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Minimum Problems on SBV with Irregular Boundary Datum.

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ABSTRACT - A lower semicontinuity result for functionals, defined on functions $u \in SBV(\Omega)$, which depend both on the «jumps» of u and on a boundary datum φ is proved. An application to minimum problems is given.

0. - Introduction.

The so-called free discontinuity problems (see [9]) constitute a broad class of problems whose main feature is the presence of a set of discontinuities which is not prescribed *a priori*. Such problems arise in the variational approach to problems originating in such different branches of applied sciences as image segmentation (see [15]), fracture mechanics (see [2] and the references therein) and liquid crystals (see [18]) among others.

The pioneering work of E. De Giorgi and L. Ambrosio on this subject (see [10] and [1]) led to the introduction of the space $SBV(\Omega)$ of special functions of bounded variation on Ω which turned out to be a natural functional-analytic environment for the existence theory for such problems.

The model case of a free discontinuity problem, in its weak formulation, is the minimum problem for the relaxed Mumford-Shah functional

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(see [15]), i.e.

$$(0.1)\qquad \min\left\{ \int_{\Omega} [\,|\nabla u|^2 + |u - \psi|^2\,]\,dx + \mathcal{H}^{N-1}(S_u) \colon u \in SBV(\Omega) \right\},$$

where Ω is a bounded open subset of $\mathbb{R}^N$, ∇u is the approximate gradient of u, S_u is the set of «jumps» of u in Ω, i.e. the set of points of approximate discontinuity of u and ψ is a given function in $L_\infty(\Omega)$. This problem has been widely investigated both from the point of view of existence (see [1], [8]) and from the point of view of regularity (see [11], [8], [3] and [4]). Here we notice that problem (0.1) features no prescribed boundary conditions on $\partial\Omega$. Whenever boundary conditions are to be taken into account, it is natural to consider the following minimization problem

$$(0.2)\qquad \min\left\{ \int_{\Omega} [\,|\nabla u|^2 + |u - \psi|^2\,]\,dx + \mathcal{H}^{N-1}(S_u^{\varphi}) \colon u \in SBV(\Omega) \right\},$$

where $\varphi \colon \partial\Omega \to \mathbb{R}$ is a given $\mathcal{H}^{N-1}$-measurable function and $S_u^{\varphi} = S_u \cup \cup \{x \in \partial\Omega \colon \gamma_{\partial\Omega} u(x) \neq \varphi(x)\}$ consists of the «jump» points of u in Ω and of those points of $\partial\Omega$ where the trace $\gamma_{\partial\Omega} u$ of u on $\partial\Omega$ differs from φ. For a smooth enough boundary datum, say $\varphi \in W^{1/2,2}(\partial\Omega)$, the minimization problem (0.2) can be handled by arguments similar to those used for problem (0.1), see for instance [8]. Indeed, once a bounded open set Ω' containing $\overline{\Omega}$ has been chosen, φ can be recovered as the trace on $\partial\Omega$ of a function $v \in W^{1,2}(\Omega' \setminus \overline{\Omega})$ so that, letting $\psi' \in L_\infty(\Omega')$ denote any extension of ψ, the minimization problem

$$\min\left\{ \int_{\Omega'} [\,|\nabla u^2 + |u - \psi'|^2\,]\,dx + \mathcal{H}^{N-1}(S_u) \colon \right.$$
$$\left. u \in SBV(\Omega'),\ u = v \ a.\,e. \ \ on\ \Omega' \setminus \overline{\Omega} \right\}$$

is equivalent to (0.2) up to an additive constant.

The aim of this short note is to contribute to the studying of minimization problems of the form (0.2) in the case of an irregular boundary datum φ. In particular, denoting the traces of u on S_u by u^+ and u^-, we shall prove, for a suitable integrand g, the lower semicontinuity of integral functionals defined on $SBV(\Omega)$ of the form

$$(0.3)\qquad G_{\varphi}(u) = \int_{S_u} g(x, |u^+(x) - u^-(x)|)\,d\mathcal{H}^{N-1}(x) +$$
$$+ \int_{\partial\Omega} g(x, |\gamma_{\partial\Omega} u(x) - \varphi(x)|)\,d\mathcal{H}^{N-1}(x),$$

along bounded sequences of functions in $SBV(\Omega)$ which converge in $L_1(\Omega)$ and whose approximate gradients are relatively weakly compact in $L_1(\Omega, \mathbb{R}^N)$ in the case of a boundary datum φ which is only assumed to be $\mathcal{H}^{N-1}$-measurable. The proof of this lower semicontinuity result is based, as the original proof by L. Ambrosio ([1]), on the properties of the one dimensional sections of functions of bounded variation. The same fruitful approach has been widely used; see for instance [6] and also [5] in the context of (vector-valued) functions of bounded deformation (see [17]).

Finally, the lower semicontinuity of the functionals (0.3) yields an existence result for the problem of minimizing

$$\int_\Omega [\,|\nabla u|^2 + |u - \psi|^2]\,dx + G_\varphi(u) \tag{0.4}$$

on $SBV(\Omega)$ in the case of a suitable integrand g provided φ is $\mathcal{H}^{N-1}$-essentially bounded on $\partial\Omega$. Indeed, the boundedness assumptions on φ and ψ ensure the existence of a minimizing sequence for the above mentioned functional which is bounded in $L_\infty(\Omega)$ and hence in $BV(\Omega)$ so that the direct method of the Calculus of Variations applies.

1. Notation and statement of the main result.

Throughout this paper, let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary and let $\mathcal{B}(\Omega)$ be the σ-algebra of all Borel subsets of Ω. For $k \geqslant 1$, we shall denote the Lebesgue measure on $\mathbb{R}^k$ by $\mathcal{L}^k$ and, for $\alpha \geqslant 0$, the α-dimensional Hausdorff measure on $\mathbb{R}^N$ by $\mathcal{H}^\alpha$.

The functional framework of this paper is $BV(\Omega)$, the space of *functions of bounded variation on* Ω, which consists of all functions $u \in L_1(\Omega)$ whose distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure with bounded total variation on Ω. We shall denote the total variation of Du by $|Du|$. It is well known that $BV(\Omega)$, endowed with the norm

$$\|u\|_{BV(\Omega)} = \|u\|_{L_1(\Omega)} + |Du|(\Omega), \qquad u \in BV(\Omega)$$

is a Banach space. Moreover, its bounded subsets are relatively compact in $L_1(\Omega)$ and the closure with respect to the topology of $L_1(\Omega)$ of any bounded subset of $BV(\Omega)$ is contained in $BV(\Omega)$ itself. We now recall some properties of functions of bounded variation that will be used in the sequel.

