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A Note on Factorized (Finite) p-Groups.

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Dedicated to Prof. Zacher, on his 70-th birthday

A group G is factorized if it has the form $G = AB$, where A and B are two proper subgroups of G. A lot of work has been done to investigate how the structure of A and B influences the structure of G, and in this direction there is a famous theorem by Ito which states that if A and B are abelian then G is metabelian [6, p. 384]. Moreover, Kegel and Wielandt proved that, in the case of finite groups, if A and B are nilpotent then G is soluble [6, pp. 379-382].

From these results originates the following:

CONJECTURE. *If A and B are finite nilpotent groups of classes α and β respectively, the derived length of G is bounded by a function of α and β, perhaps $\alpha + \beta$.*

Elisabeth Pennington proved the conjecture when A and B have coprime orders, with the function given by $\alpha + \beta$, and showed that the problem of bounding the derived length of G can be reduced to the case of p-groups [5]. But in this critical case no general result is known.

A very special situation is examined by Mc Cann, who proved that if $G = AB$ is a finite p-group, with A abelian and B extraspecial, then the derived length of G is at most 3 [2].

In this note we are able to be a little more general, with the following:

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THEOREM 2. *Let $G = AB$ be a finite p-group, where A is abelian and B' has order p. Then one of the following must occur:*

i) *$A^G B'$ is the product of two abelian groups.*

ii) *B'^G is abelian.*

In both cases, the derived length of G is at most 3.

Examples of groups satisfying the assumptions and having derived length 3 are easily constructed; for instance it is enough to consider the wreath product of a group of order p with a non-abelian group of order p^3.

Theorem 2 can be obtained by specializing a more general proposition, which bounds the derived length of a factorized finite p-group $G = AB$, where A is abelian, in terms of the order of B', the derived subgroup of B. More precisely, we have:

THEOREM 1. *Let $G = AB$ be a finite p-group, where A is abelian and $|B'| = p^n$, then the derived length of G is at most $n + 2$.*

A theorem of the same flavour was obtained by Kazarin [4], who proved that if $G = AB$ is a finite p-group, such that $|A'| = p^m$ and $|B'| = p^n$, then the derived length of G is at most $2m + 2n + 2$. Combining this with Theorem 1 it is also possible to prove:

THEOREM 3. *Let $G = AB$ be a finite p-group, such that $|A'| = p^m$ and $|B'| = p^n$, with $m \geq n$, then the derived length of G is at most $m + 2n + 2$.*

In the proof of these thorems the hypothesis that the groups considered are finite plays an essential role. Nevertheless, after having proved Theorem 1 in the finite case, it is possible to prove it for infinite groups as well. More precisely we have:

THEOREM 2′. *Let $G = AB$ be a group, where A, B are p-groups such that A is abelian and $|B'| = p^n$, then G is soluble and has derived length at most $n + 2$.*

Here the notation used is standard and in particular the symbol H^G denotes the normal closure of a subgroup H in a group G, i.e. the smallest normal subgroup of G containing H. Moreover, $Z(G)$ denotes the center of the group G, G' is the derived subgroup of G, $\langle g \rangle$ is the subgroup generated by $g \in G$ and $[x, y] = x^{-1}y^{-1}xy$ is the usual commutator, for $x, y \in G$.

To prove the theorem, two results are needed which are probably known but the author could not find them in the literature stated in the following form.

LEMMA 1. *Let $G = AB$ be a group, with A abelian and $Z(G) \neq 1$; then there exists a non-trivial normal subgroup N of G such that $N \leqslant A$ or $N \leqslant B$.*

PROOF. Consider $1 \neq ab \in Z(G)$, with $a \in A$ and $b \in B$. We may assume $a \neq 1 \neq b$. Then for each $x \in A$ we have $1 = [ab, x] = [a, x]^b [b, x] = [b, x]$, so $[b, A] = 1$.

