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Steady ideals and rings

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ABSTRACT - A (right R-) module M is *dually slender* if the covariant $\mathrm{Hom}_R(M, -)$ functor commutes with direct sums. A ring R is *right steady* provided that the dually slender modules coincide with the finitely generated ones. Rings satisfying various finiteness conditions are known to be steady. We investigate steadiness of the rings such that each two-sided ideal is countably generated. We prove e.g. that such rings are steady provided that they are von Neumann regular and with all primitive factors artinian. Also, we obtain a characterization of steadiness for arbitrary valuation rings, and an example showing that steadiness is not left-right symmetric even for countable rings.

Steady rings were introduced by Rentschler [R2] as part of an investigation of commutativity properties of covariant and contravariant Hom functors in module categories (see [W] for a survey). More recently, steady rings have played an important role in dealing with various particular problems such as investigations of homomorphisms of graded modules over group graded rings [GMN], representable equivalences of module categories [MO], [CM], [CT] et al. It is well-known that a ring is steady provided that it satisfies some of the classical finiteness conditions, e.g. provided it is noetherian [R2], perfect [CT], semiartinian of countable Loewy length [EGT] etc. Nevertheless, a ring-theoretic characterization of steady rings remains an open problem. In this paper, we investigate the steadiness of rings such that each two-sided ideal is countably generated.

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Recall that a ring R is said to be *right steady* if each dually slender (right R-) module is finitely generated. By [EGT], a module M is *dually slender* («de type $\sum$» in [R2], «small» in [GMN] et al.) provided that the covariant functor $\mathrm{Hom}_R(M, -)$ commutes with direct sums. Dually slender modules are characterized by various equivalent conditions. We mention two of them: the expression «commutes with direct sums» can be replaced by «commutes with countable direct sums». This makes clear the duality with the well-known class of slender modules [EM, Ch. III]. Also, a module is dually slender iff it is not a union of a strictly increasing countably infinite chain of its submodules [R2]. The latter characterization shows that dual slenderness is actually a chain condition which is satisfied by each finitely generated module, but is not satisfied by any countably infinitely generated module.

In general, there exists an intermediate class between the finitely generated and the dually slender modules: if κ is an infinite cardinal, then a module M is said to be *κ-reducing* provided that each $< \kappa$-generated submodule of M is contained in a finitely generated one. κ-reducing modules were used in [T1] to solve a problem of Eklof and Mekler on almost free modules over non-perfect rings. In [Ro] and [T2], they were employed to provide constructions of non-steady rings.

Our approach to steadiness is through a study of the ω-complete anti-filter, $\mathcal{S}_R$, of all right steady (two-sided) ideals of the ring R (see Definition 1). Assume that all ideals of R are countably generated. Then $\mathcal{S}_R$ has a maximal element, M. Now, simply, R is right steady iff $M = R$. Thus we prove steadiness of von Neumann regular rings such that all primitive factor-rings are artinian (Theorem 9). Also, we show that an arbitrary valuation ring S is steady iff $\mathrm{rad}(S)$ is countably generated and $S/\mathrm{rad}(S)$ contains no uncountable chains of ideals (Theorem 13). There is a couple of examples in the paper illustrating the limits of the methods employed. Moreover, Example 14 shows that the property of being a steady ring is not left-right symmetric, even for countable rings.

Let R be a ring. Then $\mathrm{Rad}(R)$ and $\mathrm{rad}(R)$ denote the Jacobson radical and the prime radical, respectively, of R. Unless otherwise stated, the term *ideal* denotes a two-sided ideal of R. In particular, an ideal I is *countably generated* if I contains a subset, G, which generates I as a two-sided ideal and $\mathrm{card}(G) \leq \omega$. A ring R is a *valuation ring* if R is commutative and the lattice of all ideals is linearly ordered by inclusion.