I) *Structural properties of functions in $BV(\Omega)$.*

For each $u \in BV(\Omega)$, the Radon-Nikodym derivative of Du with respect to $\mathcal{L}^N$ will be denoted by $\nabla u \in L_1(\Omega, \mathbb{R}^N)$ while the $\mathcal{L}^N$-singular part of Du will be denoted by $D^s u$. This latter part of Du can be further splitted in a unique way into two parts $D^s u = D^j u + D^c u$ where $D^j u$, the so-called «jump» part of $D^s u$, is an $\mathcal{H}^{N-1}$-absolutely continuous measure and $D^c u$, the Cantor part of $D^s u$, is a measure that does not charge any Borel subset of Ω of finite $\mathcal{H}^{N-1}$-measure. More precisely, $D^j u$ turns out to be concentrated on the countably $(N-1)$-rectifiable set S_u, the singular set of u, which consists of all points in Ω which are not Lebesgue points of u. For a given approximate unit normal $\nu_u : S_u \to S^{N-1}$ on S_u, we set u^+ and u^- to be the traces of u on the sides of S_u determined by ν_u and $-\nu_u$ respectively. As is well known, they are defined for $\mathcal{H}^{N-1}$-a.e. $x \in S_u$ by

$$\begin{cases} \displaystyle\lim_{\varrho \to 0_+} \frac{1}{\mathcal{L}^N(B_\varrho(x))} \int\limits_{\{y \in B_\varrho(x):\, (y-x,\, \nu_u(x)) > 0\}} |u(y) - u^+(x)|\, d\mathcal{L}^N(y) = 0\,, \\ \hfill x \in S_u\,, \\ \displaystyle\lim_{\varrho \to 0_+} \frac{1}{\mathcal{L}^N(B_\varrho(x))} \int\limits_{\{y \in B_\varrho(x):\, (y-x,\, \nu_u(x)) < 0\}} |u(y) - u^-(x)|\, d\mathcal{L}^N(y) = 0\,, \\ \hfill x \in S_u\,, \end{cases}$$

and satisfy $u^+ \neq u^-$ $\mathcal{H}^{N-1}$-a.e. on S_u. In the sequel, we agree to extend u^+ and u^- $\mathcal{H}^{N-1}$-a.e. on Ω setting them equal to the approximate limit of u. Moreover, $D^j u$ can be written as

$$D^j u(B) = \int\limits_{S_u \cap B} (u^+ - u^-)\, \nu_u\, d\mathcal{H}^{N-1}\,, \qquad B \in \mathcal{B}(\Omega)\,.$$

When Ω is an open subset of $\mathbb{R}$ and $u \in BV(\Omega)$, we simply write u' for Du, $\dot{u}$ for ∇u and

$$u' = \dot{u}\, d\mathcal{L}^1 + (u')_s$$

for the Lebesgue decomposition of u' with respect to $\mathcal{L}^1$. We refer to [12] for a detailed exposition of the properties of functions of bounded variation.

Finally, we recall that a function $u \in BV(\Omega)$ is said to be a *special function of bounded variation on Ω* if $D^c u = 0$ (see [1]) so that the following decomposition formula holds

$$Du(B) = \int_B \nabla u \, d\mathcal{L}^N + \int_{S_u \cap B} (u^+ - u^-) \nu_u \, d\mathcal{H}^{N-1}, \qquad B \in \mathcal{B}(\Omega).$$

We shall denote by $SBV(\Omega)$ the space of all such functions.

II) *Characterization of $BV(\Omega)$ by one dimensional sections.*

As proved in [1], functions of bounded variation on an open subset of $\mathbb{R}^N$ can be characterized by means of their one dimensional sections. Indeed, given any $\xi \in S^{N-1}$, let $\pi_\xi = \{y \in \mathbb{R}^N : (y, \xi) = 0\}$ be the hyperplane of codimension one orthogonal to ξ and for all subsets $E \subset \mathbb{R}^N$ set

$$E_{x,\xi} = \{t \in \mathbb{R} : x + t\xi \in E\}, \qquad x \in \pi_\xi,$$

$$E_\xi = \{x \in \pi_\xi : E_{x,\xi} \neq \emptyset\}.$$

Then, if $u: \Omega \to \mathbb{R}$ is an $\mathcal{L}^N$-measurable function, set

$$u_{x,\xi}(t) = u(x + t\xi), \qquad t \in \Omega_{x,\xi}$$

for all $\xi \in S^{N-1}$ and for all $x \in \Omega_\xi$. It turns out (see [1]) that, for a function $u \in BV(\Omega)$ and for all $\xi \in S^{N-1}$, it happens that $u_{x,\xi} \in BV\Omega_{x,\xi}$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$ and

$$\int_{\pi_\xi} |u'_{x,\xi}(\Omega_{x,\xi})| \, d\mathcal{H}^{N-1}(x) < \infty .$$

If this is the case, for all $\xi \in S^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$, we have

$$\dot{u}_{x,\xi}(t) = (\nabla u(x + t\xi), \xi), \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in \Omega_{x,\xi}, \tag{1.1}$$

$$S_{u_{x,\xi}} = (S_u)_{x,\xi}. \tag{1.2}$$

Moreover, letting ξ and x be as above, if we agree to choose the approximate unit normal to $S_{u_{x,\xi}}$ defined by

$$\nu_{u_{x,\xi}}(t) = \operatorname{sign}(\nu_u(x + t\xi), \xi), \qquad t \in S_{u_{x,\xi}}$$

we have also the following identities:

$$\begin{cases} (u_{x,\xi})^+(t) = u^+(x + t\xi), \\ (u_{x,\xi})^-(t) = u^-(x + t\xi), \end{cases} \qquad \forall t \in \Omega_{x,\xi}. \tag{1.3}$$

Finally, for all $\xi \in S^{N-1}$ and $B \in \mathcal{B}(\Omega)$, the different parts of Du can be reconstructed from the corresponding parts of the one dimensional sections as follows

$$
(1.4)\quad \begin{cases}
\displaystyle\int_B (\nabla u, \xi)\, d\mathcal{L}^N = \int_{\pi_\xi} \left(\int_{B_{x,\xi}} \dot{u}_{x,\xi}\, d\mathcal{L}^1 \right) d\mathcal{H}^{N-1}(x), \\
\displaystyle(D^j u(B), \xi) = \int_{\pi_\xi} \left(\int_{B_{x,\xi} \cap S_{u_{x,\xi}}} (u^+_{x,\xi} - u^-_{x,\xi})\, d\mathcal{H}^0 \right) d\mathcal{H}^{N-1}(x), \\
\displaystyle(D^c u(B), \xi) = \int_{\pi_\xi} D^c u_{x,\xi}(B_{x,\xi})\, d\mathcal{H}^{N-1}(x),
\end{cases}
$$

so that, in particular, for all $u \in SBV(\Omega)$ and $\xi \in S^{N-1}$, we have $u_{x,\xi} \in \in SBV(\Omega_{x,\xi})$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$ and conversely, a function $u \in BV(\Omega)$ is a special function of bounded variation on Ω provided there exist N linearly independent vectors $\xi \in S^{N-1}$ such that the sections $u_{x,\xi}$ are in $SBV(\Omega_{x,\xi})$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$.

