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On a Carathéodory's Conjecture on Umbilics: Representing Ovaloids.

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1. - Introduction.

The classical Carathéodory's Conjecture states that every smooth convex embedding of a 2-sphere in R^3 must have at least two umbilics. A well known approach to the problem is based on a «semi-local» argument. For any surface in R^3, the eigenspaces of the second fundamental form define two orthogonal line fields (principal directions) whose singularities are exactly the umbilics. To each isolated umbilic we can attach the index of either one of the two fields, which is half of an integer, and the sum of those indexes is the Euler characteristic of the surface, if this is compact and all umbilics are isolated. So, if an embedded sphere has only one umbilic, this must have index two. We just observe that, up to an inversion in R^3, we can always suppose that the curvature at a given umbilic is positive and therefore the convexity hypothesis is not relevant for this argument. Examples of umbilics of index j are known for all $j \leq 1$ and a local conjecture, known as the Loewner conjecture, states that there are not umbilics of index bigger than one. This conjecture has been asserted to be true for analytic surfaces by several authors among which H. Hamburger [Ham], G. Bol [Bol], T. Klotz [Klo] and C. J. Titus [Tit], implying therefore Carathéodory's Conjecture for analytic surfaces. Very recently we where informed that Voss and Scherbel are claming to have clarified some points in the above mentioned works. These points are explained in Scherbel's

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thesis [Sch, Appendix B]; in this respect see also [Yau, cf. p. 684] and [Lan, cf. p. 19].

In [GMS] it is proved that for each umbilic on a C^r surface it is possible to construct, under a mild non-degeneracy condition, an analytic surface with an umbilic with the same index and therefore, in those cases, a positive answer to the local C^r conjecture follows from a positive answer for the analytic case.

To attack Carathéodory's Conjecture, all existing works use the same approach which is local and depends on analyticity. In this note we provide information that may be useful to attack the referred conjecture by a global argument and for general smooth surfaces: We establish a correspondence between C^r functions $f: S^2 \to \mathbb{R}$ and C^r ovaloids. For the ovaloid associated to the function $f: S^2 \to \mathbb{R}$ we give an explicit formula (in terms of f) for the inverse of its Gauss map. By using this, we obtain an interesting ovaloid.

There are other works in which ovaloids are represented by maps from S^2 to $\mathbb{R}$ (see for instance [Fav], [Fi] and [Nir]). However, their representation is a little different from ours; they do not give the explicit inverse of the Gauss map and do not show the relationship with Carathéodory's Conjecture.

2. - Bonnet coordinates and ovaloids.

Orient the sphere $S^2 \subset \mathbb{R}^3$ so that the positive unitary normal vector at $p \in S^2$ is p itself. Let $\mathcal{S} \subset \mathbb{R}^3$ be a C^r-*ovaloid*. By this we mean that $\mathcal{S}$ is an oriented C^r-embedded surface such that its Gauss map $N: \mathcal{S} \to S^2$ is an orientation preserving diffeomorphism. This definition implies that $\mathcal{S}$ is convex, compact and that its Gaussian curvature is positive everywhere. We define the *support function of* $\mathcal{S}$ as the map $f: S^2 \to \mathbb{R}$ given by

$$f(p) = p \cdot N^{-1}(p),$$

where the dot stands for the usual inner product.

Given $\delta \in \{-, +\}$, let $\Pi^\delta: \mathbb{R}^2 \to S^2 \setminus \{(0, 0, \delta(1))\}$ be the diffeomorphism given by

$$\Pi^\delta(x, y) = \left(\frac{2x}{1 + x^2 + y^2}, \frac{2y}{1 + x^2 + y^2}, \frac{\delta(x^2 + y^2 - 1)}{1 + x^2 + y^2} \right).$$

That is, Π^δ is the inverse map of the corresponding stereographic projection. The map

$$\Phi^\delta(x, y) = (X^\delta(x, y), Y^\delta(x, y), Z^\delta(x, y)) = N^{-1} \circ \Pi^\delta(x, y),$$

defined in $\mathbb{R}^2$, provides a global C^{r-1} parametrization of $S \setminus \{N^{-1}(0, 0, \delta(1))\}$ called *bonnet chart associated to* (S, Π^δ). Given the support function f of S, associated to (S, Π^δ) we define the *Bonnet function*

$$\beta^\delta(x, y) = (1 + x^2 + y^2) f(\Pi^\delta(x, y)).$$

Let $\Lambda^\delta(x, y) = (1 + x^2 + y^2)\Pi^\delta(x, y)$; that is,

$$\Lambda^\delta(x, y) = (2x, 2y, \delta(x^2 + y^2 - 1)).$$

As $\Lambda^\delta \cdot \Phi^\delta_x = \Lambda^\delta \cdot \Phi^\delta_y = 0$, where the subindex means the partial derivative with respect to this variable, we have that $\Lambda^\delta_x \cdot \Phi^\delta = \beta^\delta_x$ and $\Lambda^\delta_y \cdot \Phi^\delta = \beta^\delta_y$. This together with $\Lambda^\delta \cdot \Phi^\delta = \beta^\delta$ can be written in matrix notation as $M^\delta \cdot \Phi^\delta = \mathcal{B}^\delta$, where