A set, $\mathcal{S}$, of ideals is an *anti-filter* provided that for all $I, J \in \mathcal{S}$, also $I + J \in \mathcal{S}$, and $K \in \mathcal{S}$ whenever K is an ideal with $K \subseteq I$. The anti-filter $\mathcal{S}$ is *w-complete* if $\mathcal{S}$ is closed under unions of countable chains.

The term *module* denotes a right R-module, and the category of all modules is denoted by $\mathrm{Mod} - R$. For a module M, $\mathrm{gen}(M)$ denotes the minimal cardinality of an R-generating subset of M. A ring R is a right *V-ring* provided that all simple modules are injective. For further notation, we refer to [AF].

We start with the basic notion of a steady ideal:

DEFINITION 1. *Let R be a ring and I an ideal. Then I is said to be* right steady *provided that each dually slender module M satisfying $MI = M$ is finitely generated.*

Denote by $\mathcal{S}_R$ the set of all right steady ideals. Clearly, $0 \in \mathcal{S}_R$, and $R \in \mathcal{S}_R$ iff R is a right steady ring.

LEMMA 2. *Let R be a ring.*

(i) *$\mathcal{S}_R$ is an ω-complete anti-filter of ideals.*

(ii) *Assume that all ideals are countably generated. Then $\mathcal{S}_R$ has a maximal element.*

(iii) *Let $J \subsetneq I$ be ideals. Assume that $J \in \mathcal{S}_R$ and $I/J \in \mathcal{S}_{R/J}$. Then $I \in \mathcal{S}_R$.*

(iv) *Let $I \in \mathcal{S}_R$. Then R/I is a right steady ring iff R is so.*

PROOF. (i) Let $I, J \in \mathcal{S}_R$ and let M be a dually slender module such that $M(I + J) = M$. Since $(M/MI)J = (MJ + MI)/MI = M/MI$ is a dually slender module, there is a finitely generated module F such that $M = F + MI$. Then $(M/F)I = M/F$, so M/F and M are finitely generated, and $I + J \in \mathcal{S}_R$.

Let $I \in \mathcal{S}_R$ and $J \subsetneq I$. Obviously, if $MJ = M$, then $MI = M$. It follows that $\mathcal{S}_R$ is an anti-filter.

Let $(I_n;\, n < \omega)$ be an increasing chain of right steady ideals. Denote $L = \bigcup_{n<\omega} I_n$ and let M be a dually slender module such that $ML = M$. Since $M = M(\bigcup_{n<\omega} I_n) = \bigcup_{n<\omega} MI_n$ and M is dually slender, there is $n < \omega$ such that $MI_n = M$. But I_n is right steady so M is finitely generated. It follows that $L \in \mathcal{S}_R$, and $\mathcal{S}_R$ is ω-complete.

(ii) An easy application of Zorn's Lemma, using (i).

(iii) Let M be a dually slender module such that $MI = M$. By the premise, M/MJ is finitely generated. So there is a finitely generated module F such that $M = F + MJ$. Since $(M/F)J = M/F$, the modules M/F and M are finitely generated.

(iv) By (iii). ■

LEMMA 3. *Let R be a ring and J be a right ideal which is either noetherian or right T-nilpotent. Then $RJ \in \mathcal{S}_R$.*

PROOF Put $I = RJ$. Then $MI = MJ$ for every $M \in$ Mod-R. If J is right T-nilpotent, then $MJ \neq M$ for every non-zero module M by [AF], Lemma 28.3.