Now, we describe the functionals on $BV(\Omega)$ whose lower semicontinuity will be investigated. To this purpose, given an $\mathcal{H}^{N-1}$-measurable function $\varphi\colon \partial\Omega \to \mathbb{R}$, for all $u \in BV(\Omega)$ we let $J_\varphi u\colon \overline{\Omega} \to \mathbb{R}$ be defined by

$$
(1.5)\qquad J_\varphi u(x) = \begin{cases} u^+(x) - u^-(x), & x \in \Omega, \\ \gamma_{\partial\Omega} u(x) - \varphi(x), & x \in \partial\Omega, \end{cases}
$$

where $\gamma_{\partial\Omega} u$ is the trace of u on $\partial\Omega$. It is clear that $J_\varphi u$ is defined $\mathcal{H}^{N-1}$-a.e. on $\overline{\Omega}$ and is an $\mathcal{H}^{N-1}$-measurable function. Moreover, as $u^+ \neq u^-$ $\mathcal{H}^{N-1}$-a.e. on S_u, it is also clear that, up to an $\mathcal{H}^{N-1}$-null set, $J_\varphi u$ is different from 0 on the set

$$S_u^\varphi = S_u \cup \{x \in \partial\Omega\colon \gamma_{\partial\Omega} u(x) \neq \varphi(x)\}$$

only. In the sequel, when no confusion may arise, we shortly write Ju instead of $J_\varphi u$.
Then, given a function $g\colon \mathbb{R}^N \times [0, \infty) \to [0, \infty]$ such that

(1.6) g is lower semicontinuous on $\mathbb{R}^N \times [0, \infty)$;

(1.7) for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$, the function $t \in [0, \infty) \to g(x, t) \in [0, \infty)$ is non decreasing and subadditive;

(1.8) $g(x, 0) = 0$ for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$;

(1.9) there exists a non decreasing function $\vartheta: [0, \infty) \to [0, \infty)$ such that

— $\lim_{t \to 0_+} t^{-1}\vartheta(t) = \infty$;

— for $\mathcal{H}^{N-1}$-a.e. $x \in \mathbb{R}^N$, $g(x, t) \geq \vartheta(t)$ for all $t > 0$;

we let $G_\varphi: BV(\Omega) \to [0, \infty)$ be the functional defined by

$$G_\varphi(u) = \int_{\bar{\Omega}} g(x, |J_\varphi u(x)|)\, d\mathcal{H}^{N-1}(x), \qquad u \in BV(\Omega). \tag{1.10}$$

Throughout this paper, we shall always assume that g satisfies hypotheses (1.6), (1.7), (1.8) and (1.9). We notice that, as $G_\varphi(u)$ depends both on the size of the «jumps» of u in Ω and on the trace of u on $\partial\Omega$, it has the meaning of a «surface» energy which simultaneously measures how far is u from the given boundary datum φ. We also point out that the model case (0.2) can be recovered by letting

$$g(x, t) = 1 - \chi_{\{0\}}(t), \qquad (x, t) \in \mathbb{R}^N \times [0, \infty)$$

so that all the previous hypotheses on g are fulfilled.

We can now state the lower semicontinuity result for G_φ that we are going to prove.

THEOREM 1.1. *Given an $\mathcal{H}^{N-1}$-measurable function $\varphi: \partial\Omega \to \mathbb{R}$, let $(u_h)_h \subset SBV(\Omega)$ and $u_\infty \in L_1(\Omega)$ be such that*

a) $u_h \to u_\infty$ in $L_1(\Omega)$;

b) the sequence $(u_h)_h$ is bounded in $BV(\Omega)$;

c) the sequence $(\nabla u_h)_h$ is weakly relatively compact in $L_1(\Omega, \mathbb{R}^N)$;

d) the sequence $(G_\varphi(u_h))_h$ is bounded.

Then, $u_\infty \in SBV(\Omega)$, $\nabla u_h \rightharpoonup \nabla u_\infty$ weakly in $L_1(\Omega, \mathbb{R}^N)$ and $G_\varphi(u_\infty) \leq \leq \liminf_{h \to \infty} G_\varphi(u_h)$.

REMARK 1.2. The above theorem remains true if we replace hypothesis (b) with

b') the sequence $(u_h)_h$ is bounded in $L_\infty(\Omega)$.

This follows from the fact that (b') together with (d) and the behavior of g near 0 (see (1.9)) imply that the total variations of the singular parts of the gradients of the functions $(u_h)_h$ are uniformly bounded. This argument will be used in the proof of the following Theorem 1.4.

REMARK 1.3. A further statement of Theorem 1.1 is the following. Drop hypothesis (*a*), replace (*d*) with

d') $\liminf_{h \to \infty} G_{u_0}(u_h) < \infty$;

and assume (*c*) and either (*b*) or (*b'*). Then, the existence of a function $u_\infty \in SBV(\Omega)$ which is the limit in $L_1(\Omega)$ of a subsequence of $(u_h)_h$ becomes part of the thesis and the remaining parts of the conclusion hold true up to a subsequence. This follows immediately from the compactness theorem for functions of bounded variation.

The previous theorem can be applied to ensure the existence of solutions to minimum problems on $SBV(\Omega)$ for functionals of the form (0.4) provided φ is $\mathcal{H}^{N-1}$-essentially bounded on $\partial\Omega$. Indeed, we shall prove the following theorem.

THEOREM 1.4. *Let $\psi \in L_\infty(\Omega)$, $\varphi \in L_\infty\, \partial\Omega$ with respect to $\mathcal{H}^{N-1}$ and set*

$$F(u) = \int_\Omega [\,|\nabla u|^2 + |u - \psi^2]\, d\mathcal{L}^N\,, \qquad u \in BV(\Omega)\,.$$

Then, the minimum problem

$$\min\{F(u) + G_\varphi(u) \colon u \in SBV(\Omega)\}$$

admits a solution.

The proofs of Theorems 1.1 and 1.4 will be given in the next section.

2. - Proof of the main result.

The proof of Theorem 1.1 will be accomplished by reduction to the one dimensional case by means of a slicing argument. Indeed, consider a bounded open interval $I = (a_1, a_2)$ of R and let $g\colon \mathrm{R} \times [0, \infty) \to [0, \infty)$ satisfy (1.6), (1.7), (1.8) and (1.9). Then, given $\varphi\colon \{a_1, a_2\} \to \mathrm{R}$ and $G_\varphi\colon BV(I) \to [0, \infty)$ defined by (1.10), Theorem 1.1 reads as follows.

LEMMA 2.1. *Let $(u_h)_h \subset SBV(I)$ and $u_\infty \in L_1(I)$ be such that*

a) $u_h \to u_\infty$ *in* $L_1(I)$;

b) the sequence $(u_h)_h$ is bounded in $BV(I)$;

c) $(\dot{u}_h)_h$ is weakly relatively compact in $L_1(I)$;

d) the sequence $(G_\varphi(u_h))_h$ is bounded.