So $\langle b \rangle^G = \langle b \rangle^{AB} = \langle b \rangle^B \leqslant B$ is a normal subgroup of G contained in B, as we wanted.

Note that Lemma 1 is no longer true if we drop the hypothesis that A is abelian, even for finite p-groups, as an example of Gillam shows [3].

LEMMA. 2. *Let $G = AB$ be a group, $V \leqslant Z(A)$, $W \leqslant Z(B)$ such that $AW = WA$ and $VB = BV$ are subgroups; then $[V, W] \leqslant Z([A, B])$.*

PROOF The proof is the same as Ito's (see [5]).

Let $a \in A$, $b \in B$, $a_1 \in V$, $b_1 \in W$, and put $b^{a_1} = a_2 b_2$, $a^{b_1} = b_3 a_3$, where $a_1 \in V$ and $b_3 \in W$. Then:

$$[a, b]^{a_1 b_1} = [a, b^{a_1}]^{b_1} = [a, a_2 b_2]^{b_1} =$$

$$= [a, b_2]^{b_1} = [a^{b_1}, b_2] = [b_3 a_3, b_2] = [a_3, b_2],$$

$$[a, b]^{b_1 a_1} = [a^{b_1}, b]^{a_1} = [b_3 a_3, b]^{a_1} =$$

$$= [a_3, b]^{a_1} = [a_3, b^{a_1}] = [a_3, a_2 b_2] = [a_3, b_2].$$

The statement now follows immediately.

Another Lemma is needed, whose proof is trivial:

LEMMA 3. *Let K be a p-group such that K' has order p^k, then the derived length of K is at most $k + 1$.*

PROOF OF THEOREM 1. The proof is by induction on n, where $p^n = |B'|$.

For $n = 0$ the theorem is true by Ito's result.

We now assume that it is true for each natural number less or equal than $n - 1$.

If $A \cap B \nleqslant Z(B)$ then $N = (A \cap B)^G = (A \cap B)^{AB} = (A \cap B)^B$ is a normal subgroup of G contained in B and such that $N \cap B' \neq 1$.

If N is abelian then $d(G) \leqslant n + 2$ because $|(B/N)'| \leqslant p^{n-1}$ and so by induction $d(G/N) \leqslant n + 1$.

If N is not abelian then $N' \trianglelefteq G$ has order p^t, $t \geqslant 1$. Moreover $|(B/N')'| = p^{n-t}$, so by induction $d(G/N') \leqslant n - t + 2$. As $d(N') \leqslant t$ the result follows. So w.m.a. $A \cap B \leqslant Z(B)$.

We now prove the following statement:

There exists $W \leqslant Z(B)$ *such that* $AW = WA$ *and* $1 \neq T = AW \cap B'$.

Choose a maximal subgroup $W \leqslant Z(B)$ such that AW is a subgroup and assume by contradiction that $AW \cap B' = 1$. Put $K = (AW)_G$, the core of AW in G. Then $G/K = (AK/K)(BK/K)$ and AK/K is core-free. By Lemma 1 there exists a minimal normal subgroup N/K of G/K such that $N \leqslant BK$ and so $N = ZK$, where $Z = B \cap N$ is not contained in K. Of course, $N/K \leqslant Z(G/K)$, as the groups considered are finite and nilpotent. $[Z, B] \leqslant K \cap B' \leqslant AW \cap B' = 1$, so $Z \leqslant Z(B)$. It follows that $AWZ = AWN$ is a subgroup, as $N \trianglelefteq G$. Moreover $Z \nleqslant W$, otherwise we would have $ZK \leqslant (AW)_G = K$. As $WZ \leqslant Z(B)$, this contradicts the maximality of W, and we have proved the statement.

We have $T \leqslant AW \cap B = (A \cap B)W \leqslant Z(B)$, and moreover $[A, T] \leqslant$ $\leqslant [A, AW] = [A, W] \leqslant Z([A, B])$ by Lemma 2.