$$(1) \quad M^\delta = \begin{pmatrix} 2x & 2y & \delta(x^2 + y^2 - 1) \\ 2 & 0 & \delta(2x) \\ 0 & 2 & \delta(2y) \end{pmatrix}, \quad \Phi^\delta = \begin{pmatrix} X^\delta \\ Y^\delta \\ Z^\delta \end{pmatrix}, \quad \mathcal{B}^\delta = \begin{pmatrix} \beta^\delta \\ \beta^\delta_x \\ \beta^\delta_y \end{pmatrix}.$$

As N is of class C^{r-1}, Φ^δ is also of class C^{r-1}. Therefore, $M^\delta \cdot \Phi^\delta = \mathcal{B}^\delta$ implies that β^δ is of class C^r. Since, for all $(x, y) \in \mathbb{R}^2$, the determinant of M^δ is $-(\delta)4(1 + x^2 + y^2) \neq 0$, we may write $\Phi^\delta = (M^\delta)^{-1} \cdot \mathcal{B}^\delta$. From this, using a symbolic computer system we can obtain the first and second fundamental form of Φ^δ and therefore the proof of proposition below (see [GMS]).

PROPOSITION 2.1. *Let $S \subset \mathbb{R}^3$ be a C^r ovaloid, $r \geq 3$. Then the support function f of S is of class C^r and the differential equation of the principal lines of curvature of S in its Bonnet chart, associated to (S, Π^δ), is given by*

$$(2) \qquad \beta^\delta_{xy} dx^2 + (\beta^\delta_{yy} - \beta^\delta_{xx}) dx\, dy - \beta^\delta_{xy} dy^2 = 0$$

where

$$\beta^\delta(x, y) = (1 + x^2 + y^2) f((\Pi^\delta(x, y)).$$

THEOREM 2.2. *Let $\mathcal{S} \subset \mathrm{R}^3$ be a C^r ovaloid, $r \geq 3$. Then the inverse $N^{-1}: S^2 \to \mathcal{S}$, of the Gauss map N, can be written as follows:*

$$N^{-1}(u, v, w) = f(u, v, w) \cdot (u, v, w) + \mathcal{A}(u, v, w) \cdot \nabla f(u, v, w)$$

where $f: S^2 \to \mathrm{R}$ denotes the support function of $\mathcal{S}$, ∇f its gradient vector field and

$$\mathcal{A}(u, v, w) = \begin{pmatrix} v^2 + w^2 & -uv & -uw \\ -uv & u^2 + w^2 & -vw \\ -uw & -vw & u^2 + v^2 \end{pmatrix}. \tag{3}$$

Conversely, given a C^r function $f: S^2 \to \mathrm{R}$, $r \geq 3$, there exists a constant $c > 0$ such that $f + c$ is the support function an ovaloid of class C^r.

PROOF Let $\mathcal{S} \subset \mathrm{R}^3$ be a C^r ovaloid, $r \geq 3$ and let $f: S^2 \to \mathrm{R}$ be its support function. Consider an arbitrary extension $F = F(u, v, w)$ of f defined in a neighborhood of S^2.

Given $\delta \in \{-, +\}$, the function f determines the function

$$\beta^\delta(x, y) = (1 + x^2 + y^2) f(\Pi^\delta(x, y))$$

and so the column vector $\mathcal{B}^\delta$ as in (1).

Observe that, for all $(u, v, w) \in S^2 \setminus \{(0, 0, \delta(1))\}$,

$$(x, y) = (\Pi^\delta)^{-1}(u, v, w) = \left(\frac{u}{1 - \delta w}, \frac{v}{1 - \delta w} \right).$$

Therefore, if we denote by f, β^δ, $F_u \ldots$, the functions $f = f(u, v, w)$, $\beta^\delta \circ (\Pi^\delta)^{-1}(u, v, w)$, $F_u(u, v, w)$, $\ldots$, respectively, we will have that for all $(u, v, w) \in S^2 \setminus \{(0, 0, \delta(1))\}$,

$$\mathcal{B} = \begin{pmatrix} \beta^\delta \\ \beta^\delta_x \\ \beta^\delta_y \end{pmatrix} = \frac{2}{1 - \delta w} \begin{pmatrix} f \\ uf + (1 - w - u^2) F_u - uvF_v + u(1 - \delta w) F_w \\ vf - uvF_u + (1 - w - v^2) F_v + v(1 - \delta w) F_w \end{pmatrix}.$$

