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Minimal Non-Totally Minimal Topological Rings.

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ABSTRACT - We establish the existence of minimal non-totally minimal topological rings with a unit answering a question of Dikranjan. The Pontryagin duality and a generalization of Ursul's «semidirect product type» construction play major roles in the construction.

Introduction.

A Hausdorff topological ring R is called *minimal* if its topology is minimal in the sense of Zorn among all Hausdorff ring topologies on R. If R/J is minimal for every closed ideal J, then R is called *totally minimal* [2].

The induced topology of a nontrivial valuation on a field is (totally) minimal (see [10, 6]). Some generalizations and related results in the context of fields or divisible rings may be found in [11, 13, 14]. For more general cases we refer to [1, 2, 3, 9]. Recall [2, 3] for instance that the class of all minimal rings with a unit is closed under forming topological products, direct sums and matrix rings. If P is a non-zero prime ideal of finite index in a Dedekind ring, then the P-adic topology is minimal.

The question about existence of minimal non-totally minimal rings with a unit is discussed by Dikranjan in [2, 3].

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Conventions and preliminaries.

As usual $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the set of all natural, integer and real numbers, respectively. The unit circle group $\mathbb{R}/\mathbb{Z}$ will be denoted by $\mathbb{T}$ and the n-element cyclic ring by $\mathbb{Z}_n$.

All rings are assumed to be associative. A ring R is *unital* if it has a unit. The zero-element will be denoted by 0. By $\operatorname{char}(R)$ we indicate the minimal natural number (if it exists) n such that $nx = 0$ for every $x \in R$. Otherwise we write $\operatorname{char}(R) = 0$. Clearly, $\operatorname{char}(R) = n > 0$ iff R is a (left) $\mathbb{Z}_n$-algebra in a natural way $(k, x) \mapsto x + x + \ldots + x$ (k terms) for each $(k, x) \in \mathbb{Z}_n \times R$.

For a locally compact Abelian group G, denote by G^* the dual group $H(G, \mathbb{T})$ of all continuous characters endowed with the compact open topology. If R is a locally compact ring, then R^* is a topological (R, R)-bimodule [12].

If P is a subgroup of a topological group (G, τ), then $\tau|_P$ will denote the relative topology on P, and τ/P will be the quotient topology on the left coset space G/P. The following useful result is well known.

MERZON'S LEMMA [8] (See also [4], Lemma 7.2.3 for a proof). *Let P be a subgroup of a group G, and let τ' and τ be (not necessarily Hausdorff) group topologies on G with the properties: $\tau' \subseteq \tau$, $\tau'|_P = \tau|_P$ and $\tau'/P = \tau/P$. Then $\tau' = \tau$.*

Main results.

Recall a construction from [12]. Let R be a topological ring and X a topological (R, R)-bimodule. On the product $R \times X$ of topological groups R and X, consider the multiplication

$$(r_1, x_1)(r_2, x_2) = (r_1 r_2, r_1 x_2 + x_1 r_2), \qquad r_1, r_2 \in R, \quad x_1, x_2 \in X.$$

Then $R \times X$ becomes a topological ring which is denoted by $R \leftthreetimes X$. For details and a particular case of $\mathbb{R} \leftthreetimes \mathbb{R}^*$ see Ursul [12].

Now we generalize this construction in two directions. The first change is minor. Let K be a commutative unital Hausdorff topological ring, (R, τ) a topological K-algebra, and (S, ν) be a topological K-module. Instead of $R^* = H_{\mathbb{Z}}(R, \mathbb{T})$, consider the K-module $H_K(R, S)$ of all continuous K-homomorphisms $R \to S$. As in the case of R^*, the left and right multiplications in R induce the (R, R)-bimodule structure in $H_K(R, S)$. The second modification is more essential. We add to $R \leftthreetimes H_K(R, S)$ a supplementary coordinate. Denote by $M_K(R, S)$ the product $R \times H_K(R, S) \times S$ of K-modules. The multiplication we define by the rule:

$$(r_1, f_1, s_1)(r_2, f_2, s_2) = ((r_1 r_2, r_1 f_2 + f_1 r_2, f_2(r_1) + f_1(r_2))$$

where $r_1, r_2 \in R$, $f_1, f_2 \in H_K(R, S)$ and $s_1, s_2 \in S$. Simple computations show that $M_K(R, S)$ becomes a K-algebra. Let $H_K(R, S)$ carry a K-module topology σ such that its (R, R)-bimodule structure is topological too. Moreover, suppose that the evaluation mapping

$$\omega: H_K(R, S) \times R \to S, \qquad \omega(f, r) = f(r)$$

is continuous with respect to the triple (σ, τ, ν) of Hausdorff topologies. Then $(M_K(R, S), \gamma)$ is a Hausdorff topological K-algebra with respect to the product topology γ. In particular, if R is a locally compact ring, $S = \mathbb{T}$ and $K = \mathbb{Z}$, then one gets a locally compact topological ring $M_{\mathbb{Z}}(R, \mathbb{T}) = R \times R^* \times \mathbb{T}$ which will be denoted by $M(R)$.

Furthermore, we identify R, $H_K(R, S)$ and $H_K(R, S) \times S$ with the corresponding subsets of $M_K(R, S)$. We will keep below our assumptions about $(M_K(R, S), \gamma)$.

PROPOSITION 1. *Let γ' be a new ring topology on $M_K(R, S)$ such that the canonical group retraction $q: H_K(R, S) \times S \to S$ is continuous for the topologies $\gamma'|_{H_K(R, S) \times S}$ and ν. Then the evaluation mapping ω is continuous with respect to the triple of topologies $\gamma'|_{H_K(R, S)}$, $\gamma'/H_K(R, S) \times S$ and ν.*

PROOF. Fix $\varphi_0 \in H_K(R, S)$, $r_0 \in R$ and a ν-neighborhood O at $\varphi_0(r_0)$ in S. By the continuity of q, we may choose a γ'-neighborhood U of the element $z_0 = (0, \varphi_0 r_0, \varphi_0(r_0)) \in M_K(R, S)$, such that $q(U \cap \cap (H_K(R, S) \times S)) \subseteq O$.

By our assumption, the ring multiplication is γ'-continuous. Therefore, there exist γ'-neighborhoods V, W of the elements $(0, \varphi_0, 0)$ and $(r_0, 0, 0)$ respectively, such that $V \cdot W$ is contained in the chosen γ'-neighborhood U of $z_0 = (0, \varphi_0, 0)(r_0, 0, 0)$.

For every $\varphi \in V \cap H_K(R, S)$ and every $(r, f, s) \in W$, we have

$$(0, \varphi, 0)(r, f, s) = (0, \varphi r, \varphi(r)) \in U \cap (H_K(R, S) \times S).$$

Clearly, $\varphi(r) = \omega(\varphi, r) \in \omega(V \cap H_K(R, S), \operatorname{pr}(W))$, where pr denotes the projection $M_K(R, S) \to R$ on the first coordinate.

Then,

$$\varphi(r) \in \omega(V \cap H_K(R, S), \operatorname{pr}(W)) \subseteq q(U \cap (H_K(R, S) \times S)) \subseteq O.$$

Since $\operatorname{pr}(W)$ is a $\gamma'/H_K(R, S) \times S$-neighborhood of the point r_0 and

$V \cap H_K(R, S)$ is a $\gamma'|_{H_K(R,S)}$-neighborhood of the point φ_0, then the continuity of ω at (φ_0, r_0) is proved. ■