Then, $u_\infty \in SBVI$, $\dot{u}_h \rightharpoonup \dot{u}_\infty$ *weakly in* $L_1(I)$ *and* $G_\varphi(u_\infty) \leqslant \leqslant \liminf_{h \to \infty} G_\varphi(u_h)$.

The proof is an immediate consequence of the lower semicontinuity result on $SBV(I)$ with respect to convergence in $L_1(I)$ for autonomous functionals defined on the set of jumps (see [1] and [7]). Indeed, set $I' = (a_1', a_2')$ where $a_1' = a_1 - 1$ and $a_2' = a_2 + 1$, extend each function u_h, $h \in \mathbb{N}_+ \cup \{\infty\}$ to a function v_h on I' by setting

$$v_h(t) = \begin{cases} \varphi(a_1) & a_1' < t \leqslant a_1 , \\ u_h(t) & a_1 < t < a_2 , \\ \varphi(a_2) & a_2 \leqslant t < a_2' , \end{cases} \qquad h \in \mathbb{N}_+ \cup \{\infty\} .$$

and notice that $v_h \in SBV(I')$ for $h \geqslant 1$, $v_\infty \in L_1(I')$ and that $(v_h)_h$ and v_∞ satisfy hypotheses (a), (b) and (c) of Lemma 2.1 with respect to I'. Moreover, it is clear that

$$G_\varphi(u_h) = \sum_{t \in S_{v_h}} g(t, |v_h^+(t) - v_h^-(t)|)$$

so that the functional $v \in BV(I') \to \sum_{t \in S_{v_h}} g(t, |v_h^+(t) - v_h^-(t)|)$ is bounded along the sequence $(v_h)_h$. Now, this latter functional is independent of any boundary datum. Hence, it belongs to the class of functionals considered in [1] where its lower semicontinuity along bounded sequences of functions in $SBV(I')$ converging in $L_1(I')$ whose approximate gradients are relatively weakly compact in $L_1(I')$ is proved, at least in the case of autonomous functionals. The case where an explicit dependence on t is allowed can be dealt with similar arguments. Thus, the conclusion follows.

It is clear that the previous lemma extends trivially to the case of a finite union of bounded open intervals of $\mathbb{R}$ with pairwise disjoint closures.

Before inferring the proof of Theorem 1.1 from its one dimensional version, we prove some technical results.

LEMMA 2.2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz continuous boundary. Then,*

a) $\mathcal{H}^{N-1}(\partial\Omega) < \infty$;

b) *for all* $\xi \in S^{N-1}$ *and for* $\mathcal{H}^{N-1}$*-a.e.* $x \in \pi_\xi$, *the open set* $\Omega_{x,\xi}$ *has finitely many connected components with pairwise disjoint closures;*

c) *for all* $\xi \in S^{N-1}$, *we have* $\partial\Omega_{x,\xi} = (\partial\Omega)_{x,\xi}$ *for* $\mathcal{H}^{N-1}$*-a.e.* $x \in \pi_\xi$.

PROOF. By assumption, $\partial\Omega$ can be covered by finitely many graphs of real-valued Lipschitz continuous functions defined on bounded open subsets of $\mathbb{R}^{N-1}$ so that, by the area formula, (a) holds true.

As far as (*b*) is concerned, choose $\xi \in S^{N-1}$ and notice that Corollary 2.10.11 of [13] applied to the orthogonal projection p_ξ of $\mathbb{R}^N$ onto π_ξ yields

$$\int_{\pi_\xi} \mathcal{H}^0((\partial\Omega)_{x,\xi})\, d\mathcal{H}^{N-1}(x) \leq \mathcal{H}^{N-1}(\partial\Omega) < \infty .$$

Hence, the set $(\partial\Omega)_{x,\xi}$ is finite for $\mathcal{H}^{N-1}$-a.e. $x \in \pi_\xi$ so that $\Omega_{x,\xi}$ is the union of finitely many disjoint intervals. Next, let N_ξ be the set of those points $x \in \pi_\xi$ such that there exists a connected component of $\mathbb{R}\setminus\Omega_{x,\xi}$ which reduces to a singleton. We claim that $\mathcal{H}^{N-1}(N_\xi) = 0$. In order to estimate the size of N_ξ, let us notice that $N_\xi \subset p_\xi(N') \cup \cup p_\xi(N''_\xi)$ where the sets N' and N''_ξ are defined as follows:

— N' is the set of those points $y \in \partial\Omega$ such that there is no tangent hyperplane to $\partial\Omega$ at y;

— N''_ξ is the set of those points $y \in \partial\Omega$ such that the tangent hyperplane to $\partial\Omega$ at y exists and the unit normal vector $\nu(y)$ to $\partial\Omega$ at y is orthogonal to ξ.

We claim that both $p_\xi(N')$ and $p_\xi(N''_\xi)$ are $\mathcal{H}^{N-1}$-null sets. Indeed, as Ω has Lipschitz continuous boundary, N' itself is an $\mathcal{H}^{N-1}$-null set and hence the same property holds true for its projection on π_ξ. As far as the set $p_\xi(N''_\xi)$ is concerned, notice that N''_ξ is an $\mathcal{H}^{N-1}$-measurable subset of the countably $(N-1)$-rectifiable set $\partial\Omega$ so that the area formula (see [16], pg. 68) yields

$$\int_{N''_\xi} J_{\partial\Omega} p_\xi \, d\mathcal{H}^{N-1} = \int_{\pi_\xi} \mathcal{H}^0(N''_\xi \cap p_\xi^{-1}(x))\, d\mathcal{H}^{N-1}(x),$$

where $J_{\partial\Omega} p_\xi$ is the Jacobian of $p_\xi \colon \partial\Omega \to \mathbb{R}^N$ with respect to the countably $(N-1)$-rectifiable set $\partial\Omega$ (see again [16]). It is then easy to check that $J_{\partial\Omega} p_\xi$ vanishes on N''_ξ so that $N''_\xi \cap p_\xi^{-1}(x) = \emptyset$ for

$\mathcal{H}^{N-1}$-a.e. $x \in \pi_\xi$. Thus, $\mathcal{H}^{N-1}(p_\xi(N''_\xi)) = 0$ and the claim is proved so that (b) holds true.

