HOWARD SMITH
JAMES WIEGOLD

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Groups which are Isomorphic to their Nonabelian Subgroups.

HOWARD SMITH (*) - JAMES WIEGOLD (**)

ABSTRACT - Let $\mathcal{X}$ denote the class of groups G which are isomorphic to all of their nonabelian subgroups and which are such that not all proper subgroups are abelian. It is shown that every insoluble $\mathcal{X}$-group is centre-by-simple. A complete characterisation is given of the soluble groups in $\mathcal{X}$.

1. - Introduction.

Let $\mathcal{Y}$ denote the class of nonabelian groups G in which all proper subgroups are abelian. Finite $\mathcal{Y}$-groups were classified by Miller and Moreno in [4]. Infinite $\mathcal{Y}$-groups do exist, and we refer the reader to [6] for results on such groups. However, an infinite locally graded $\mathcal{Y}$-group is abelian [1; Corollary 2]—recall that a group G is locally graded if every nontrivial finitely generated subgroup of G has a nontrivial finite image. In this paper, we consider the related class $\mathcal{X}$ of groups G which contain proper nonabelian subgroups, all of which are isomorphic to G. Clearly, every $\mathcal{X}$-group is infinite and 2-generator. There is a satisfactory classification of soluble groups in the class $\mathcal{X}$ and, although we do not know whether there exist insoluble $\mathcal{X}$-groups, nevertheless we are able to show that such groups would have to be very restricted in structure. Our main results are as follows.

(*) Indirizzo dell'A.: Department of Mathematics, Bucknell University, Lewisburg, PA 17837, U.S.A.

(**) Indirizzo dell'A.: School of Mathematics, University of Wales College of Cardiff, Cardiff CF2 4YH, Wales, U.K..

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THEOREM 1. *Let G be an insoluble $\mathfrak{X}$-group, and let Z denote the centre of G. Then G is 2-generator and G/Z is infinite simple. Moreover, Z is contained in every nonabelian subgroup of G.*

THEOREM 2. *Let G be a soluble group.*

(*a*) *If $G \in \mathfrak{X}$ then G contains an abelian normal subgroup of prime index.*

(*b*) *If G is nilpotent then $G \in \mathfrak{X}$ if and only if G is isomorphic to a group having one of the following presentations* (*where* «nil-2» *denotes the pair of relations* $[a, b, b] = 1$, $[a, b, a] = 1$, *p is an arbitrary prime and k is an arbitrary positive integer*).

(i) $\langle a, b \mid \text{nil-2},\ [a, b]^p = 1\rangle$,

(ii) $\langle a, b \mid \text{nil-2},\ [a, b]^p = b^{p^k} = 1\rangle$,

(iii) $\langle a, b \mid \text{nil-2},\ [a, b]^2 = 1,\ b^{2^k} = [a, b]\rangle$,

(iv) $\langle a, b \mid \text{nil-2},\ [a, b]^3 = 1,\ b^{3^k} = [a, b]\rangle$.

(*c*) *If G is not nilpotent then $G \in \mathfrak{X}$ if and only if* either

(v) $G = \langle A, x\rangle$, *where A is a finite elementary abelian p-subgroup of order p^n which is minimal normal in G, x is of infinite order and has order $q \bmod Z(G)$, where p, q are distint primes, and for each k in the interval $1 \leq k \leq q-1$, $\bar{x}$ is conjugate to $\bar{x}^k$ or $\bar{x}^{-k}$ in $GL(n, p)$, where now $\bar{x}$ denotes the image of x under the natural map from $\langle x\rangle$ to $GL(n, p)$*; or

(vi) *G has a normal abelian subgroup $B = A \times \langle b\rangle$, where $A = \langle a_1\rangle \times \ldots \times \langle a_{p-1}\rangle$ is free abelian of rank $p-1$ and normal in G, b is of infinite order or of order p^k (for some nonnegative integer k) and is central in G, and $G = A]\langle x\rangle$ for some x, where $x^p = b$, $a_i^x = a_{i+1}$ for $i = 1, \ldots, p-2$ and $a_{p-1}^x = (a_1 \ldots a_{p-1})^{-1}$, where p is a prime at most* 19.