Proving indirectly, assume that J is a noetherian ideal and M is an infinitely generated module such that $MJ = M$. Let C be a countably infinitely generated submodule of M and D be a maximal submodule of M such that $C \cap D = 0$. Then $M_0 = C \oplus D$ is an essential submodule of M which is not dually slender. By [CT, Lemma 1.2] there exists $\varphi \in \operatorname{Hom}_R(M_0, \bigoplus_{i<\omega} E_i)$ such that E_i are injective modules for all $i < \omega$ and $\varphi(M_0) \not\subseteq \bigoplus_{i<n} E_i$ for each $n < \omega$. Define a set

$$U = \{(N, \psi);\, M_0 \subseteq N \subseteq M,\ \psi \in \operatorname{Hom}_R(N, \bigoplus_{i<\omega} E_i),\ \psi \restriction M_0 = \varphi\}$$

and a partial order $\leqslant$ on U by $(N_1, \psi_1) \leqslant (N_2, \psi_2)$ provided $N_1 \subseteq N_2$ and $\psi_2 \restriction N_1 = \psi_1$. By Zorn's Lemma there is a maximal element, (N_0, ψ_0), of U.

Assume there exists $m \in M \setminus N_0$. By the premise, $mR \subseteq \sum_{i \in F} m_i J$ for a finite set F and $m_i \in M$, $i \in F$. Since J is noetherian, so is mR. In particular, $mR \cap N_0$ is finitely generated. Then $\psi_0(mR \cap N_0) \subseteq \bigoplus_{i<n} E_i$ for some $n < \omega$. By injectivity, it is possible to extend $\psi_0 \restriction (mR \cap N_0)$ to a homomorphism ν_0 defined on all the module mR. Now, we define $\psi \in \operatorname{Hom}_R(N_0 + mR, \bigoplus_{i<\omega} E_i)$ by $\psi(x + y) = \psi_0(x) + \nu_0(y)$, for all $x \in N_0$ and $y \in mR$. Clearly, ψ properly extends ψ_0, in contradiction with the maximality of (N_0, ψ_0).

This proves that $N_0 = M$. Since $\psi_0(M) \not\subseteq \bigoplus_{i<n} E_i$ for each $n < \omega$, M is not dually slender. ■

Taking $J = R$ in Lemma 3, we obtain the well-known fact that each right noetherian ring is right steady [R2, 7^0], [CM, Proposition 1.9].

Similarly, taking $J = \operatorname{Rad}(R)$, we get that each right perfect ring is right steady [CT, Corollary 1.6]. Moreover, if R is a ring such that $\operatorname{Rad}(R)$ is right T-nilpotent, then R is right steady iff $R/\operatorname{Rad}(R)$ is such (cf. [R1, Lemma 2.3.2)]. For example, let R be the upper triangular $n \times n$ matrix ring over a ring S. Then R is (left, right) steady iff S is such.

PROPOSITION 4. *Let R be a ring such that all ideals are countably generated. Assume that every factor-ring of R contains a non-zero right ideal which is either noetherian or right T-nilpotent. Then R is right steady.*

PROOF. By Lemma 2 (ii), there is a maximal element $J \in \mathcal{S}_R$. Assume that $J \neq R$. By Lemma 3, there is an ideal $I \in R$ such that $J \subsetneqq I$ and $I/J \in \mathcal{S}_{R/J}$. So $I \in \mathcal{S}_R$ by Lemma 2(iii), a contradiction. ■

LEMMA 5. *Let R be a ring such that each dually slender submodule of any cyclic module is finitely generated. Then each dually slender submodule of any countably generated module is finitely generated.*

PROOF. Let M be a countably generated module and N be a dually slender submodule of M.

(i) Assume $\operatorname{gen}(M) < \omega$. By induction on $\operatorname{gen}(M)$, we prove that N is finitely generated. The base step ($\operatorname{gen}(M) = 1$) is our assumption. Assume $n \geq 1$ and the assertion holds for all $M' \in$ Mod-R with $\operatorname{gen}(M') \leq n$. Let $\operatorname{gen}(M) = n + 1$, and $M = \sum_{i \leq n} m_i R$ for some $m_i \in M$, $i \leq n$. Put $M' = \sum_{i < n} m_i R$ and $N' = M' \cap N$. Then $N/N' \cong (N + M')/M' \subseteq M/M'$, whence N/N' is isomorphic to a dually slender submodule of the cyclic module M/M'. So there is a finitely generated module F such that $N = F + N'$. Then $N/F = (F + N')/F \cong N'/(F \cap N') \subseteq M'/(F \cap N')$. So N/F is isomorphic to a dually slender submodule of a $\leq n$-generated module. By the induction premise, N/F and N, are finitely generated.

