) = (\tau_M^{-1}(\pi^{-1}(U))) \cap V(M)$ and

$$(\tau_M^{-1}(\pi^{-1}(U)) \cap V(M), \beta, (E \times F \times \{0\} \times F, \Delta \circ p_1))$$

is a chart of $V(M)$ where $\beta(y, v) = ((\varphi \times 1_F) \circ \psi^{-1}(y), 0, p_2 \circ (\theta^y_{c_t})^{-1}(v))$. In fact this chart is the one induced on $V(M)$ by c_t and the vector chart on $V(M)$, $(\pi^{-1}(U), (\psi_{c_t})_{|\pi^{-1}(U) \times \{0\} \times F}, \{0\} \times F)$.

PROPOSITION 5.2. *There is a unique π-morphism of class $p - 1$, $\varrho: V(M) \to M$ such that for every chart $c = (U, \varphi, (E, \Delta))$ of B and every vector chart $t = (U, \psi, F)$ of (M, B, π), $\varrho(x, v) = \psi_{\pi(x)} \circ p_2 \circ (\theta^x_{c_t})^{-1}(v)$ for every $(x, v) \in V(M) \cap (\sum_{x \in \pi^{-1}(U)} T_xM)$ ($c_t = (\pi^{-1}(U), (\varphi \times 1_F) \circ \psi^{-1}, (E \times F, \Delta \circ p_1))$). Moreover ϱ is an inyective map.*

PROOF. For other charts,

$$\psi'_{\pi(x)} \circ p_2 \circ (\theta^x_{c'_{t'}})^{-1}(v) = \psi_{\pi(x)} \circ p_2 \circ (\theta^x_{c_t})^{-1}(v)$$

because of (see (4))

$$\psi^{-1}_{\pi(x)} \circ \psi'_{\pi(x)} \circ p_2 \circ (\theta^x_{c'_{t'}})^{-1} \circ \theta^x_{c_t}(v_1, v_2) =$$

$$= (\mu(\pi(x)))^{-1} \circ D(\mu \circ \varphi^{-1})(\varphi(\pi(x)))(v_1)(p_2 \psi^{-1}(x)) + v_2 = v_2,$$

where $(\theta^x_{c_t})^{-1}(v) = (v_1, v_2)$ and $v_1 = 0$. Thus ϱ is a well defined map.

We know that $(\pi^{-1}(U), (\psi_{c_t})_{|\pi^{-1}(U) \times \{0\} \times F}, 0 \times F)$ is a vector chart of $V(M)$.

Now the map $\mu': \pi^{-1}(U) \to L(F, F)$ defined by $\mu'(b) = 1_F$ for every $b \in \pi^{-1}(U)$ verifies that $\varrho_b((\psi_{c_t})_{|\pi^{-1}(U) \times \{0\} \times F})_b = \psi_{\pi(b)}\mu'(b)$ for every $b \in \pi^{-1}(U)$. ∎

PROPOSITION 5.3. *Let $\xi = (M, B, \pi)$ be a vector bundle of class p.*

(i) *The map η: $M \times (\mathbb{R} - \{0\}) \to M$ defined by $\eta(x, r) = rx$ is a*

C^p-action on the right of the Lie group $\mathbb{R}-\{0\}$ on M. In particular, $\eta_r\colon M\to M$, $\eta_r(x)=rx$, is a C^p-diffeomorphism for every $r\in\mathbb{R}-\{0\}$.

(ii) *The map $\alpha\colon M\times_B M\to M$ defined by $\alpha(x,y)=x+y$, is a C^p-submersion that preserves the index, where $(M\times_B M, M, \pi^*)$ is the pullback of ξ by π.*

(iii) *If $(M\times_B M, M, \pi^*)$ is the pullback of ξ by π, then for every $(x,y)\in M\times_B M$,*

$$(T_{(x,y)}\pi^*, T_{(x,y)}\pi_2)\colon T_{(x,y)}(M\times_B M)\to T_xM\times_{T_{\pi(x)}B}\times T_yM$$

is a linear homeomorphism and the linear continuous map

$$T_{(x,y)}\alpha\circ(T_{(x,y)}\pi^*, T_{(x,y)}\pi_2)^{-1}\colon T_xM\times_{T_{\pi(x)}B}T_yM\to T_{x+y}M$$

will be denoted by $T^*_{(x,y)}\alpha$, ($\pi_2\colon M\times_B M\to M$ is the C^p-map defined by $\pi_2(x,y)=y$).