Finally, it is easy to check that $\partial\Omega_{x,\xi} \subset (\partial\Omega)_{x,\xi}$ for all $\xi \in S^{N-1}$ and for all $x \in \pi_\xi$ and, for a given $\xi \in S^{N-1}$, the set of those points $x \in \pi_\xi$ for which the previous inclusion is strict is easily seen to be contained in N_ξ. Thus, (*c*) follows from (*b*). ∎

LEMMA 2.3. *Let M be a countably $(N-1)$-rectifiable set such that $\overline{M} \subset \Omega$, let ν_M be an approximate unit normal to M and let $u \in BV(\Omega)$. Then, for every $\xi \in S^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $x \in M_\xi$,*

a) the set $M_{x,\xi}$ consists of finitely many points and $(\nu_M(x + t\xi), \xi) \neq 0$ for every $t \in M_{x,\xi}$;

b) letting $\gamma^{\pm}_{M_{x,\xi}} u_{x,\xi}$ be the traces on the sides of $M_{x,\xi}$ determined by the approximate unit normal $\nu_{M_{x,\xi}}(t) = \operatorname{sign}(\nu_M(x + t\xi), \xi)$, $t \in M_{x,\xi}$, we have

$$\begin{cases} (\gamma^{+}_{M} u)_{x,\xi}(t) = \gamma^{+}_{M_{x,\xi}} u_{x,\xi}(t), \\ (\gamma^{-}_{M} u)_{x,\xi}(t) = \gamma^{-}_{M_{x,\xi}} u_{x,\xi}(t), \end{cases} \qquad t \in M_{x,\xi}. \tag{2.1}$$

PROOF. Suppose first that M is a $\mathcal{C}^1$-submanifold of codimension one whose closure is contained in Ω and let ν_M be a continuous orientation of M. Fix $\xi \in S^{N-1}$ and notice that, since the set $\{y \in M: (\nu_M(y), \xi) = 0\}$ is closed with respect to M and has $\mathcal{H}^{N-1}$-null projection on π_ξ, we can assume for instance that $(\nu_M(y), \xi) > 0$ for all $y \in M$. Then, recalling that $\mathcal{H}^{N-1}$ restricted to M is σ-finite, we see that claim (*a*) can be proved by the very same argument of Lemma 2.2. Then, we decompose M as a countable union of possibly overlapping open (with respect to M) sets $(M_h)_h$ of the form $M_h = \{x + \eta_h(x)\xi: x \in (M_h)_\xi\}$ where $\eta_h: \pi_\xi \to \mathbb{R}$ is continuously differentiable. We have $(M_h)_{x,\xi} = \{\eta_h(x)\}$ for every h and for every $x \in (M_h)_\xi$ and the sets $(M_h)_h$ can be chosen in such a way that for every h there exists $\varepsilon'_h > 0$ such that

$$M_h = M \cap \{x + t\xi: x \in (M_h)_\xi, \ |\eta_h(x) - t| \leq \varepsilon'_h\}.$$

It is clear that it is enough to prove (*b*) for $\mathcal{H}^{N-1}$-a.e. $x \in (M_h)_\xi$ and for all h. Therefore, choose h and, in order not to overburden the notation, drop the index h everywhere from now on. We are left to prove that

$$(\gamma^{\pm}_{M} u)_{x,\xi}(\eta(x)) = \gamma^{\pm}_{M_{x,\xi}} u_{x,\xi}(\eta(x))$$

for $\mathcal{H}^{N-1}$-a.e. $x \in M_\xi$. Now, by Tonelli's theorem, the functions $v_\varepsilon : M_\xi \to \mathbb{R}$ defined by $v_\varepsilon(x) = u(x + (\eta(x) + \varepsilon)\xi)$, $x \in M_\xi$ belong to $L_1(M_\xi)$ with respect to $\mathcal{H}^{N-1}$ for $\mathcal{L}^1$-a.e. $\varepsilon \in (0, \varepsilon')$ and satisfy

$$\int_{M_\xi} |v_{\varepsilon_2} - v_{\varepsilon_1}| \, d\mathcal{H}^{N-1} \leq |Du|(\{x + (\eta(x) + t)\xi : x \in M_\xi, \varepsilon_1 < t < \varepsilon_2\})$$

for $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon'$ avoiding a suitable $\mathcal{L}^1$-null set (see [14]). Hence, $(v_\varepsilon)_\varepsilon$ converges in $L_1 M_\xi$ with respect to $\mathcal{H}^{N-1}$ as $\varepsilon \to 0_+$ avoiding a suitable $\mathcal{L}^1$-null set and its limit is

$$v(x) = \gamma_M^+ u(x + \eta(x)\xi) = (\gamma_M^+ u)_{x,\xi}(\eta(x)), \qquad x \in M_\xi .$$

Now, choose a sequence $\varepsilon_k \downarrow 0_+$ such that $v_{\varepsilon_k} \in L_1(M_\xi)$ with respect to $\mathcal{H}^{N-1}$ for all $k \geq 1$ and $v_{\varepsilon_k} \to v$ $\mathcal{H}^{N-1}$-a.e. on M_ξ. Let $x \in M_\xi$ be such a point with the further property that $u_{x,\xi} \in BV(\Omega_{x,\xi})$. Then, we have

$$v(x) = \lim_{k \to \infty} v_{\varepsilon_k}(x) = \lim_{k \to \infty} u_{x,\xi}(\eta(x) + \varepsilon_k) = \gamma_{M_{x,\xi}}^+ u_{x,\xi}(\eta(x)) .$$

and this holds true for $\mathcal{H}^{N-1}$-a.e. $x \in M_\xi$. This proves the first equality in (2.1) and, as the proof of the remaining one is completely analogous, claim (*b*) holds true for a smooth manifold.
Finally, we notice that, as claim (*b*) is of pointwise nature, it remains true even if we let the orientation ν_M be only $\mathcal{H}^{N-1}$-measurable so that the statement for countably $(N-1)$-rectifiable sets can be proved by a standard argument. ∎

LEMMA 2.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz continuous boundary and let $u \in BV(\Omega)$. Then, for all $\xi \in S^{N-1}$ and for $\mathcal{H}^{N-1}$-a.e. $x \in \pi_\xi$, we have*

$$(2.2) \qquad (\gamma_{\partial\Omega} u)_{x,\xi}(t) = \gamma_{\partial\Omega_{x,\xi}} u_{x,\xi}(t), \qquad t \in \partial\Omega_{x,\xi} = (\partial\Omega)_{x,\xi} .$$

PROOF. Let Ω' be a bounded open set containing the closure of Ω and let $v \in BV(\Omega')$ be the function which agrees with u on Ω and vanishes on $\Omega' \setminus \Omega$. As $\partial\Omega$ is a closed countably $(N-1)$-rectifiable subset of Ω' that we orientate by the inward normal, the conclusion follows from the previous lemmas. ∎

We can now prove the lower semicontinuity result stated as Theorem 1.1. Its proof follows the lines of [1] (see also the proof of Theorem 6.1 in [5]).

PROOF OF THEOREM 1.1. We split the proof into two parts.