(ii) Let $\operatorname{gen}(M) = \omega$. Then $M = \bigcup_{n < \omega} M_n$, where $(M_n \mid n < \omega)$ is a strictly increasing chain of finitely generated submodules of M. In particular, $N = \bigcup_{n < \omega} (N \cap M_n)$. So there is $n < \omega$ with $N \subseteq M_n$. By part (i), N is finitely generated. ■

LEMMA 6. *Let R be an abelian regular ring. Assume that for every factor-ring, S, of R and every infinitely generated ideal, I, of S there is a strictly increasing chain of ideals $(I_n; n < \omega)$ of S such that*

(1) $I = \bigcup_{n < \omega} I_n$ *and*

(2) I_n *is not essential in I for each $n < \omega$.*

Then R is steady.

PROOF. We prove the assertion in two steps.

Step (I): We show that if R' is a factor-ring of R and N' is a faithful module such that $N'J + L = N'$ for an infinitely generated ideal J of R' and for a finitely generated submodule L of N' with $N'e \nsubseteq L$ for each non-zero idempotent $e \in R'$, then N' is not dually slender. Indeed, by the premises (1) and (2), there is a strictly increasing chain of ideals $(J_n;\, n < \omega)$ of R' such that $J = \bigcup_{n<\omega} J_n$ and $J_n \not\trianglelefteq J$ for each $n < \omega$. So for each $n < \omega$ there is a non-zero idempotent $e_n \in J$ such that $e_n R' \cap \cap J_n = 0$. Since $N' e_n \not\subseteq L$, the chain of modules $((N' J_n + L)/L;\, n < \omega)$ contains a strictly increasing subchain and $N'/L = \bigcup_{n<\omega} (N' J_n + L)/L$. This proves that N' is not small.

Step (II): Assume R is not steady and let $M \in$ Mod-R be an infinitely generated dually slender module. Note that by (1), Lemma 5 applies to R. Define a set $I \subseteq R$ by

$$I = \{r \in R;\ \mathrm{gen}(MrR) < \omega\}.$$

Since $M(r+s)R \subseteq (MrR + MsR)$ for every $r, s \in I$, $M(r+s)R$ is finitely generated by Lemma 5. So $r + s \in I$. Similarly, $MirR \subseteq MiR$ for every $i \in I$, $r \in R$, so $ir \in I$. This proves that I is an ideal.

Let $S = R/I$ and $N = M/MI \in$ Mod-S. We show that the module Ne is infinitely generated for every non-zero idempotent $e \in S$. Indeed, if $f \in R$ is an idempotent and $e = \bar{f} \in S$ is such that $\mathrm{gen}(Ne) < \omega$, then there is a finitely generated module $K \subseteq Mf$ with $Mf + MI = K + MI$. Since R is abelian regular, we get $MfI + K \supseteq MIf + Kf \supseteq Mf$, whence $(Mf/K)I = Mf/K$. By Step (I) applied to the setting of $R' = = R/\mathrm{Ann}_R(Mf/K)$, $N' = Mf/K$, $J = I/(I \cap \mathrm{Ann}_R(Mf/K))$, and $L = 0$, we infer that J is finitely generated. So $I = (I \cap \mathrm{Ann}_R(Mf/K)) + gR$ for an idempotent $g \in I$. In particular, $\mathrm{gen}(Mg) < \omega$ and $(Mf/K)g = Mf/K$, whence $Mf = (Mg)f + K$ is finitely generated. This proves that $f \in I$, i.e. $e = 0$.