Moreover,

$$M^\delta \circ (\Pi^\delta)^{-1} = \frac{2}{1-\delta w}\begin{pmatrix} u & v & \delta w \\ 1-\delta w & 0 & u \\ 0 & 1-\delta w & v \end{pmatrix}$$

and

$$(M^\delta)^{-1} \circ (\Pi^\delta)^{-1} = \frac{1}{2(1-\delta w)}\begin{pmatrix} u(1-\delta w) & 1-u^2-\delta w & -uv \\ v(1-\delta w) & -uv & 1-v^2-\delta w \\ -\delta(1-\delta w)^2 & \delta u(1-\delta w) & \delta v(1-\delta w) \end{pmatrix}.$$

Recall that $\Phi^\delta = N^{-1} \circ \Pi^\delta$ and that $\Phi^\delta = (M^\delta)^{-1} \cdot \mathcal{B}^\delta$. It follows that

$$N^{-1} = \Phi^\delta \circ (\Pi^\delta)^{-1} = ((M^\delta)^{-1} \cdot \mathcal{B}^\delta) \circ (\Pi^\delta)^{-1}.$$

Under these conditions, it may be seen that, when restricted to S^2, N^{-1} has the required form (see the remark following this proof).

Conversely, Let $f: S^2 \to \mathrm{R}$ be an arbitrary C^r function and let $F: \mathrm{R}^3 \to \mathrm{R}$ be an arbitrary C^r extension of f. Given $\sigma \in (0, \infty)$, we define $H: \mathrm{R}^3 \to \mathrm{R}^3$ and $G: \mathrm{R}^3 \to \mathrm{R}^3$ by

$$H(u, v, w) = \sigma \cdot (u, v, w) + G(u, v, w)$$

and

$$G(u, v, w) = F(u, v, w) \cdot (u, v, w) + \mathcal{A}(u, v, w) \cdot \nabla F(u, v, w).$$

Let $Id: \mathrm{R}^3 \to \mathrm{R}^3$ be the identity and let DG denote the derivative of G. As S^2 is compact, we obtain that if σ is large enough, for all $(u, v, w) \in S^2$,

$$Id + \frac{1}{\sigma} DG(u, v, w)$$

is invertible. This implies that $H|_{S^2}: S^2 \to H(S^2)$ is an immersion. By definition of H and the first part of this proof, $H \circ \Pi^\delta = \Phi^\delta$ and therefore, $N^{-1} = H$. This implies that the Gauss map must be not only a local diffeomorphism but, in fact, a global one. According to [Hop], these conditions imply that $H(S^2)$ is an ovaloid. ■

REMARK. Let $f: S^2 \to \mathrm{R}$ be a C^r function, $r \geq 3$, and let $F = F(u, v, w)$ be an arbitrary extension of f to a neighborhood of S^2. As

$A(u, v, w) \cdot (u, v, w) \equiv 0$, it follows that

$$\mathcal{A}(u, v, w) \cdot \nabla f(u, v, w) = \mathcal{A}(u, v, w) \cdot \nabla F(u, v, w)$$

where ∇f and ∇F are the gradient vector fields of f and of F, respectively.

EXAMPLE 1. If f: $S^2 \to \mathbb{R}$ is identically the constant $r > 0$, then the ovaloid associated to f is the sphere $r \cdot S^2$.

EXAMPLE 2. Let $\varepsilon > 0$, φ: $\mathbb{R} \to (-\varepsilon, \varepsilon)$ be an orientation preserving smooth diffeomorphism and f: $S^2 \setminus \{(0, 0, 1)\} \to \mathbb{R}$ be given by

$$f(u, v, w) = (1 - w)\varphi\left(\frac{u}{1 - w}\right)\varphi\left(\frac{v}{1 - w}\right).$$

Observe that f is smooth and extends continuously to $\{(0, 0, 1)\}$ by defining $f(0, 0, 1) = 0$. Let

$$\beta(x, y) = (1 + x^2 + y^2) f(\Pi(x, y)) = \varphi(x) \cdot \varphi(y).$$

We have that, for all $(x, y) \in \mathbb{R}^2$,

$$\beta_{xy}(x, y) = \varphi'(x) \cdot \varphi'(y) \neq 0$$

It follows from Proposition 1.1 that if g: $S^2 \to \mathbb{R}$ is a smooth map which is a perturbation of f supported in an arbitrarily small neighbohood W of $\{(0, 0, 1)\}$ and $c > 0$ is an appropiate constant, we will have that the smooth ovaloid associated to $g + c$ has no umbilics in the complement of $N^{-1}(W)$, where N is the Gauss map of the ovaloid.

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