As may be seen from Theorem 1, the class of soluble $\mathfrak{X}$-groups is precisely that of locally graded $\mathfrak{X}$-groups—the factor group of a finitely generated locally graded group by its centre cannot be infinite simple (see [8]). Note that, in the case where $k = 0$, the group G described in part (vi) of Theorem 2 is precisely the central factor group of the wreath product $\mathbb{Z} \operatorname{wr} C_p$. The proof of Theorem 2 requires the following result on wreath products, which is of independent interest. Its proof, in turn, depends on a substantial result from Number Theory, and we are grateful to L. G. Kovács for pointing out the connection. We are grateful also to M. W.Liebeck for the observation that there do indeed exist pairs of primes p, q for which the conditions in (v) hold, provided that either $n+1$ is an odd prime or n is odd and $2n+1$ is prime: if q is the prime $n + 1$ (respectively $2n + 1$) then there exists a prime p of order n modulo q, and the pair (p, q) can be shown to satisfy the extra hypothesis on conjugates.

THEOREM 3. *Let p be a prime and let G be the central factor group of the wreath product of an infinite cyclic group by a group of order p. Then every normal subgroup of G contained in the image of the base group is the normal closure of a single element if and only if p is at most* 19.

2. - Proof.

We begin with a very easy result.

LEMMA 1. *Suppose that $G \in \mathfrak{X}$ and G is abelian-by-finite. Then G is metabelian.*

PROOF. If G is centre-by-finite then G' is finite and therefore abelian. Otherwise, there exists a noncentral normal abelian subgroup A and then, for some $g \in G$, we have $\langle A, g \rangle$ nonabelian and therefore isomorphic to G. The result follows.

PROOF OF THEOREM 1. Let G and Z be as stated, and let A denote the Hirsch-Plotkin radical of G. Certainly G is not locally nilpotent, and so A is abelian. In fact $A = Z$, otherwise $G \cong \langle A, g \rangle$ for some $g \in G$, giving the contradiction that G is soluble. By Lemma 1, G/Z is infinite. Suppose, for a contradiction, that there exists a normal subgroup N of G such that $Z < N < G$. For some $g \in G \backslash N$ we have $\langle N, g \rangle$ nonabelian and hence isomorphic to G, and so G has a nontrivial finite image. It follows that G is locally graded and hence, by Lemma 1 of [8], that G/Z is locally graded. Now Z is also the Hirsch-Plotkin radical of N and, since $N \cong G$, we deduce that $N/Z \cong G/Z$, that is, G/Z is isomorphic to all of its nontrivial normal subgroups. Since G/Z has a nontrivial finite image, we may apply the main result of [3] to obtain the contradiction that G/Z is cyclic. Thus G/Z is simple and $G'Z = G$, and so $G' = G''$. But $G \cong G'$ and so G is perfect. Thus if H is any nonabelian subgroup of G then we have $HZ = (HZ)' = H' = H$, and the proof of Theorem 1 is complete.

We now consider the nilpotent case. It is convenient to state our conclusion here in the form of a lemma.

LEMMA 2. *Let G be a nilpotent group. Then $G \in \mathfrak{X}$ if and only if G has one of the presentations* (i)-(iv) *of Theorem* 2.

PROOF. First assume that G is a nilpotent $\mathfrak{X}$-group. Certainly G has class exactly 2 and is generated by two elements a and b, say. Suppose that $[a, b]$ has infinite order; then G is free nil-2 and $G' = Z(G)$. For each $n > 1$, set $H_n = \langle a^n, b, [a, b] \rangle$. Then $H_n' = \langle [a, b]^n \rangle$ and $Z(H_n) = \langle [a, b] \rangle$, so $Z(H_n)/H_n'$ is cyclic of order n, and we even have that G contains infinitely many pairwise nonisomorphic nonabelian subgroups. By this contradiction, $[a, b]$ has finite order m, say. If $m = rs$ for some $r, s > 1$, then the subgroup $H = \langle a^r, b \rangle$ has its commutator subgroup of order $s \neq |G'|$, another contradiction. So $|G'| = p$, a prime, and there are just the following cases to consider.