Let J be a maximal infinitely generated ideal of S. Then S/J is completely reducible, and there is a finitely generated right S-module L such that $N = L + NJ$. By Step (I) for $R' = S$ and $N' = N$, we infer that N is not dually slender, a contradiction. ∎

COROLLARY 7. *Let R be an abelian regular ring such that all ideals are countably generated. Then R is steady.*

PROOF. All factor-rings of an abelian regular ring are also abelian regular. Moreover, if I is an ideal in an abelian regular ring S such that $\mathrm{gen}(I) = \omega$, then there is a set of orthogonal idempotents, $\{e_i;\, i < \omega\}$, in S such that $I = \bigcup_{i<\omega} e_i S$. The result now follows by Lemma 6. ∎

The following example shows that the converse of Proposition 4 does not hold for abelian regular rings.

EXAMPLE 8. There exists an abelian regular non-commutative ring R such that all ideals are countably generated but R does not contain any non-zero right ideal which is either noetherian or right T-nilpotent.

PROOF. Take two injective mapping $x, y\colon \omega \to \omega$ such that $x(\omega) \cap \cap\, y(\omega) = \emptyset$ and $x(\omega) \cup y(\omega) = \omega$. Denote by M the submonoid of the monoid of all mappings on ω generated by the set $\{x, y\}$. Clearly, M is a free monoid with the free basis $\{x, y\}$, and every $g \in M$ is an injective mapping with $\operatorname{card}(g(\omega)) = \omega$. Let K be a division ring. For each $g \in M$ define $e_g \in K^\omega$ such that $e_g(\alpha) = 1_K$ for each $\alpha \in g(\omega)$, and $e_g(\alpha) = 0_K$ for each $\alpha \notin g(\omega)$. Finally, denote by R the K-subalgebra of K-algebra K^ω generated by $\{e_g;\, g \in M\}$. Then the elements of R are piecewise constant functions from ω to K (Each $r \in R$ is of the form $r = \sum_{i<n} k_i\, e_{g_i}$ where $k_i \in K$, $g_i \in M$, and $g_i(\omega) \cap g_j(\omega) = \emptyset$ for all $i \neq j < n$). Moreover, R is a subring of the (non-commutative) ring K^ω and all ideals in R are countably generated. For every $u \in R$ define $v \in K^\omega$ by $v(\alpha) = (u(\alpha))^{-1}$ provided that $u(\alpha) \neq 0_K$, and $v(\alpha) = 0_K$ otherwise. Clearly, $v \in R$ and $uvu = u$, and R is abelian regular. By Corollary 7, R is a steady ring.

Finally, let $0 \neq r = \sum_{i<n} t_i\, g_i \in R$ for some $t_i \in K$ and $g_i \in M$; $0 < i < < n < \omega$. Since $x^j y(\omega) \cap x^l y(\omega) = \emptyset$ for all $j \neq l < \omega$, we have $\bigoplus_{j<\omega} e_{g_1 x^j y} R \subseteq rR$, and rR is not noetherian. Since R is abelian regular, R does not contain any non-zero nilpotent elements.

Now, we generalize Corollary 7 to regular rings with primitive factors artinian:

THEOREM 9. *Let R be a regular ring such that all ideals are countably generated. If all primitive factor-rings of R are artinian, then R is right steady.*

PROOF By Lemma 2 (ii), there is a maximal element $I \in \mathcal{S}_R$.

Assume that $I \neq R$. If R/I is a simple ring then it is artinian, and right steady, by Corollary 4. So $R \in \mathcal{S}_R$ by Lemma 2 (iii), in contradiction with the maximality of I. If R/I is not simple then, by [G, Theorem 6.6], there is a non-zero central idempotent $e \in R/I$ such that $e(R/I) = = (eR + I)/I$ is isomorphic to a full matrix ring over an abelian regular ring. By Corollary 7 and [EGT, Lemma 1.7], $e(R/I)$ is right steady. By Lemma 2 (iii), $eR + I \in \mathcal{S}_R$, a contradiction.