Let $(F, \sigma) \cdot (E, \tau)$, (S, ν) be Abelian Hausdorff groups. A continuous mapping $\omega: F \times E \to S$ is called biadditive if the induced mappings $\omega_x: F \to S$, $\omega_f: E \to S$ are homomorphisms for every $x \in E$ and every $f \in F$. We say that a coarser pair $(\sigma', \tau') \leqslant (\sigma, \tau)$ of group topologies is ω-compatible if ω remains continuous with respect to the triple (σ', τ', ν). If ω is separated (i.e., if the annihilators of E and F are both zero), then the Hausdorff property of ν implies that every ω-compatible pair (σ', τ') is necessarily Hausdorff. Following [7], we say that ω is *minimal* if for every ω-compatible pair $(\sigma', \tau') \leqslant (\sigma, \tau)$, we have necessarily $\sigma' = \sigma$, $\tau' = \tau$.

LEMMA 2 [7, Proposition 1.10]. *For every Hausdorff locally compact Abelian group G, the evaluation mapping* $G^* \times G \to \mathbb{T}$ *is minimal.*

Another example of a minimal biadditive mapping is the canonical duality $E^* \times E \to \mathbb{R}$ for a normed space E.

PROPOSITION 3. *Let the evaluation mapping*

$$\omega: (H_K(R, S), \sigma) \times (R, \tau) \to (S, \nu)$$

be minimal, and let $\gamma' \subseteq \gamma$, *be a coarser Hausdorff ring topology on* $M_K(R, S)$ *such that* γ' *and* γ *coincide on* $H_K(R, S) \times S$. *Then* $\gamma' = \gamma$.

PROOF. Because γ' and γ agree on $H_K(R, S) \times S$, then, in particular, the mapping

$$q: H_K(R, S) \times S \to S$$

is continuous with respect to the pair $(\gamma'|_{H_K(R,S) \times S}, \nu)$. So, we can apply Proposition 1. Then $\gamma'|_{H_K(R,S)}$, $\gamma'/H_K(R, S) \times S$ is a ω-compatible pair of group topologies. The minimality of ω implies $\gamma'/H_K(R, S) \times \times S = \tau = \gamma/H_K(R, S) \times S$. Now Merzon's Lemma finishes the proof. ■

As a corollary we get

PROPOSITION 4. *Let the evaluation mapping* ω *be minimal and let* S *and* $H_K(R, S)$ *be compact. Then* $M_K(R, S)$ *is a minimal ring.*

THEOREM 5. *Let R be a discrete ring. Then the topological ring $M(R) = R \times R^* \times \mathbb{T}$ is minimal. Hence, every (commutative) discrete ring is a continuous ring retract of a minimal (commutative) locally compact ring.*

PROOF. By Pontryagin's Theorem, R^* is compact iff R is discrete. Now the minimality of $M(R)$ follows from Lemma 2 and Proposition 4. The canonical retraction pr: $M(R) \to R$ is the desired one. ■

The ring $M(R)$ from Theorem 5 is not unital. In order to «improve» this, we use a well known unitalization procedure. Let R be a topological K-algebra. Consider a new K-algebra

$$R_+ = \{r + \alpha 1_+ \mid r \in R,\ \alpha \in K\}$$

adjoining a unit 1_+. More precisely, R_+ is a topological K-module sum $R \oplus K$, and we identify $(r, \alpha) = r + \alpha 1_+$. A multiplication on R_+ is defined in the following manner:

$$(a + \alpha 1_+)(b + \beta 1_+) = ab + \alpha b + \beta a + \alpha\beta 1_+$$

where $\alpha, \beta \in K$ and $a, b \in R$. The following lemma is trivial.

LEMMA 6. *If J is a (closed) ideal in R, then J is a (closed) ideal in R_+ and $R_+/J = (R/J)_+$.*

In the following result we use a method familiar from the theory of minimal topological groups (see, for example, [5]).