Consider $A^G = A(A^G \cap B)$. If $T \leqslant (A^G \cap B)'$, then $T \leqslant [A, B]$ because $A[A, B]/[A, B] \cong A/(A \cap [A, B])$ is abelian. It follows that $T^G =$ $= T[A, T]$ is abelian. By induction $d(G/T^G) \leqslant n + 1$, so $d(G) \leqslant n + 2$

Now assume $T \nleqslant (A^G \cap B)'$ and consider $A^G T$, which is normal in G as $T \leqslant Z(B)$.

We have $A^G T = A(A^G T \cap B) = A(A^G \cap B)T$, so $((A^G \cap B)T)' =$ $= (A^G \cap B)'$, as $T \leqslant Z(B)$. Let $p^m = |((A^G \cap B)T)'|$. We have $|A^G T \cap$ $\cap B'| = p^{m+t}$, where $t \geqslant 1$ because $T \leqslant A^G T \cap B'$ and $T \nleqslant ((A^G \cap$ $\cap B)T)'$. By induction $d(A^G T) \leqslant m + 2$. The derived subgroup of $(G/A^G T) \cong (B/(B \cap A^G T))$ has order $p^{n-(m+t)}$, so $d(G/A^G T) \leqslant n -$ $- (m + t) + 1$ by Lemma 3. It follows that $d(G) \leqslant n + 2 - t + 1 \leqslant n + 2$ and the proof is complete.

PROOF OF THEOREM 2. It is enough to go through the proof of Theorem 1 once more, considering $n = 1$ and keeping in mind that the subgroup T considered is actually B'.

PROOF OF THEOREM 3. The proof is by induction on $n + m$ and follows exactly the same steps as Kazarin's proof [4].

PROOF OF THEOREM 2′. Note that by Corollary 7.3.10 of [1] G there is a soluble normal subgroup M of G such that G/M is finite. As AM/M and BM/M are finite p-groups, $G/M = (AM/M)(BM/M)$ is nilpotent, thus G is soluble. From this follows that G is locally finite. Let $g \in G^{(n+2)}$, the $(n+2)$-th term of the derived series of G. There exists a finite set $X = \{a_i, b_j \mid a_i \in A, b_j \in B, i = 1, \ldots, r; j = 1, \ldots, s\}$ such that $g \in \langle X \rangle^{(n+2)}$.

Let $A_0 = \langle a_i \mid i = 1, \ldots, r \rangle$, $B_0 = \langle B', b_j \mid j = 1, \ldots, s \rangle$. Then A_0 and B_0 are finite and $\langle X \rangle \leq \langle A_0, B_0 \rangle$. By Lemma 1.2.3 of [1] there exists a finite subgroup H of G such that $\langle A_0, B_0 \rangle \leq H$ and $H = (H \cap A)(H \cap B)$. Applying Theorem 2 to H we obtain that $H^{(n+2)} = 1$ and thus $G^{(n+2)} = 1$, as g was an arbitrary element of $G^{(n+2)}$.

The proof is now complete.

REFERENCES

[1] B. Amberg - S. Franciosi - F. de Giovanni, *Products of Groups*, Oxford Mathematical Monographs, Clarendon Press, Oxford (1992).

[2] B. Mc Cann, *A note on the derived length of products of certain p-groups*, Proc. R. Ir. Acad., **93A**, No 2 (1993), pp. 185-187.

[3] J. D. Gillam, *A finite p-group $P = AB$ with core(A) = core(B) = $\{1\}$*, Rocky Mountain J. Math., **3**, No 1 (1973), pp. 15-17.

[4] L. S. Kazarin, *The product of two nilpotent groups* (in Russian), Problems in group theory and homological algebra, Yaroslavl Gos. Univ., Yaroslavl (1982), pp. 47-49.

[5] E. A. Pennington, *On products of finite nilpotent groups*, Math. Z., **134** (1973), pp. 81-83.

[6] W. R. Scott, *Group Theory*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey (1964).

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