This proves that $I = R$, i.e. R is right steady. ∎

EXAMPLE 10. (i) The converse of Theorem 9 does not hold. Namely, there exist right steady non-artinian primitive regular rings such that each ideal is countably generated. To see this, take an ω-dimensional right linear space, L, over a field K. Let B be a basis of L. Let R be the subring of $\mathrm{End}(L_K)$ generated by all endomorphisms whose matrix with respect to B is row- and column- finite, and by all scalar multiplications. Then R is a semiartinian regular ring of Loewy length 2, so R is right steady by [EGT, Theorem 2.2], whence R is the required example.

(ii) Also, Theorem 9 is not true for arbitrary regular rings with all primitive factors artinian. By [EGT, Theorem 2.6], for each uncountable ordinal σ, there is a commutative regular semiartinian ring of Loewy length $\sigma + 1$ which is not steady.

(iii) By [G, Proposition 6.18], for any regular ring R, all primitive factors of R are artinian iff all homogenous semisimple modules are injective. It is an open problem whether Theorem 9 extends to regular right V-rings such that each ideal is countably generated.

Nevertheless, further extension to countable unit regular rings is not possible: by [T2, Example 2.8], each simple countable non-completely reducible unit regular ring is non-steady.

The following result was proved in the particular case when R is commutative in [R1, Lemma 2.3.6]:

LEMMA 11. *Let R be a ring such that all ideals are countably generated. Assume R is not right steady. Then there exists a prime ideal I such that the ring $S = R/I$ is not right steady and there is an infinitely generated dually slender module $M \in \text{Mod-}S$ with $\mathrm{gen}(M/MsS) < \omega$ for each non-zero $s \in S$.*

PROOF Let N be an infinitely generated dually slender module. Define a set of ideals by $\mathfrak{I} = \{I \subsetneq R;\ \mathrm{gen}(N/NI) > \omega\}$. If $(I_\alpha;\ \alpha < \kappa)$ is a strictly increasing chain of elements $\mathfrak{I}$, then it is countable, and by the dual slenderness of N, $\bigcup_{\alpha < \kappa} I_\alpha \in \mathfrak{I}$. By Zorn's Lemma, there is a maximal element, I, of $\mathfrak{I}$. Let $M = N/NI$. Clearly, M/MrR is finitely generated for each $r \in R \setminus I$. Similarly, for each ideal $J \supsetneq I$ there is a finitely generated submodule, F, of M with $(M/F)J = M/F$. It follows that for all right ideals J_1, $J_2 \nsubseteq I$ there is a finitely generated module $G \subsetneq M$ such that $(M/G)J_1J_2 = (M/G)J_2 = M/G \neq 0$. In particular, $J_1J_2 \nsubseteq I$, and I is prime. ∎

COROLLARY 12. *Let R be a ring such that all ideals are countably generated. Then R is right steady iff all prime factor-rings of R are right steady.*

Since all prime rings are indecomposable (as rings), Corollary 12 and [G, Corollary 6.7] give an alternative proof of Theorem 9.

Now, we turn to arbitrary valuation rings:

THEOREM 13. *Let R be a valuation ring. Then R is steady iff $\operatorname{rad}(R)$ is countably generated and the ring $R/\operatorname{rad}(R)$ contains no uncountable chains of ideals.*

PROOF ($\rightarrow$) Since R is a valuation ring, $\operatorname{rad}(R)$ is the least prime ideal in R [FS, Ch.I]. If $\operatorname{rad}(R)$ is not countably generated, then there is an ω_1-generated ω_1-reducing ideal contained in $\operatorname{rad}(R)$, a contradiction. Further, assume that the (steady) valuation domain $R/\operatorname{rad}(R)$ contains an uncountable chain of ideals. Denote by Q the quotient field of $R/\operatorname{rad}(R)$. Then Q contains an ω_1-generated ω_1-reducing $R/\operatorname{rad}(R)$-submodule, a contradiction.