Case 1. G/G' is free abelian—so G has the presentation (i).

Case 2. $G/G' = \langle aG' \rangle \times \langle bG' \rangle$, where $|aG'|$ is infinite $|bG'|$ is finite. Suppose here that bG' has order $p^k l$, where $(p, l) = 1$, and set $K = \langle a, b^l \rangle$. Then $K' = G'$ and b^l has order $p^k \bmod G'$, and $K \cong G$ implies $l = 1$. Thus b has order $p^k \bmod G'$, for some positive integer k, and either $b^{p^k} = 1$, in which case we have the presentation (ii), or $b^{p^k} = [a, b]^s$, for some integer s prime to p, and we now assume that this relation holds. Let H be a nonabelian subgroup of G. Then $H' = G'$ and $H/H' = \langle bH' \rangle \times \langle a^\alpha H' \rangle$ for some (arbitrary) integer α prime to p. Thus $H = \langle a^\alpha, b \rangle$. Suppose that θ: $G \to H$ is an isomorphism. Then $a\theta = a^{\alpha\varepsilon} b^r$, $b\theta = b^t$, where $\varepsilon = \pm 1$ and r, t are integers, with $(t, p) = 1$. Now we have $[a, b]^{st} = b^{p^k t} = (b^{p^k})\theta = [a^{\alpha\varepsilon} b^r, b^t]^s = [a, b]^{\alpha\varepsilon st}$, and so p divides $st(\alpha\varepsilon - 1)$.

Thus $\alpha\varepsilon \equiv 1 \bmod p$, that is, $\alpha \equiv \pm 1 \bmod p$. But this must hold for all α prime to p, and so $p = 2$ or 3, and we have the presentations (iii), (iv)—note that replacing a by a^{-1} allows us to assume that $s = 1$ in the case where $p = 3$.

It remains only to show that a group G having one of the presentations (i)-(iv) is an $\mathfrak{X}$-group. We shall retain the appropriate notation at each stage.

In case (i), an arbitrary nonabelian subgroup H of G satisfies $H' = G'$ and H/H' free abelian, and so $H \cong G$. In case (ii), every nonabelian subgroup H is of the form $\langle a^\alpha, b \rangle$, where $(p, \alpha) = 1$, and the map $a \to a^\alpha$, $b \to b$ extends to an isomorphism from G to H. Each nonabelian subgroup H is also of this type in the remaining two cases. The map $a \to a^{\alpha\delta}$, $b \to b$, with $\delta = 1$ if $\alpha \equiv 1 \bmod p$ and $\delta = -1$ if $\alpha \equiv -1 \bmod p$ (where $p = 2$ or 3) again extends to an isomorphism θ from G to H, as the following calculations show: $(b\theta)^{p^k} = [a\theta, b\theta]^s$ iff $b^{p^k} = [a^{\alpha\varepsilon}, b]^s$ iff

$[a, b]^s = [a, b]^{\alpha\varepsilon s}$, which is true and so all relations are satisfied, and θ extends to a homomorphism onto H. Also, $(a^m b^n)\theta = 1$ implies $a^{\alpha\varepsilon m} b^n = 1$, which implies that p divides n and $m = 0$ and so θ is injective. The lemma is thus proved.

Next, we deal with the centre-by-finite $\mathfrak{X}$-groups. Again it is convenient to isolate this part of the argument.

LEMMA 3. *Let G be a nonnilpotent, centre-by-finite $\mathfrak{X}$-group. Then G has the structure described in part* (v) *of Theorem* 2.