(iv) *$T^*_{(x,y)}\alpha(V_x(M)\times V_y(M))=V_{x+y}(M)$ for every $(x,y)\in M\times_B M$ and $T^*_{(x,y)}\alpha=\theta^{x+y}_{c_t}\sigma(\theta^x_{c_t}\times\theta^y_{c_t})^{-1}_{|T_xM\times_{T_{\pi(x)}B}T_yM}$ where $\sigma((z,u)(z,v))=(z,u+v)$, for every c, t charts of B and ξ respectively, $\pi(x)\in V$, $(x,y)\in M\times_B M$*

PROOF. (i) It is easy to be checked.

(ii) Let $c=(U,\varphi,(E,\Delta))$ be a chart of B and $t=(U,\psi,F)$ a vector chart of (M,B,π). Then $c_t=(\pi^{-1}(U), \varphi_{ct}=(\varphi\times 1_F)\circ\psi^{-1}, (E\times F,\Delta\circ p_1))$ is a chart of M and $t^*=(\pi^{-1}(U),\psi^*,F)$, where $\psi^*\colon\pi^{-1}(U)\times F\to(\pi^*)^{-1}(\pi^{-1}(U))$ is the map defined by $\psi^*(x,v)=(x,\psi(\pi(x),v))$, is a vector chart of $(M\times_B M, M,\pi^*)$. Therefore $(c_t)_{t^*}=((\pi^*)^{-1}(\pi^{-1}(U)), \varphi_{c_t,t^*}=((\varphi\times 1_F)\circ\psi^{-1})\times 1_F)\circ\circ(\psi^*)^{-1}, (E\times F\times F,\Delta\circ p_1))$ is a chart of the C^p-manifold $M\times_B M$. Since $\alpha((\pi^*)^{-1}(\pi^{-1}(U))=\pi^{-1}(U)$ and $\overline{\alpha}=\varphi_{c_t}\circ\alpha\circ\varphi^{-1}_{c_tt^*}\colon\varphi(U)\times F\times F\to\varphi(U)\times F$ verifies $\overline{\alpha}(z,u,v)=(z,u+v)$, we have that α is a C^p-map.

Note that $D\overline{\alpha}(z,u,v)(z_1,u_1,v_1)=(z_1,u_1+v_1)$ for every $(z,u,v)\in\varphi(U)\times F\times F$ and $(z_1,u_1,v_1)\in E\times F\times F$, which proves that α is a submersion and preserves the index.

(iii) With the notations established in the proof of (ii), we have that the map

$$\overline{\pi^*}=\varphi_{c_t}\circ\pi^*\circ\varphi^{-1}_{c_tt^*}\colon\varphi(U)\times F\times F\to\varphi(U)\times F$$

verifies $\overline{\pi^*}((z,u,v))=(z,u)$ and the map $\overline{\pi_2}=\varphi_{c_tt^*}\circ\pi_2\circ\varphi^{-1}_{c_tt^*}$:

$\varphi(U) \times F \times F \to \varphi(U) \times F$ verifies $\overline{\pi}_2((z, u, v)) = (z, v)$. Then

$(T_{(x,y)}\pi^*, T_{(x,y)}\pi_2) =$

$$= \theta^x_{c_t} \times \theta^y_{c_t} \circ \delta \circ (\theta^{(x,y)}_{(c_t)t^*})^{-1}\colon T_{(x,y)}(M \times_B M) \to T_x M \times_{T\pi(x)B} T_y M$$

where $\delta\colon E \times F \times F \to (E \times F) \times_E (E \times F)$ is the map defined by $\delta((z, u, v)) = ((z, u), (z, v))$, is a linear homeomorphism.