($\leftarrow$) Since $\operatorname{rad}(R)$ is a countably generated nil ideal, it is a countable union of nilpotent ideals. By Lemmas 2 (i) and 3, $\operatorname{rad}(R) \in \mathcal{S}_R$. Assume R is not steady. By Lemma 2 (iv), the ring $R/\operatorname{rad}(R)$ is not steady, so Lemma 11 applies to it. W.l.o.g., we assume that $\operatorname{rad}(R) = I = 0$. There is a sequence of non-zero elements of R, $\{x_n\}_{n<\omega}$, such that for each $n < \omega$, $x_{n+1}R \subsetneq x_nR$, and for every non-zero $r \in R$ there exists $n < \omega$ such that $x_n \in rR$. Moreover, in the notation of Lemma 11, for every $n < \omega$ there is a finitely generated module F_n such that $Mx_n + F_n = M$. Then $P = M/(\sum_{n<\omega} F_n)$ is an infinitely generated dually slender module, and $Pr = P$ for each non-zero $r \in R$. Define an increasing chain of submodules of P by $P_n = \{m \in P;\ mx_n = 0\}$ for each $n < \omega$. Since $P/P_n \cong P$, we have $\operatorname{gen}(P/P_n) > \omega$, whence $\operatorname{gen}(P/\bigcup_{n<\omega} P_n) > \omega$. Since $\bigcup_{n<\omega} P_n$ is the torsion part of P, the module $N = P/\bigcup_{n<\omega} P_n$ is torsion-free. As $Nr = N$ for each non-zero $r \in R$, we infer that N is a dually slender, hence finitely generated, module over the quotient field Q of R. The module Q is generated by $\{x_n^{-1};\ n < \omega\}$, so $\operatorname{gen}(N) \leq \omega$, a contradiction. ■

Theorem 13 does not extend to non-commutative domains with the property that all (right) ideals are linearly ordered by inclusion. We prove this using the counter-examples of Jategaonkar [J], also showing that the property of being a steady ring is not left-right symmetric:

EXAMPLE 14. By the right-hand version of [J, Theorem 4.6], for each infinite cardinal κ there is a non-commutative principal right ideal domain R_κ such that the right ideals of R_κ coincide with the two-sided ones, and contain a cofinal chain which is anti-isomorphic to the interval of ordinals $[0, \kappa]$. In particular, R_κ is a right noetherian, hence a right steady, domain. On the other hand, by [T3, Example 2.11], R_κ contains a free left R_κ-module of rank κ. By [T2], Example 2.8, R_κ is not left steady. Moreover, by [J, Theorem 3.1], R_ω can be taken to be countable.

PROPOSITION 15. *Let R be a countable ring. Assume that each non-zero right ideal of R/I contains a non-zero left ideal, for each prime ideal I. Then R is right steady.*

PROOF. Suppose R is not steady. Then Lemma 11 and its notation apply. W.l.o.g., we assume that $I = 0$, i.e. $R = S$ is a prime ring. Denote by $\{r_n\}_{n<\omega}$ the sequence of all non-zero elements of R. For each $n < \omega$, let $I_n = Rr_nRr_{n-1}\dots Rr_0R (\neq 0)$. Define a sequence of modules by $M_n = \{m \in M;\ mI_n = 0\}$, $n < \omega$. If $\operatorname{gen}(M/M_n) < \omega$ for some $n < \omega$, then $M = M_n + F$ for a finitely generated module F. So $MI_n = FI_n$ is countable, and $\operatorname{gen}(M/MI_n) < \omega$ by Lemma 11, whence $\operatorname{gen}(M) \leq \omega$, a contradiction. This shows that $\operatorname{gen}(M/M_n) > \omega$ for all $n < \omega$, whence $\operatorname{gen}(M/\bigcup_{n<\omega} M_n) > \omega$.