I) *$u_\infty \in SBV(\Omega)$ and $G_\varphi(u_\infty) \leq \liminf_{h\to\infty} G_\varphi(u_h)$.*

First, we notice that $u_\infty \in BV(\Omega)$ and that it is not restrictive to assume that $(G_\varphi(u_h))_h$ is a convergent sequence. Then, let $C_1 < \infty$ be such that

$$G_\varphi(u_h) \leq C_1\,, \qquad h \geq 1 \tag{2.3}$$

and, recalling hypotheses (*b*) and (*c*), let $\vartheta: [0, \infty) \to [0, \infty)$ be a Nagumo function such that

$$\max\left\{\int_{(\Omega)} \vartheta(|\nabla u_h|)\, d\mathcal{L}^N,\ |D^j u_h|(S_{u_h})\right\} \leq C_2\,, \qquad h \geq 1 \tag{2.4}$$

for some constant $C_2 < \infty$. Next, recall that $\mathcal{H}^{N-1}$ coincides with the $(N-1)$-dimensional integral-geometric measure on every countably $(N-1)$-rectifiable subset M of $\mathbb{R}^N$ (see [13], Theorem 2.10.15), so that, for every non negative $\mathcal{H}^{N-1}$-measurable function $f: M \to [0, \infty)$ we have

$$\int_M f\, d\mathcal{H}^{N-1} = \alpha_N \int_{S^{N-1}} \left\{ \int_{\pi_\xi} \left[\int_{M_{x,\xi}} f(x+t\xi)\, d\mathcal{H}^0(t)\right] d\mathcal{H}^{N-1}(x)\right\} d\mathcal{H}^{N-1}(\xi)\,,$$

where $\alpha_N = (2\omega_{N-1})^{-1}$ and ω_m is the $\mathcal{L}^m$-measure of the unit ball of $\mathbb{R}^m$ for $m \geq 1$ and $\omega_0 = 1$. Starting from the previous formula and taking into account (1.2), (1.3) and Lemmas 2.2 and 2.4, it is easy to establish, for all $u \in BV(\Omega)$, the following formulae

$$|D^j u|(S_{u_h}) = \alpha_N \int_{S^{N-1}} \left\{ \int_{\pi_\xi} |D^j u_{x,\xi}|(S_{u_{x,\xi}})\, d\mathcal{H}^{N-1}(x)\right\} d\mathcal{H}^{N-1}(\xi)\,, \tag{2.5}$$

$$G_\varphi(u) = \tag{2.6}$$

$$= \alpha_N \int_{S^{N-1}} \left\{ \int_{\pi_\xi} \left[\int_{(\overline{\Omega})_{x,\xi}} g_{x,\xi}(t, |Ju_{x,\xi}(t)|)\, d\mathcal{H}^0(t)\right] d\mathcal{H}^{N-1}(x)\right\} d\mathcal{H}^{N-1}(\xi)$$

where $Ju_{x,\xi} = J_{\varphi_{x,\xi}} u_{x,\xi}$ and where, for all $\xi \in S^{N-1}$ and $x \in \pi_\xi$, we have set $g_{x,\xi}(t, s) = g(x + t\xi, s)$ for $(t, s) \in \mathbb{R} \times [0, \infty)$. In order to simplify the notation, for $\xi \in S^{N-1}$ and $u \in BV(\Omega)$, set

$$
(2.7)\qquad \begin{cases} I^1_{x,\xi}(u) = \displaystyle\int_{(\Omega)_{x,\xi}} \vartheta(|\dot{u}_{x,\xi}|)\, d\mathcal{L}^1 , \\ I^2_{x,\xi}(u) = |D^j u_{x,\xi}|(S_{u_{x,\xi}}), \\ I^3_{x,\xi}(u) = \displaystyle\int_{(\overline{\Omega})_{x,\xi}} g_{x,\xi}(t, |Ju_{x,\xi}(t)|)\, d\mathcal{H}^0(t), \end{cases} \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \pi_\xi ,
$$

and notice that the functions $x \in \pi_\xi \dashrightarrow I^i_{x,\xi}(u) \in [0, \infty]$, $i = 1, 2, 3$ are well defined up to an $\mathcal{H}^{N-1}$-null set and are $\mathcal{H}^{N-1}$-measurable on π_ξ. Moreover, Tonelli's theorem together with (2.4), (1.1) and the monotonicity of ϑ yields that

$$
(2.8)\qquad \int_{\pi_\xi} I^1_{x,\xi}(u_h)\, d\mathcal{H}^{N-1}(x) \leq \int_\Omega \vartheta(|\nabla u_h|)\, d\mathcal{L}^N \leq C_2 , \quad h \geq 1,\ \xi \in S^{N-1} .
$$

In particular, all functions $I^1_{x,\xi}(u_h)$, $h \geq 1$ are $\mathcal{H}^{N-1}$-a.e. finite-valued on π_ξ. Now, we turn to ${}^2I_{x,\xi}(u_h)$ and $I^3_{x,\xi}(u_h)$. Taking into account (2.5) and (2.6), the estimates (2.3) and (2.4) and using Fatou's lemma, we see that, for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$, there exists $C_3(\xi) < \infty$ such that

$$
(2.9)\qquad \liminf_{h\to\infty} \int_{\pi_\xi} (I^2_{x,\xi}(u_h) + I^3_{x,\xi}(u_h))\, d\mathcal{H}^{N-1}(x) \leq C_3(\xi) .
$$

Moreover, hypothesis (a) and Tonelli's theorem together yield for each $\xi \in S^{N-1}$

$$
(2.10)\qquad \int_{\pi_\xi} \left(\int_{(\Omega)_{x,\xi}} |u_{h,x,\xi} - u_{\infty,x,\xi}|\, d\mathcal{L}^1 \right) d\mathcal{H}^{N-1}(x) = \\ = \|u_h - u_\infty\|_{L_1(\Omega)} \to 0 , \qquad h \to \infty .
$$

Therefore, given $\xi \in S^{N-1}$ such that (2.9) holds and $0 < \varepsilon < 1$, there exists a subsequence $u_k = u_{h_k}$, $k \geq 1$, which depends on ξ and ε such that

$$
(2.11)\qquad \lim_{k\to\infty} \int_{\pi_\xi} [\varepsilon(I^1_{x,\xi}(u_k) + I^2_{x,\xi}(u_k)) + I^3_{x,\xi}(u_k)]\, d\mathcal{H}^{N-1}(x) = \\ = \liminf_{h\to\infty} \int_{\pi_\xi} [\varepsilon(I^1_{x,\xi}(u_h) + I^2_{x,\xi}(u_h)) + I^3_{x,\xi}(u_h)]\, d\mathcal{H}^{N-1}(x) \leq C_2 + C_3(\xi),
$$

$$
(2.12)\qquad u_{k,x,\xi} \to u_{\infty,x,\xi} \quad \text{in } L_1(\Omega_{x,\xi}) \text{ for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega_\xi .
$$