PROOF. As in the proof of Lemma 1, G' is finite and therefore abelian. Since G' is not central it has a noncentral Sylow p-subgroup, and we may write $G = A]\langle x\rangle$, where A is a finite normal abelian p-subgroup of G and x has infinite order. Now $G' = [A, \langle x\rangle]$ and so $[a, x, x] \neq 1$ for some element a of A, and we have $\langle [a, x], x\rangle \cong G$. But $[a, x]^p = [a^p, x] = 1$, since $\langle A^p, x\rangle$ is certainly not isomorphic to G. It follows that A has exponent p. Suppose that x has order $n \bmod Z(G)$. If $n = rs$, where $r, s > 1$, then $\langle A, x^r\rangle$ is not abelian and is therefore isomorphic to G. But this easily gives a contradiction, and so $n = q$, a prime. Certainly $q \neq p$, since G is not nilpotent. Further, if A contains a proper nontrivial G-invariant subgroup B then, by Maschke's Theorem, we have $A = B \times C$, where C is also nontrivial and G-invariant. Now either $\langle B, x\rangle$ or $\langle C, x\rangle$ is isomorphic to G, another contradiction. Finally, if q does not divide k then $\langle A, x^k\rangle$ is isomorphic to G and so x^k acts like $x^{\pm 1}$ on A and the conjugacy condition follows.

Now suppose that G is a group having the structure indicated, and let H be a nonabelian subgroup of G. Then H contains a nontrivial element b of A and an element of the form $g = ux^\lambda$, where $u \in A$ and $\lambda \not\equiv 0 \bmod q$. Since A is minimal normal we have $\langle b\rangle^{\langle g\rangle} = A$, and so H is normal in G and $H = \langle A, x^\mu\rangle$, for some μ which is not a multiple of q. Clearly then $H \cong G$, and the result follows.

We now proceed to the proof of part (a) of Theorem 2.

LEMMA 4. *Let G be a soluble $\mathfrak{X}$-group. Then G has a normal abelian subgroup of prime index.*

PROOF. By Lemma 2, a nilpotent $\mathfrak{X}$-group G has a normal abelian subgroup, namely $\langle a^p, b\rangle G'$ in the notation there employed, of prime index p. Assuming G is not nilpotent, we see that the Hirsch-Plotkin radical A of G is abelian and self-centralizing, and it is clear that if G/A is finite then it is of prime order. Thus we may assume that G is not abelian-by-finite, and hence that $G = A]\langle x\rangle$ for some element x of infinite order.

Suppose that A contains a noncentral element a of finite order. Then $H = \langle a, x\rangle$ is isomorphic to G. Now $H \cap A = \langle a\rangle^{\langle x\rangle}$ is the Hirsch-Plotkin radical of H, else $\langle a, x^n\rangle$ is locally nilpotent and hence abelian for some $n > 0$, giving the contradiction that H is abelian-by-finite. Thus $A \cong H \cap A$ and A has finite exponent. We may now argue as in the proof of Lemma 3 to deduce that A has prime exponent n, say. If A is finite then G is centre-by-finite, a contradiction. It follows that G is (isomorphic to) the wreath product $\langle a\rangle \mathrm{wr} \langle x\rangle$. But the nonabelian subgroup $A\langle x^n\rangle$ is not even 2-generator. By this contradiction, the torsion subgroup T of A is central in G. If G/T is abelian-by-finite then G is nilpotent-by-finite, another contradiction. Let H/T be a nonabelian subgroup of G/T. Then of course H is nonabelian and T is the maximal torsion subgroup of H, and so $H/T \cong G/T$ and G/T belongs to $\mathfrak{X}$.