(iv) It is easy to be checked. ■

Definition 5.1. *Let $\xi = (M, B, \pi)$ be a vector bundle of class p. A linear connection on ξ is a $(\mathbb{R} - \{0\})$-connection $H(M)$ on M associated to R_π such that*

$$(T^*_{(x,y)}\alpha)(H_x(M) \times_{T_{\pi(x)}B} H_y(M)) = H_{x+y}M$$

for every $(x, y) \in M \times_B M$ (see Proposition 5.3 (iii)).

Let $\xi = (M, B, \pi)$ be a vector bundle of class $p \geqslant 1$ and $H(M)$ a linear connection on ξ. Then by Proposition 3.2 there is a unique M-morphism of class $p - 1$ $\phi\colon TM \to V(M)$ such that $\phi \circ j = 1_{V(M)}$, $\phi_{rx} \circ T_x\eta_r = T_x\eta_r \circ \phi_x$ for every $(x, r) \in M \times (\mathbb{R} - \{0\})$ and $\ker\phi = H(M)$. Then by Proposition 5.2, we have the π-morphism of class $p - 1$, $K = \varrho \circ \phi\colon TM \to M$.

Definition 5.2. *Let $\xi = (M, B, \pi)$ be a vector bundle of class p and $H(M)$ a linear connection on ξ. The map $K = \varrho \circ \phi\colon TM \to M$, constructed above, will be called the connector of $H(M)$.*

The following proposition collects the main properties of connectors of linear connections.

Proposition 5.4. *Let $\xi = (M, B, \pi)$ be a vector bundle of class p, $H(M)$ a linear connection on ξ and $K = \varrho \circ \phi\colon TM \to M$ the connector of $H(M)$. Then:*

(i) *K is a π-morphism of class $p - 1$ from (TM, M, τ_M) into (M, B, π) and $\ker K = H(M)$.*

(ii) *If $c = (U, \varphi, (E, \Delta))$ is a chart of B and $t = (U, \psi, F)$ is a vector chart of ξ, then there exists a C^{p-1}-map $\Gamma_{c,t}\colon \varphi(U) \to L(E, F; F)$ (bilinear continuous maps), that will be called the Christoffel symbols of the linear connection $H(M)$, such that $K_{c,t}(x, y, w) = (x, y' + \Gamma_{c,t}(x)(x', y))$, where $K_{c,t} = (\varphi \times 1_F) \circ \psi^{-1} \circ (K_{|T(\pi^{-1}(U))}) \circ T(\psi(\varphi^{-1} \times 1_F))$ and $w = \theta^{(x,y)}_{c_{\varphi(U)} \times c_F}(x', y')(c_{\varphi(U)} = (\varphi(U), i, (E, \Delta))$, for every $(x, y) \in \varphi(U) \times F$ and every $w \in T_{(x,y)}(\varphi(U) \times F) = \theta^{(x,y)}_{c_{\varphi(U)} \times c_F}(E \times F)$.*

(iii) *K is a τ_B-morphism of class $p-1$ from $(TM, TB, T\pi)$ into (M, B, π).*

(iv) *Let r be an element of $\mathbb{R} - \{0\}$ and $\eta_r \colon M \to M$ the C^p-diffeomorphism defined by $\eta_r(x) = rx$. Then $K \circ T\eta_r = \eta_r \circ K$, η_r is a B-isomorphism and $(\eta_r)^{-1} = \eta_{1/r}$.*

PROOF. (i) Since $K = \varrho \circ \phi \colon TM \to M$, the result follows of Proposition 3.2 and Proposition 5.2.

(ii) We have that

$$T(\psi(\varphi^{-1} \times 1_F))(x, y, w) = (\psi(\varphi^{-1}(x), y), \theta_{c_t}^{\psi(\varphi^{-1}(x), y)}(\theta_{c_{\varphi(U)} \times c_F}^{(x, y)})^{-1}(w))$$

for every $(x, y, w) \in \varphi(U) \times F \times T_{(x, y)}(\varphi(U) \times F)$. Then

$$K_{c, t}(x, y, w) = (x, y' + p_2 \circ (\theta_{c_t}^{\psi(\varphi^{-1}(x), y)})^{-1} \circ \phi_{\psi(\varphi^{-1}(x), y)} \theta_{c_t}^{\psi(\varphi^{-1}(x), y)}(x', 0)).$$

Thus $\Gamma_{c, t}(x)(x', y) = p_2 \circ (\theta_{c_t}^{\psi(\varphi^{-1}(x), y)})^{-1} \circ \phi_{\psi(\varphi^{-1}(x), y)} \theta_{c_t}^{\psi(\varphi^{-1}(x), y)}(x', 0)$ for every $(x, x', y) \in \varphi(U) \times E \times F$.