THEOREM 7. *Let R be a complete K-algebra such that (R, τ), (K, σ) are minimal topological rings. Then the K-unitalization R_+ is a minimal topological ring.*

PROOF. Denote by γ the given product topology on R_+ and suppose that $\gamma' \subseteq \gamma$ is a new Hausdorff ring topology. Since (R, τ) is a minimal ring, $\gamma'|_R = \gamma|_R = \tau$. By our assumption, (R, τ) is complete. Therefore, R is a closed ideal in (R_+, γ'). Consider the *Hausdorff* ring topology γ'/R on K. Since $\gamma'/R \subseteq \gamma/R = \sigma$ and (K, σ) is a minimal ring, then $\gamma'/R = \gamma/R$. By Merzon's Lemma we get $\gamma' = \gamma$. ■

COROLLARY 8. *Let R be a minimal complete ring with* $\mathrm{char}(R) = n > 0$. *Then the $\mathbb{Z}_n$-unitalization R_+ of R is a minimal ring.*

THEOREM 9. *Let R be a discrete ring with* $\mathrm{char}(R) = n > 0$. *Then the $\mathbb{Z}_n$-unitalization R_+ of R is a continuous ring retract of a minimal locally compact unital ring M_+.*

PROOF. Apply our construction for the situation $S = K = \mathbb{Z}_n$ and consider the $\mathbb{Z}_n$-algebra $M: = M_{\mathbb{Z}_n}(R, \mathbb{Z}_n) = R \times H_{\mathbb{Z}_n}(R, \mathbb{Z}_n) \times \mathbb{Z}_n$. Denote by M_+ the $\mathbb{Z}_n$-unitalization of M. Since $\mathrm{char}(R) = n > 0$, then every character $\xi: R \to \mathbb{T}$ can actually be considered as a restricted homomorphism $R \to \mathbb{Z}_n \subset \mathbb{T}$ identifying $\mathbb{Z}_n$ with the n-element cyclic subgroup of $\mathbb{T}$. It is also clear that every homomorphism $R \to \mathbb{Z}_n$ is even a morphism of $\mathbb{Z}_n$-algebras. Therefore, $H_{\mathbb{Z}_n}(R, \mathbb{Z}_n)$ and $R^* = H(R, \mathbb{T})$ coincide algebraically. Endow $H_{\mathbb{Z}_n}(R, \mathbb{Z}_n)$ with the compact topology σ of R^*. Eventually, the mapping $\omega: (R, \tau) \times (H_{\mathbb{Z}_n}(R, \mathbb{Z}_n), \sigma) \to \mathbb{Z}_n \subset \mathbb{T}$ is minimal, because of Lemma 2. By Proposition 4, the ring M is minimal. Since M is a $\mathbb{Z}_n$-algebra, then Corollary 8 and Lemma 6 complete the proof. ■

COROLLARY 10. *For every nonnegative integer n which is not equal to* 1 *there exists a minimal non-totally minimal separable metrizable locally compact unital ring with* $\mathrm{char}(R) = n$.

PROOF. Fix a natural number $n \geqslant 2$. Let $F_i = \mathbb{Z}_n$ for every $i \in \mathbb{N}$. Consider the topological ring product $\left(\prod_{i \in \mathbb{N}} F_i, \sigma\right)$ and the dense countable topological subring $\left(\sum_{i \in \mathbb{N}} F_i, \tau\right)$. Denote by τ_d the discrete topology on $R: = \sum_{i \in \mathbb{N}} F_i$. Clearly, the $\mathbb{Z}_n$-unitalization R_+ of (R, τ_d) is not a minimal ring because we can take on R_+ the (strictly coarser) ring topology of the $\mathbb{Z}_n$-unitalization for (R, τ). On the other hand, by Theorem 9, the discrete non-minimal ring R_+ is a continuous ring retract of a minimal ring M_+. Eventually, M_+ is the desired ring.

For the case $n = 0$, consider the ring product $\mathbb{R} \times M_+$, where M_+ is a minimal ring constructed for the case $n \geqslant 2$, and use the productivity of the class of minimal unital rings [3].

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