Factoring by T if necessary, we may assume that G is torsionfree. Let Z denote the centre and C the second centre of G. If $C > Z$ then $\langle C, x\rangle$ is nonabelian and therefore isomorphic to G, giving the contradiction that G is nilpotent. So Z is the hypercentre of G; also A/Z is torsionfree. Let $D = C_A(x^n)$, where n is some positive integer. Then D is normal in G and $\langle D, x\rangle$ is abelian-by-finite and hence abelian, giving $D = Z$. Suppose that H/Z is a nonabelian subgroup of G/Z. Certainly H is nonabelian, and so it contains an element of the form ax^n, for some positive integer n and element a of A. Also, of course, $H \cap A \neq 1$, and it follows that $C_G(H \cap A) = A$ and hence that $Z(H) = Z$. Thus $H/Z \cong G/Z$, and we have $G/Z \in \mathfrak{X}$. Factoring as before, we may assume that $Z = 1$ and hence that $C_A(x^n) = 1$ for all positive integers n. We claim that G has finite rank; assuming this to be false, we have from [2] that G contains a section L/K which is isomorphic to $C_p \mathrm{wr}\, C_\infty$ for some prime p. Since L is isomorphic to G, we may as well write $G = L$. Let B/K denote the base group of G/K. Then B is not isomorphic to G, since it is not finitely generated. Hence B is abelian and, since G/B is infinite cyclic, we see that $B = A$. But then $\langle A, x^p\rangle$ is not 2-generator (mod K) and we have the contradiction that $\langle A, x^p\rangle$ is abelian. This establishes the claim.

Next, suppose that A is finitely generated, of rank r, say, and let $\{a_1, \ldots, a_r\}$ be a $\mathbb{Z}$-basis for A. Relative to this basis, the action of x on A may be represented by an invertible $r \times r$ matrix X with integer entries. Set $H = \langle A, x^2\rangle$, and let θ be an isomorphism from G to H. Since A is the Hirsch-Plotkin radical of both G and H, it is fixed by θ, which is therefore determined by some assignment $a_i \to b_i$ $(i = 1, \ldots, r)$, $x \to ax^{\pm 2}$, where $\{b_1, \ldots, b_r\}$ is also a basis for A and a is some element of A. By taking the composite with the isomorphism $b_i \to b_i$ $(\forall i)$, $ax^{\pm 2} \to$

$\to x^{\pm 2}$, we may assume that $a_i \to b_i$, $x \to x^{\pm 2}$. Suppose M represents the change of basis $\{a_1, \ldots, a_r\} \to \{b_1, \ldots, b_r\}$; then θ restricted to A is represented by M and, since θ is an isomorphism, we have $MX^{\pm 2} = XM$, or $M^{-1}XM = X^{\pm 2}$. Consider the subgroup $K = \langle X, M \rangle$ of $GL(r, \mathbb{Z})$; this is a homomorphic image of either $U = \langle a, b \mid a^b = a^2 \rangle$ or $V = \langle a, b \mid a^b = a^{-2} \rangle$ via the assignment $a \to X$, $b \to M$. Now each of U and V is an extension of the dyadic rationals by the infinite cyclic group $\langle b \rangle$ and so K is soluble and hence polycyclic (see Chapter 2 of [7], for example). But, in every polycyclic image of U or of V, the image of the subgroup $\langle a \rangle^{\langle b \rangle}$ is finite, and so X has finite order n, say. This gives the contradiction that $[A, x^n] = 1$. Thus A is not finitely generated and, since $C_A(x) = 1$, we see that A contains no nontrivial finitely generated G-invariant subgroups. Since A has rank r, there exists an r-generator subgroup A_0 of A such that A/A_0 is periodic. Further, since $\langle A_0, x \rangle$ is not abelian, we may assume $A_0^{\langle x \rangle} = A$. Write $A_1 = A_0 A_0^x A_0^{x^{-1}}$. Then $|A_1 : A_0| = m$, where m is an integer greater than 1, since A_0 is not normal in G. Setting $A_2 = A_1 A_1^x A_1^{x^{-1}}$, we note that each of the indices $|A_1^x : A_0^x|$ and $|A_1^{x^{-1}} : A_0^{x^{-1}}|$ is also m and hence that $|A_2 : A_0|$ divides m^3. We deduce that A/A_0 is a π-group for some finite set π of primes, namely those dividing m. Since A has finite rank but is not finitely generated, it has a subgroup A^* such that $A/A^* \cong C_{p^\infty}$ for some prime q. Let $\mathcal{L}$ be the set of all subgroups L of A such that $A/L \cong C_{q^\infty}$. It is easy to see that no member L of $\mathcal{L}$ can be isomorphic to A, and it follows from the $\mathfrak{X}$-property that, for each L in $\mathcal{L}$, $L^{\langle x \rangle} = A$. Choose $B \in \mathcal{L}$ such that the index $|BB^xB^{x^{-1}} : B| = q^\beta$, say, is minimal. Since B is not normal in G, we have $\beta > 0$ and thus, for all $L \in \mathcal{L}$, $|LL^{x^2}L^{x^{-2}} : L| > q^\beta$ (here we are using the fact that A/L is locally cyclic). Now let $J = \langle A, x^2 \rangle$.