It is obvious that the map $\Gamma_{c, t}(x)(\cdot, y) \colon E \to F$ is linear for every $(x, y) \in \varphi(U) \times F$. Let r be an element of $\mathbb{R} - \{0\}$. Then $\phi_{r\psi(\varphi^{-1}(x), y)} T_{\psi(\varphi^{-1}(x), y)}(\eta_r) = T_{\psi(\varphi^{-1}(x), y)}(\eta_r) \phi_{\psi(\varphi^{-1}(x), y)}$. On the other hand $T_{\psi(\varphi^{-1}(x), y)}(\eta_r) = \theta_{c_t}^{r\psi(\varphi^{-1}(x), y)} D\overline{\eta_r}(x, y)(\theta_{c_t}^{\psi(\varphi^{-1}(x), y)})^{-1}$, where $\overline{\eta_r} \colon \varphi(U) \times F \to \varphi(U) \times F$ is the map defined by $\overline{\eta_r}(\alpha, \beta) = (\alpha, r\beta)$ and therefore

$$D(\overline{\eta}_r)(x, y)(u, v) = (u, rv).$$

Thus $\Gamma_{c, t}(x)(x', ry) = r\Gamma_{c, t}(x)(x', y)$.

Let y_1, y_2 be elements of F. The conditions

$$T^*_{(\psi(\varphi^{-1}(x), y_1), \psi(\varphi^{-1}(x), y_2))} \alpha(H_{\psi(\varphi^{-1}(x), y_1)}(M) \times_{T_{\varphi^{-1}(x)}B} H_{\psi(\varphi^{-1}(x), y_2)}(M)) = H_{\psi(\varphi^{-1}(x), y_1 + y_2)}(M)$$

and (iv) of Proposition 5.3 imply that

$$\Gamma_{c, t}(x)(x', y_1 + y_2) = \Gamma_{c, t}(x)(x', y_1) + \Gamma_{c, t}(x)(x', y_2).$$

By localization on the corresponding vector bundles one proves that $\Gamma_{c, t}(x) \in L(E, F; F)$ and that $\Gamma_{c, t} \colon \varphi(U) \to L(E, F; F)$ is of class $p-1$.

(iii) Let $c = (U, \varphi, (E, \Delta))$ be a chart of B, $t = (U, \psi, F)$ a vector chart of (M, B, π) and $\sigma = (\tau_B^{-1}(U), \psi_2, F \times F)$ the induced vector chart of $(TM, TB, T\pi)$. Consider the map $\lambda \colon \tau_B^{-1}(U) \to L(F \times F, F)$ defined by $\lambda(x, v) = \Gamma_{c,t}(\varphi(x))((\theta_c^x)^{-1}(v), \cdot)p_1 + p_2$. Then λ is a C^{p-1}-map and $K \circ (\psi_2)_{(x,v)} = \psi_x \circ \lambda(x, v)$ for every $(x, v) \in \tau_B^{-1}(U)$.

(iv) Let $c = (U, \varphi, (E, \Delta))$ be a chart of B, $t = (U, \psi, F)$ a vector chart of (M, B, π) and $c_t = (\pi^{-1}(U), (\varphi \times 1_F) \circ \psi^{-1}, (E \times F, \Delta \circ p_1))$ the associated chart of M. Then

$$(\varphi \times 1_F) \circ \psi^{-1} \circ K_{|T(\pi^{-1}(U))} T((t_M)_{|\pi^{-1}(U)}) T(\psi(\varphi^{-1} \times 1_F))(x, y, w) =$$

$$= (\varphi \times 1_F) \circ \psi^{-1} \circ t_M \circ K_{|T(\pi^{-1}(U))} \circ T(\psi(\varphi^{-1} \times 1_F))(x, y, w),$$

where $\theta^{(x,y)}_{c_{\varphi(U)} \times c_F}(v_1, v_2) = w$, and $K \circ T(t_M) = t_M \circ K$. ■

Now we prove that the properties $\pi \circ K = \pi \circ \tau_M$ of (ii) of Proposition 5.4 characterize the connector of linear connections.