Denote by $N = M/\bigcup_{n<\omega} M_n$. If $mr \in \bigcup_{n<\omega} M_n$ for some $m \in M$ and $0 \neq r \in R$, then there is $k < \omega$ such that $mrI_k = 0$. Since R is prime, $rI_k \neq 0$. By the assumption there is $p < \omega$ such that $Rr_p \subseteq rI_k$. So $mRr_p = 0$ and $m \in M_p$. It follows that the module N is torsion-free and R is a domain. Moreover, for all $0 \neq a, b \in R$ there is $0 \neq d \in aR$ such that $Rd \subseteq aR$, so $0 \neq bd \in aR$, and R satisfies the right Ore Condition. Denote by K the right quotient ring of R. Since N is torsion-free, [C, 0.6.1.] shows there is a module embedding $N \to N \otimes_R K$. Moreover, $N \otimes_R K \cong K^{(\kappa)}$ (as right K-modules) for a cardinal κ. Since N is dually slender, there exists $n < \omega$ and an embedding $N \to K^{(n)}$. In particular, N is countable, a contradiction. ∎

Proposition 15 implies e.g. that each countable commutative ring is steady (cf. [R2]).

There exist countable non-noetherian rings satisfying the assumptions of Proposition 15 and possessing distinct lattices of left, and of right, ideals:

EXAMPLE 16. By [J], Theorem 3.1, we can take R_ω from Example 14 to be a countable ring. Replacing K by R_ω in the construction of

Example 8, and proceeding as in that construction, we obtain a countable ring R. Since each element of R is a piecewise constant function from ω to R_ω, each right ideal of R is two-sided. On the other hand, the lattice of all left ideals of R_ω embeds in that of R. A similar argument to the one of Example 8 shows that R has no noetherian right ideals.

There is a construction of commutative semiartinian steady rings of arbitrary non-limit Loewy length [EGT, Theorem 2.5]. The construction is a special case of the following one, for $\mathfrak{F}$ = the Fréchet filter on κ.

DEFINITION 17. Let κ be a cardinal, $\mathfrak{F}$ be a filter containing the Fréchet filter on κ and let S and R_α, $\alpha < \kappa$ be rings such that S is subring of all R_α for $\alpha < \kappa$. Define $R(\mathfrak{F})$ as the subring of $P = \prod_{\alpha<\kappa} R_\alpha$ generated by the set $\{r \in P; (\exists F \in \mathfrak{F})\, r(F) = 0\} \cup \{r \in P; (\exists s \in S)(\forall \alpha < \kappa)\, r(\alpha) = s\}$.

We show that the Fréchet case is the only one for which this construction yields a steady ring:

PROPOSITION 18. *The ring $R(\mathfrak{F})$ is right steady if and only if $\mathfrak{F}$ is the Fréchet filter and all the rings S, R_α, $\alpha < \kappa$ are right steady.*

PROOF The reverse implication follows from [EGT, Proposition 2.3].

Assume $\mathfrak{F}$ is not the Fréchet filter on κ. So there is a set $G \in \mathfrak{F}$ such that $\kappa \setminus G$ is not finite. Define an ideal $I = \{r \in R(\mathfrak{F}); (\forall \alpha \in (\kappa \setminus G)) \colon r(\alpha) = 0\}$. Denote by $\pi \colon \prod_{\alpha<\kappa} R_\alpha \to \prod_{\alpha \in (\kappa \setminus G)} R_\alpha$ the natural projection. Define a homomorphism $\psi \colon R(\mathfrak{F})/I \to \prod_{\alpha \in (\kappa\setminus G)} R_\alpha$ by $\psi(r + I) = \pi(r)$. It is easy to prove that ψ is a ring isomorphism. By [T2, Theorem 2.5], $R(\mathfrak{F})$ is not steady.

This proves that $\mathfrak{F}$ is the Fréchet filter on κ. Then each of the rings S, R_α, $\alpha < \kappa$, is a factor-ring of $R(\mathfrak{F})$, and hence is right steady. ■

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