There exists an isomorphism ϕ from G onto J and, as for our previous isomorphism θ, we may assume that $x\phi = x^{\pm 2}$. Also, $A\phi = A$ and so the set $\mathcal{L}$ is invariant under ϕ. Thus $|BB^xB^{x^{-1}} : B| = |(B\phi)(B\phi)^{x^{\pm 2}}(B\phi)^{x^{\mp 2}} : (B\phi)| > q^\beta$, a contradiction which concludes the proof of Lemma 4.

Proof of Theorem 2. All that remains here is to show that a nonnilpotent group G which has an abelian normal subgroup of prime index, but which is not centre-by-finite, is an $\mathfrak{X}$-group if and only if it is of one of the types described in part (vi) of the theorem. We shall use the result of Theorem 3. Firstly, we make an elementary observation: let $W = \langle u \rangle \mathrm{wr} \langle v \rangle$, where u has infinite order and v has prime order p. Viewing the base group D of W in the natural way as the additive group of the group ring $\mathbb{Z}\langle v \rangle$, we may regard the centre C of W as the ideal of D generated by the element $f(v) = 1 + v + \ldots + v^{p-1}$. Since f is irreducible over

$\mathbb{Z}$, and hence over $\mathbb{Q}$, we have that every W-invariant subgroup of D which properly contains C is of finite index in D.

Now let $G = A]\langle x \rangle$ be as in (vi), and let H be an arbitrary nonabelian subgroup of G. Then $H = \langle H \cap A, ax^r \rangle$, where $a \in A$ and $(p, r) = 1$. We have $(H \cap A)^G = (H \cap A)^{\langle x \rangle} = (H \cap A)^{\langle ax^r \rangle}$, so that $H \cap A$ is normal in G. Applying Theorem 3 to the group $G/\langle b \rangle$, we deduce that $H \cap A$ is the normal closure in G (and hence in H) of a single element b, say. Now $H \cap A$ has finite index in A and therefore has rank $p-1$, while $bb^{(ax^r)} \dots b^{(ax^r)^{p-1}} = 1$, $(ax^r)^p = x^{rp}$, and $H \cap B = (H \cap A) \times \langle x^{rp} \rangle$. It follows easily that the assignment $a_1 \to b$, $x \to ax^r$ determines an isomorphism from G onto H.

Now assume that G is an abelian-by-finite $\mathfrak{X}$-group which is neither nilpotent nor centre-by-finite. We may write $G = B\langle x \rangle$ for some x, where B is abelian and normal of prime index p in G. Then $x^p \in Z = Z(G)$. Consider first the case where $x^p = 1$. There is a positive integer k such that B^k is torsionfree and normal in G. We have $G \cong \langle B^k, x \rangle$ and so we may write $G = A]\langle x \rangle$, where A is torsionfree. We claim that $Z = 1$. Clearly $Z \leq A$, and if $[A, \langle x \rangle] \leq Z$ then we have the contradiction that G is nilpotent. Thus $\langle [A, \langle x \rangle], x \rangle$ is isomorphic to G, and it follows that the rank of $[A, \langle x \rangle]$ is the same as that of A. If $Z \neq 1$ there must be a nontrivial element z in $[A, \langle x \rangle] \cap Z$. It is easy to see that z must be of the form $[a, x]$, for some a in A; but then $G \cong \langle a, x \rangle$, which is nilpotent, and we have a contradiction which establishes the claim.