PROPOSITION 5.5. *Let $\xi = (M, B, \pi)$ be a vector bundle of class p and K: $TM \to M$ a map such that*

(i) $\pi \circ K = \pi \circ \tau_M$.

(ii) *For every chart $c = (U, \varphi, (E, \Delta))$ of B and every vector chart $t = (U, \psi, F)$ of (M, B, π), there is a C^{p-1}-map $\Gamma_{c,t}$: $\varphi(U) \to L(E, F; F)$ such that*

$$K c, t(x, y, w) = (x, y' + \Gamma_{c,t}(x)(x', y))$$

where $w = \theta^{(x,y)}_{c_{\varphi(U)} \times c_F}(x', y')$ ($c_{\varphi(U)} = (\varphi(U), i, (E, \Delta))$ for every $(x, y) \in \varphi(U) \times F$ and every $w \in T_{(x,y)}(\varphi(U) \times F)(T_{(x,y)}(\varphi(U) \times F) = \theta^{(x,y)}_{c_{\varphi(U)} \times c_F}(F \times F))$. Note that $K_{c,t} = (\varphi \times 1_F) \circ \psi^{-1} \circ K \circ T(\psi(\varphi^{-1} \times 1_F))$.

Then there is a M-morphism of class $p-1$, V: $TM \to VM$ such that $V \circ j = 1_{V(M)}$ and $\varrho \circ V = K$. Moreover $V = H(M)$ is a linear connection on ξ, $K = H(M)$ and the connector of $H(M)$ is K.

PROOF. Let $c = (U, \varphi, (E, \Delta))$ be a chart of B, $t = (U, \psi, F)$ a vector chart of (M, B, π) and $c_t = (\pi^{-1}(U), (\varphi \times 1_F) \circ \psi^{-1}, (E \times F, \Delta \circ p_1))$ the induced chart on M.

For every $(x, v) \in T(\pi^{-1}(U))$, consider the element

$$w(x, v) = \left(x, \theta^x_{c_t}((0, p_2 \psi^{-1} K(x, v)))\right) \in TM.$$

It is clear that $T_x\pi(p_2 w(x, v)) = 0$ and consequently $w(x, v) \in V(M)$. For other charts, if $(x, v) \in \sum_{y \in \pi^{-1}(U \cap U')} T_y M$, then $w(x, v) =$

$= (x, \theta^x_{c_t'}((0, p_2 \circ \psi'^{-1} K(x, v)))$ because of

$\theta^x_{c_t'} \theta^x_{c_t}((0, p_2 \psi^{-1} K(x, v))) = (0, \mu(\pi(x)) p_2 \psi^{-1} K(x, v)) =$

$$= (0, p_2 (\psi')^{-1} K(x, v)).$$

(see (3)). Then we can defined V locally as follows: $V(x, v) = w(x, v)$ for every $(x, v) \in T(\pi^{-1}(U))$.

We know that $(\pi^{-1}(U), \psi_{c_t}, E \times F)$, where

$$\psi_{c_t}\colon \pi^{-1}(U) \times E \times F \to \tau_M^{-1}(\pi^{-1}(U)),$$

$\psi_{c_t}(y, v_1, v_2) = (y, \theta^y_{c_t}(v_1, v_2))$, is a vector chart of (TM, M, τ_M) and

$$(\pi^{-1}(U), (\psi_{c_t})_{|\pi^{-1}(U) \times \{0\} \times F}, 0 \times F),$$