Using (2.11) and Fatou's lemma again, for $\mathcal{H}^{N-1}$-a.e. $x \in \pi_\xi$, we find $C_4(x, \xi) < \infty$ such that

$$\liminf_{k \to \infty} [\varepsilon(I^1_{x,\xi}(u_k) + I^2_{x,\xi}(u_k)) + I^3_{x,\xi}(u_k)] \leq C_4(x, \xi). \tag{2.13}$$

Moreover, recalling the properties of the one dimensional sections of functions of bounded variation and Lemma 2.2, we notice that, for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$, the functions $u_{k,x,\xi}$ belong to $SBV(\Omega)_{x,\xi}$ for all $k \geq 1$ where $\Omega_{x,\xi}$ is a bounded open set consisting of finitely many connected components with pairwise disjoint closures such that $(\partial\Omega)_{x,\xi} = \partial\Omega_{x,\xi}$. Now, choose such a point $x \in \Omega_\xi$ with the further property that both (2.12) and (2.13) hold true. Then, there exists a subsequence $u_j = u_{k_j}$, $j \geq 1$, which depends on ξ, ε and x such that

$$\lim_{j \to \infty} [\varepsilon(I^1_{x,\xi}(u_j) + I^2_{x,\xi}(u_j)) + I^3_{x,\xi}(u_j)] =$$

$$= \liminf_{k \to \infty} [\varepsilon(I^1_{x,\xi}(u_k) + I^2_{x,\xi}(u_k)) + I^3_{x,\xi}(u_k)].$$

As $g_{x,\xi}$ and the sequence $(u_{j,x,\xi})_j$ satisfy all the hypotheses of Lemma 2.1, we get that $u_{\infty,x,\xi} \in SBV(\Omega_{x,\xi})$. This holds true for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$ and for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$ so that, as $u_\infty \in BV(\Omega)$, we conclude that $u_\infty \in SBV(\Omega)$. Moreover, the lower semicontinuity of $I^3_{x,\xi}$ along the sequence $(u_{j,x,\xi})_j$ follows from Lemma 2.1 again so that

$$\begin{aligned} I^3_{x,\xi}(u_\infty) &\leq \liminf_{j \to \infty} I^3_{x,\xi}(u_j) \leq \\ &\leq \lim_{j \to \infty} [\varepsilon(I^1_{x,\xi}(u_j) + I^2_{x,\xi}(u_j)) + I^3_{x,\xi}(u_j)] = \\ &= \liminf_{k \to \infty} [\varepsilon(I^1_{x,\xi}(u_k) + I^2_{x,\xi}(u_k)) + I^3_{x,\xi}(u_k)]. \end{aligned} \tag{2.14}$$

Such an estimate holds true for $\mathcal{H}^{N-1}$-a.e. $x \in \pi_\xi$ with $(u_k)_k$ independent of $x \in \pi_\xi$. Therefore, integrating both sides of (2.14) on π_ξ with respect to $\mathcal{H}^{N-1}$, using Fatou's lemma once more and (2.11), we get

$$\int_{\pi_\xi} I^3_{x,\xi}(u_\infty)\, d\mathcal{H}^{N-1}(x) \leq$$

$$\leq \lim_{k \to \infty} \int_{\pi_\xi} [\varepsilon(I^1_{x,\xi}(u_k) + I^2_{x,\xi}(u_k)) + I^3_{x,\xi}(u_k)]\, d\mathcal{H}^{N-1}(x) =$$

$$= \liminf_{h \to \infty} \int_{\pi_\xi} [\varepsilon(I^1_{x,\xi}(u_h) + I^2_{x,\xi}(u_h)) + I^3_{x,\xi}(u_h)]\, d\mathcal{H}^{N-1}(x).$$

Again, this formula holds true for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$ with $(u_h)_h$ independent of ξ. Thus, on account of (2.5), (2.6) and (2.4), the very same argument previously used yields

$$G_\varphi(u_\infty) \leq \liminf_{h \to \infty} G_\varphi(u_h) + \varepsilon(1 + N\alpha_N \omega_N) C_2 .$$

As $0 < \varepsilon < 1$ is arbitrary, the lower semicontinuity of G_φ along the sequence $(u_h)_h$ follows.

II) $\nabla u_h \rightharpoonup \nabla u_\infty$ *weakly in* $L_1(\Omega, \mathbb{R}^N)$.

We prove this by showing that, for every subsequence $u_k = u_{h_k}$, $k \geq 1$, whose gradients are weakly convergent in $L_1(\Omega, \mathbb{R}^N)$, there exists a basis $B \subset S^{N-1}$ of vectors of $\mathbb{R}^N$ with the property that, for each $\xi \in B$, there exists a further subsequence $u_i = u_{k_i}$, $i \geq 1$, depending on ξ, such that $(\nabla u_i, \xi) \rightharpoonup (\nabla u_\infty, \xi)$ weakly in $L_1(\Omega)$.
Therefore, let $u_k = u_{h_k}$, $k \geq 1$ be a subsequence such that $(\nabla u_h)_h$ is weakly convergent in $L_1(\Omega, \mathbb{R}^N)$. Notice that

$$\alpha_N \int_{S^{N-1}} \left\{ \int_{\pi_\xi} [I^2_{x,\xi}(u_k) + I^3_{x,\xi}(u_k)] \, d\mathcal{H}^{N-1}(x) \right\} d\mathcal{H}^{N-1}(\xi) \leq C_1 + C_2 , \quad k \geq 1$$

due to (2.5), (2.6) and (2.3), (2.4) so that Fatou's lemma yields

$$\liminf_{k \to \infty} \int_{\pi_\xi} (I^2_{x,\xi}(u_k) + I^3_{x,\xi}(u_k)) \, d\mathcal{H}^{N-1}(x) < \infty \tag{2.15}$$

for $\mathcal{H}^{N-1}$-a.e. $\xi \in S^{N-1}$. Let $B = \{\xi_1, \ldots, \xi_N\} \subset S^{N-1}$ be a basis of vectors of $\mathbb{R}^N$ such that (2.15) holds true for each $\xi \in B$. Then, choose $\xi \in B$ and a subsequence $u_i = u_{k_i}$, $i \geq 1$ depending on ξ such that

$$\int_{\pi_\xi} (I^2_{x,\xi}(u_i) + I^3_{x,\xi}(u_i)) \, d\mathcal{H}^{N-1}(x) \leq C_5(\xi), \qquad i \geq 1 \tag{2.16}$$

for some constant $C_5(\xi) < \infty$. Now, we claim that

$$\int_\Omega |(\nabla u_\infty, \xi) - w| \, d\mathcal{L}^N \leq \liminf_{i \to \infty} \int_\Omega |(\nabla u_i, \xi) - w| \, d\mathcal{L}^N \tag{2.17}$$

for all $w \in L_1(\Omega)$. This is equivalent to proving that $(\nabla u_i, \xi) \rightharpoonup (\nabla u_\infty, \xi)$ by Proposition 4.4 in [1]. In order to prove the claim, choose $w \in L_1(\Omega)$ and a subsequence $u_l = u_{i_l}$, $l \geq 1$ depending on ξ and w such that

$$\lim_{l \to \infty} \int_\Omega |(\nabla u_l, \xi) - w| \, d\mathcal{L}^N = \liminf_{i \to \infty} \int_\Omega |(\nabla u_i, \xi) - w| \, d\mathcal{L}^N . \tag{2.18}$$