Now, for arbitrary nontrivial a in A, we have $aa^x \dots a^{x^{p-1}}$ central in G and hence trivial, and $\langle a, x \rangle$ is isomorphic to G. We may assume that $G = \langle a, x \rangle$, for some a in A. It follows that G is a homomorphic image of the central factor group of $\mathbb{Z} \operatorname{wr} C_p$ and is therefore isomorphic to this central factor group (since G is infinite and nonabelian). Let N be an arbitrary nontrivial G-invariant subgroup of A, and let $H = \langle N, x \rangle$. Then $H \cong G$ and it follows that $N = \operatorname{Fitt} H$ is isomorphic to $A = \operatorname{Fitt} G$ and thus that N is the normal closure in H of a single element b of A. But H contains some element cx, where $c \in A$, so that $N = \langle b \rangle^G$. Since N was arbitrary, Theorem 3 tells us that p is at most 19.

Next, consider the more general case where x has finite order $p^r l$, say, where $(p, l) = 1$. As before, we may write $G = A]\langle x \rangle$, where A is torsionfree abelian. Since $x^p \in Z$, $\langle A, x^l \rangle$ is nonabelian and therefore isomorphic to G. It follows that $l = 1$ and x has order p^r. Now $\langle x^p \rangle$ is the torsion subgroup of Z and of the centre of every nonabelian subgroup H containing x^p. It follows easily that $G/\langle x^p \rangle \in \mathfrak{X}$. If $G/\langle x^p \rangle$ is nilpotent then so is G, a contradiction. Also, if $G/\langle x^p \rangle$ is centre-by-finite then G' is finite and G is centre-by-finite, another contradiction. As for the first case, $G/\langle x^p \rangle$ has trivial centre and so G has the structure indicated in the theorem.

Suppose then that x has infinite order. Again we write $G = A\langle x\rangle$, where this time A is torsionfree abelian and x has order $p^r \bmod A$. As above, $[A, \langle x\rangle] \cap Z = 1$ and $G \cong \langle [A, \langle x\rangle], x\rangle$, and so we may write $G = D\,]\langle x\rangle$, where D is torsionfree abelian and $D \cap Z = 1$.

Thus $\langle x^p\rangle = Z$, and we deduce that $G/\langle x^p\rangle \in \mathfrak{X}$. If d is any nontrivial element of D then $[d, x] \notin Z$, else $\langle d, x\rangle$ is nilpotent and not abelian, and hence isomorphic to G. It follows once more that $G/\langle x^p\rangle$ has trivial centre, and the previous argument now shows that G is of the form specified.

PROOF OF THEOREM 3. Let W be the wreath product $\mathbb{Z}\,\mathrm{wr}\langle x\rangle$, where x has prime order p, and let Z and B respectively denote the centre and base group of W.

If $p = 2$ then W/Z is the infinite dihedral group, which clearly belongs to $\mathfrak{X}$. Assume, then, that p is odd, and let ω be a primitive p-th root of unity. The ring R of integers of $\mathbb{Q}(\omega)$ is just $\mathbb{Z}[\omega]$ (see Theorem 3.5 of [9]), and there is a natural correspondence between B/Z and R. (Recall that Z may be viewed as the ideal of $\mathbb{Z}\langle x\rangle$ generated by a cyclotomic polynomial $f(x)$.) Under this correspondence, W-invariant subgroups of B/Z are associated with ideals of R. It is routine to show that W/Z belongs to $\mathfrak{X}$ if and only if every W-invariant subgroup of B/Z is the normal closure of a single element—the details here are similar to those which appear in the related part of the proof of Theorem 2. Now R is a principal ideal domain if and only if $\mathbb{Q}(\omega)$ has class number 1 (see Chapter 9 (in particular Proposition 9.8) of [9]). But, by a result of Uchida and Montgomery, this is true if and only if $p \leqslant 19$—for further details and appropriate references the reader may consult the survey article by Masley in [5]. Theorem 3 is thus proved.

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