is a vector chart of $(V(M), M, (\tau_M)_{|V(M)})$. The map $\bar{\mu}\colon \pi^{-1}(U) \to L(E \times \times F, F)$, defined by $\bar{\mu}(b) = p_2 + \Gamma_{c,t}(\varphi(\pi(b)))(\cdot, p_2 \psi^{-1}(b)) p_1$ is a map of class $p - 1$ and

$$V_b((\psi_{c_t}))_b = (\psi_{c_t})_{|\pi^{-1}(U) \times \{0\} \times F})_b \circ \bar{\mu}(b)$$

because of

$$v_2 + \Gamma_{c,t}(\varphi(\pi(b)))(v_1, p_2 \circ \psi^{-1}(b)) = p_2 \circ \psi^{-1} K(b, \theta^b_{c_t}(v_1, v_2))..$$

Thus V is a M-morphism of class $p - 1$.

Notice that for every

$(x, v) \in T(\pi^{-1}(U)), V(x, v) = \big(x, \theta^x_{c_t}((0, p_2 \circ K_{c,t}((\varphi \times 1_F) \circ \psi^{-1}(x), w)))\big)$

where $w = \theta^{(\varphi \times 1_F)\psi^{-1}(x)}_{c_{\varphi(U)} \times C_F}(v_1, v_2), (\theta^x_{c_t})^{-1}(v) = (v_1, v_2)$.

It is clear, from the definitions, that $\varrho \circ V = K$ and $V \circ j = 1_{V(M)}$. Finally (V) is a C^{p-1}-vector subbundle of (TM, M, τ_M) because of V_x is a surjective map and $\ker(V_x) \oplus_T \ker(T_x \pi) = T_x X$.

We have

$(\varphi \times 1_F) \circ \psi^{-1} \circ K_{|T(\pi^{-1})(U))} \circ T(\eta_{r_{|\pi^{-1}(U)}}) \circ T(\psi(\varphi^{-1} \times 1_F))(x, y, w) =$

$$= (\varphi \times 1_F) \circ \psi^{-1} \circ \eta_r \circ K_{|T(\pi^{-1}(U))} \circ T(\psi(\varphi^{-1} \times 1_F))(x, y, w)$$

where $\theta^{(x,y)}_{c_{\varphi(U)} \times c_F}(v_1, v_2) = w$ and $K \circ T\eta_r = \eta_r \circ K$, $(r \neq 0)$. Thus, for every $(x, y) \in T(\pi^{-1}(U))$, $r \neq 0$, $rK_x(v) = K_{r.x} \theta^{rx}_{c_t}(v_1, rv_2)$ where $(\theta^x_{c_t})^{-1}(v) = = (v_1, v_2)$, $T_x \eta_r(v) = \theta^{rx}_{c_t}(v_1, rv_2)$. Finally $rK_x(v) = K_{rx} T_x \eta_r(v)$. Thus by Proposition 3.2, $H(M) = V$ is a $(\mathbb{R} - \{0\})$-connection associated to R_π.

Finally $T^*_{(x,y)} \alpha(H_x(M) \times_{T_{\pi(x)}B} H_y(M)) = H_{x+y}(M)$ for every $(x, y) \in$

$\in M \times_B M$. Indeed, let (u, v) be an element of $H_x(M) \times_{T_{\pi(x)}B} H_y(M)$. Then

$$K(x + y, T^*_{(x,y)}\alpha(u, v)) =$$

$$= K(x + y, \theta^{x+y}_{c_t}\sigma(\theta^x_{c_t})^{-1}(u), (\theta^y_{c_t})^{-1}(v)) = KT(\psi(\varphi^{-1} \times 1_F))(a, b, c) = \lambda$$

where $a = \varphi(\pi(x))$, $b = p_2\psi^{-1}(x + y)$ and $c = (\theta^{(\varphi \times 1_F)\psi^{-1}(x+y)}_{c_{\varphi(U)} \times c_F} \cdot$ $\cdot(\sigma(\theta^x_{c_t})^{-1}(u), \theta^y_{c_t})^{-1}(v))$. The aim is to prove that $\lambda = 0 \in \pi^{-1}(\pi(x))$ or that $(\varphi \times 1_F)\psi^{-1}(\lambda) = (\varphi\pi(x), 0)$. Now $(\varphi \times 1_F)\psi^{-1}(\lambda) = K_{c,t}(a, b, c) =$ $= (a, b' + \Gamma_{c,t}(a)(a', b))$, where