Then, for $u \in BV(\Omega)$, consider the function

$$I^4_{x,\xi}(u, w) = \int_{\Omega_{x,\xi}} |\dot{u}_{x,\xi} - w_{x,\xi}|\, d\mathcal{L}^1$$

defined for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$. It is $\mathcal{H}^{N-1}$-measurable and satisfies

$$\int_{\Omega_\xi} I^4_{x,\xi}(u, w)\, d\mathcal{H}^{N-1}(x) = \int_{\Omega} |(\nabla u, \xi) - w|\, d\mathcal{L}^N$$

by Tonelli's theorem. Now, choosing $0 < \varepsilon < 1$, relying on (2.8), (2.16) and applying Fatou's lemma as in (I), we get

$$\liminf_{l \to \infty} \left[\varepsilon \sum_{1 \leqslant i \leqslant 3} I^i_{x,\xi}(u_l) + I^4_{x,\xi}(u_l, w) \right] < \infty$$

for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$ so that, choosing such a point $x \in \Omega_\xi$ with the further properties that $u_{l,x,\xi} \in SBV(\Omega_{x,\xi})$ for all $l \in \mathbb{N}_+ \cup \{\infty\}$ and $u_{l,x,\xi} \to u_{\infty,x,\xi}$ in $L_1(\Omega_{x,\xi})$, we find a further subsequence $u_j = u_{l_j}$, $j \geqslant 1$ depending on ξ, w, ε and x itself such that

$$(2.19) \qquad \lim_{j \to \infty} \left[\varepsilon \sum_{1 \leqslant i \leqslant 3} I^i_{x,\xi}(u_j) + I^4_{x,\xi}(u_j, w) \right] = \\ = \liminf_{l \to \infty} \left[\varepsilon \sum_{1 \leqslant i \leqslant 3} I^i_{x,\xi}(u_l) + I^4_{x,\xi}(u_l, w) \right] < \infty .$$

Thus, $\dot{u}_{j,x,\xi} \rightharpoonup \dot{u}_{\infty,x,\xi}$ weakly in $L_1(\Omega_{x,\xi})$ by Lemma 2.1 so that the weak lower semicontinuity of the norm and (2.19) yield

$$I^4_{x,\xi}(u_\infty, w) \leqslant \liminf_{j \to \infty} I^4_{x,\xi}(u_j, w) \leqslant \\ \leqslant \liminf_{l \to \infty} \left[\varepsilon \sum_{1 \leqslant i \leqslant 3} I^i_{x,\xi}(u_l) + I^4_{x,\xi}(u_l, w) \right].$$

This estimate holds true for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega_\xi$ with $(u_l)_l$ independent of x so that, integrating both sides of the previous inequality on Ω_ξ with respect to $\mathcal{H}^{N-1}$, using Fatou's lemma once more and taking into account (2.8), (2.16) and (2.18), we get that

$$\int_{\Omega} |(\nabla u_\infty, \xi) - w|\, d\mathcal{L}^N \leqslant \liminf_{i \to \infty} \int_{\Omega} |(\nabla u_i, \xi) - w|\, d\mathcal{L}^N + \varepsilon[C_2 + C_5(\xi)].$$

As this holds true for all $0 < \varepsilon < 1$ with $(u_i)_i$ independent of ε, claim (2.17) is proved and this completes the proof. ■

The proof of Theorem 1.4 is now immediate.

PROOF OF THEOREM 1.4. Assume that $\inf\{F(u) + G_\varphi(u): u \in SBV(\Omega)\} < \infty$ otherwise there is nothing to prove and let $(v_h)_h \subset SBV(\Omega)$ be a minimizing sequence for $(F + G_\varphi)$ along which the functional is bounded. Set $c = \|\varphi\|_{L_\infty \partial\Omega} \vee \|\psi\|_{L_\infty(\Omega)}$ and let $\eta_c: \mathbb{R} \to \mathbb{R}$ the Lipschitz continuous function defined by

$$\eta_c(t) = (t \wedge c) \vee (-c), \qquad t \in \mathbb{R}.$$

The functions $u_h = \eta_c \circ v_h$, $h \geq 1$ are uniformly bounded in $L_\infty(\Omega)$ and still belong to $SBV(\Omega)$.
We claim that $(u_h)_h$ is still a minimizing sequence for $(F + G_\varphi)$. Indeed, as ∇u_h vanishes $\mathcal{L}^N$-a.e. on $\{x \in \Omega: \nabla u_h \neq \nabla v_h\}$ for $h \geq 1$ and $|u_h - \psi| \leq |v_h - \psi|$ $\mathcal{L}^N$-a.e. on Ω for $h \geq 1$, we conclude that $F(u_h) \leq F(v_h)$ for all $h \geq 1$. Moreover, for all $h \geq 1$ we have $|u_h^+(x) - u_h^-(x)| \leq |v_h^+(x) - v_h^-(x)|$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Omega$ and similarly $|\gamma_{\partial\Omega} u_h(x) - u_0(x)| \leq |\gamma_{\partial\Omega} v_h(x) - u_0(x)|$ for $\mathcal{H}^{N-1}$-a.e. $x \in \partial\Omega$ so that the monotonicity assumption (1.7) on g yields $G_\varphi(u_h) \leq G_\varphi(v_h)$ for all $h \geq 1$ as well.
Now, recalling (1.9), choose $\alpha > 0$ such that

$$g(x, t) \geq \vartheta(t) \geq \alpha t, \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \mathbb{R}^N, \qquad 0 \leq t \leq 2c,$$

so that, as $|J_\varphi u_h| \leq 2c$ $\mathcal{H}^{N-1}$-a.e. on $\overline{\Omega}$ for $h \geq 1$, we see that the jump parts of $(Du_h)_h$ have uniformly bounded total variation on Ω. Thus, $(u_h)_h$ is bounded in $BV(\Omega)$ and hence there does exist $u_\infty \in BV(\Omega)$ and a subsequence $u_k = u_{h_k}$, $k \geq 1$ such that $u_k \to u_\infty$ in $L_1(\Omega)$. Moreover, $(\nabla u_k)_k$ is relatively weakly compact in $L_1(\Omega, \mathbb{R}^N)$. Therefore, Theorem 1.1 ensures that $u_\infty \in SBV(\Omega)$ and that $\nabla u_h \rightharpoonup \nabla u_\infty$ weakly in $L_1(\Omega, \mathbb{R}^N)$ so that the lower semicontinuity of G_φ and F follows from Theorem 1.1 again and from a classical result respectively. Thus, the direct method of the Calculus of Variations applies.

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