$$(a', b') = \sigma((\theta^x_{c_t})^{-1}(u), (\theta^y_{c_t})^{-1}(v)) =$$

$$= (p_1(\theta^x_{c_t})^{-1}(u), p_2(\theta^x_{c_t})^{-1}(u) + p_2(\theta^y_{c_t})^{-1}(v)),$$

$$b' + \Gamma_{c,t}(a)(a', b) = p_2(\theta^x_{c_t})^{-1}(u) + p_2(\theta^y_{c_t})^{-1}(v) +$$

$$+ \Gamma_{c,t}(a)(a', \psi^{-1}_{\pi(x)}(x)) + \Gamma_{ct}(a)(a', \psi^{-1}_{\pi(x)}(y)) =$$

$$= p_2K_{ct}(\varphi(\pi(x)), \psi^{-1}_{\pi(x)}(x), \theta^{(\varphi(\pi(x)), \psi^{-1}_{\pi(x)}(x)))}_{c_{\varphi(U)} \times c_F} \circ (\theta^x_{c_t})^{-1}(u)) +$$

$$+ p_2K_{c,t}(\varphi\pi(x), \psi^{-1}_{\pi(x)}(y), \theta^{\varphi\pi(x), \psi^{-1}_{\pi(x)}(y))}_{c_{\varphi(U)} \times c_F} \circ (\theta^y_{c_t})^{-1}(v)) =$$

$$= p_2(\varphi \times 1_F) \circ \psi^{-1}K(x, u) + p_2(\varphi \times 1_F)\psi^{-1}K(y, v) = 0$$

because of $V(x, u) = 0$, $V(y, v) = 0$. Thus we have proved that

$$T^*_{(x,y)}\alpha(H_x(M) \times_{T_{\pi(x)}B} H_y(M)) \subseteq H_{x+y}(M).$$

Conversely if $u \in H_{x+y}(M)$ by Proposition 5.3 (iii) there exists $(a, b) \in$ $\in T_xM \times_{T_{\pi(x)}B} T_yM$ such that $T^*_{(x,y)}\alpha(a, b) = u$. Now $(a, b) = (a_V, b_V) +$ $+ (a_H, b_H)$ where $a_V + a_H = a$ and $b_V + b_H = b$, $(a_H, b_H) \in H_xM \times$ $\times_{T_{\pi(x)}B} H_yM$ and $T^*_{(x,y)}\alpha(a_H, b_H) = u$ (see Proposition 5.3 (iv)).

Finally we consider the problem of existence of linear connections.

PROPOSITION 5.6. *Let $\xi = (M, B, \pi)$ be a vector bundle of class p. Suppose that B admits partitions of unity of class $p - 1$. Then there is a linear connection on ξ.*

PROOF. Let $\mathcal{C} = \{c_i = (U_i, \varphi_i, (E_i, \Delta_i))\}_{i \in I}$ be a C^p-atlas of B, $\{t_i =$ $= (U_i, \psi_i, F_i)\}_{i \in I}$ a family of vector charts of ξ and $\mathcal{F} = \{s_i\}_{i \in I}$ a partition of unity of class $p - 1$ of B subordinate to $\{U_i\}_{i \in I}$. For every $i \in I$, let K_i: $\tau_M^{-1}\pi^{-1}(U_i) \to \pi^{-1}(U_i)$ be the map defined by:

$$(\varphi_i \times 1_F) \circ \psi_i^{-1} \circ K_i \circ T(\psi_i(\varphi_i^{-1} \times 1_F))(x, y, x', y') =$$
$$= (x, y'), (x, y, x', y') \in \varphi_i(U_i) \times F_i \times F_i \times F_i .$$

Consider the map K: $TM \to M$ defined by $K(m, v) = = \sum_{i \in I} s_i(\pi(m)) K_i(m, v)$, for every $(m, v) \in TM$. It is easy to check that K is the connector of a linear connection on ξ.

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